What do you want to See Covered/Not Covered Next Time I teach This
Please email me a list of which topics you would have wanted to see covered. You can list as many as you want, and you can give commentary.

VDW Topics

1. **Poly VDW Thm** For all \(p_1, \ldots, p_k \in \mathbb{Z}[x]\) such that \(p_i(0) = 0\) for all \(i \in [k]\), and \(c \in \mathbb{N}\), there exists \(W = W(p_1, \ldots, p_k; c)\) such that for all \(\text{COL}: [W] \to [c]\) there exists \(a, d\)

\[
a, a + p_1(d), \ldots, a + p_k(d) \text{ all the same color}
\]

Uses VDW.

2. **Can VDW** For all \(k\) there exists \(W = W(k)\) such that for any \(\text{COL}: [W] \to [\omega]\) there exists \(a, d\) such that either

\[
a, a + d, \ldots, a + (k - 1)d \text{ are all the same color}
\]

or

\[
a, a + d, \ldots, a + (k - 1)d \text{ are all different colors}
\]

Can VDW uses 2-dim VDW. Extends to \(a\)-dim VDW uses \(2a\)-dim VDW

That might be on a HW or midterm.

3. **LEGIT APPLICATION To Number Theory**

- For all \(k\) there exists \(p_o\) such that for all primes \(p \geq p_o\) there are \(k\) consecutive squares mod \(p\).
- For all \(k\) there exists \(p_o\) such that for all primes \(p \geq p_o\) there are \(k\) consecutive non-squares mod \(p\).

4. **Folkman’s Thm** For all \(k, c\) there exists \(N = N(k, c)\) such that for all \(\text{COL}: [N] \to [c]\) there exists \(x_1, \ldots, x_k\) such that ALL non-empty sums of the \(x_i\)’s are the same color.

Uses VDW.
5. **Hilbert’s Cube Lemma** For all $k, c$ there exists $H = H(k, c)$ such that for all $\text{COL}: [H] \rightarrow [c]$ there exists $x_0, x_1, \ldots, x_k$ such that
\[
\{x_0 + \sum_{i=1}^{k} b_ix_i : b_i \in \{0, 1\}\}
\]
is monochromatic.
Can prove from VDW or directly.

6. **LEGIT APP TO REAL MATH!** EARLIEST Ramseyian Theorem Ever! Use HCL to prove

**H Irreducibility Thm** (2 var case). If $p(x, y) \in \mathbb{Q}[x, y]$ is irred then there exists $a \in \mathbb{Z}$ such that $p(x, a) \in \mathbb{Q}[x]$ is irred.

7. **Roth’s Thm** Every set of upper positive density has a 3-AP.

There are three proofs of this: Combinatorial, Analytic, and Computer-assisted.

The idea of doing the Analytic Proof appeals to me since it would be a DIFFERENT proof technique then others that you seen. Non-Elementary proof, but not that hard.

8. **APPLICATION TO NUMBER THEORY** Schur’s Theorem is a special case or Rado’s Theorem.

**Schur’s Thm** For all $c$ there exists $S = S(c)$ such that for all $\text{COL}: [S] \rightarrow [c]$ there exists $x, y, z$ same color such that $x + y = z$.

**FLT** For all $n \geq 3$ there does not exists $x, y, z \in \mathbb{N}$ such that $x^n + y^n = z^n$. (The $n = 4$ case was done by Fermat.)

Thm (Schur’s Thm + FLT(4) implies there are an infinite number of primes.

9. **Rado’s Theorem Over the Reals: True? False? A Matter of taste?** The following are equivalent

- For all $\text{COL}: \mathbb{R} \rightarrow \mathbb{N}$ there exists mono $w, x, y, z$ with $w+x = y+z$.
- There is a cardinality between countable and the reals.
So the statement is Ind of ZFC.

10. **Hindman’s Thm** For any finite coloring of $\mathbb{N}$ there exists an infinite $A$ such that all finite sums of elements of $A$ are the same color.
   Proof uses Ultrafilters so would be a DIFFERENT proof.

11. **Gallai-Witt Thm** Multi-dim VDW. I can prove this without using Hales-Jewitt.

12. **Hales-Jewitt Thm** A Very General Thm from which we can derive cleanly VDW and Gallai-Witt.
Ramsey Topics

1. **Ramsey Theory With Other Graphs** \( R(C_k) \) is least \( n \) such that for all 2-coloring of \( \binom{\binom{n}{2}}{2} \) there exists monochromatic \( k \)-cycle.

\[
R(C_k) = \begin{cases} 
6 & \text{if } k = 3 \text{ or } k = 4 \\
2k - 1 & \text{if } k \geq 5 \text{ and } k \equiv 1 \pmod{2} \\
\frac{3k}{2} - 1 & \text{if } k \geq 4 \text{ and } k \equiv 0 \pmod{2}
\end{cases} \tag{1}
\]

Would need to read the proofs to see how interesting they are.

2. **Ramsey Games** Example: Parameter \( k, n \). Players RED and BLUE alternate coloring the edges of \( K_n \). RED goes first. The first player to get a \( C_k \) in their color wins. For which \( n \) does RED have a winning strategy? Active Research.

