An Application of Ramsey’s Theorem to Logic

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The Language of Graphs

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**Example**

$$(\exists x)(\forall y)[x \neq y \implies E(x, y)]$$

There is a vertex $x$ that has an edge to EVERY other vertex. For all $n \geq 1$ there is $G$ with $n$ vertex that satisfies this sentence.

**Example**

$$(\exists x_1, x_2)(\forall y)[y \neq x_1 \land y \neq x_2 \implies (E(x_1, y) \land \neg E(x_2, y))]$$

There are $x