3. **Thm** \( R_3(k) \leq 2^{24k} \).

Better is known:

**Thm** \( R_3(k) \leq 2^{22k} \).

Would need to reread the proofs to see if I really could do it.

4. **Thm** \( CR(k) \leq 2^{O(k^2 \log k)} \).

Proof is Mileti-style.

5. **Large Can Ramsey** For all \( k \) there exists \( n = n(k) \) such that for all \( \text{COL}: \binom{\binom{n}{2}}{a} \to [\omega] \) there is a large set that is either homog, min-homog, max-homog, rainbow.

I have a general theorem that has inf-can, fin-can, large-can as corollaries. Its a mild reworking of the proof of Can Ramsey using 4-hypergraph Ramsey.

6. **\( a \)-ary Can Ramsey** Thm For all \( a, k \in \mathbb{N} \) there exist \( C = C(a, k) \) such that for all \( \text{COL}: \binom{[\binom{k}{2}]}{a} \to [\omega] \) there exists a set \( H, |H| = k \) and \( 1 \leq i_1 < \cdots < i_L \leq a \) such that for all \( p_1 < \cdots < p_a \in H \) and \( q_1 < \cdots < q_a \in H \)

\[
\text{COL}(p_1, \ldots, p_a) = \text{COL}(q_1, \ldots, q_a) \text{ iff } (p_{i_1}, \ldots, p_{i_L}) = (q_{i_1}, \ldots, q_{i_L})
\]

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Similar to the proof on graphs, but messier. Some interesting upper bounds for \( a = 3 \) case, might just do that.

7. **Euclidean Ramsey Theory Sample Theorems**

- If you \( c \)-color the plane there exists 2 points an inch apart same color. True for \( c = 2, 3, 4 \) (result for 4 is recent and computer-assisted). False for \( c = 7 \). Lots of literature on this.

- Let \( T \) be a triangle with a 30, 90, or 150 degree angle. For every 2-coloring of \( \mathbb{R}^2 \) there exists three points that form triangle \( T \) (note-actually form \( T \), not just similar to \( T \)) that are monochromatic.

8. **Ramsey Multiplicity**

**Thm** For all 2-col of \( K_n \), exists \( \frac{n^3}{24} - O(n^2) \) mono \( K_3 \)'s.

This is the first thm in a field called **Ramsey Multiplicity**

Here is the second thm

**Thm** Fix \( k \). For large \( n \), for all 2-colorings of \( K_n \) there exists \( \frac{n^2}{4^{2(k+o(1))}} \) mono \( K_k \)'s.
1. **Def** Let $L$ be a language. Game:
   - Alice is Poly time and she has $x$, $|x| = n$.
   - Bob is all powerful and he has nothing.
   - They cooperate to determine if $x \in L$. Alice could just send Bob $x$. That takes $n$ bits.

   Let $L$ be the set of all 3-colorable graphs (or any NPC graph problem). Note size is $O(n^2)$. If there is a protocol in $O(n^2 - \epsilon)$ bits then $PH = \Sigma_p^2$.

   Proof used large 3-free set.

2. **Thm** For every computable $\text{COL}: \binom{N}{2} \to [2]$ there is a $\Pi_2$-homogenous set. There is a computable coloring such that no homog set is $\Sigma_2$.

   Could do the direction: Given a computable coloring there is a $\Pi_2$-homog set.

   Really could not do the construction of a coloring, takes us too far afield.

3. **Def** $G \to (H_1, H_2)$ means that for every 2-coloring of the edges of $G$ there is either a RED $H_1$ or a BLUE $H_2$.

   Marcus Schaefer proved the following.

   Thm $\{(G, H_1, H_2) : G \to (H_1, H_2)\}$ is $\Pi_p^2$-complete.

4. **Grid Color Extension (GCE)** is the set of tuples $(n, m, c, \chi)$ such that the following hold:

   - $n, m, c \in \mathbb{N}$. $\chi$ is a partial $c$-coloring of $[n] \times [m]$ that is rectangle-free.
   - $\chi$ can be extended to a rectangle-free coloring of $[n] \times [m]$.

   **Thm** $GCE$ is NP-complete

5. **Def** Resolution proofs are a proof system to show that a Boolean Formula is NOT satisfiable. It is of interest to find a class of non-satisfiable formulas $\phi_n$ that require (say) $(1.5)^n$ long Res Proofs.
Def A graph is $c$-random if it does not contain a clique or ind set of size $c \log n$.

Def $\phi_{n,c}$ is a Boolean Formula that says every graph on $n$ vertices is $c$-random. (This is false for $c$ around $\frac{1}{2}$.)

Lauria, Pudlak, Rodl, Thapen proved:

Thm For appropriate $c$, any resolution proof for $\phi_{n,c}$ requires length $n^{\Omega(\log n)}$.

6. Def If for all COL: $\binom{\kappa}{2}$ there is a homog set of size $\kappa$ then $\kappa$ is Ramsey.

I could look into this and see what other theorems of interest follow from the existence of Ramsey Cardinals.

7. Borel Colorings Restrict the type of coloring of the reals and you can get some results.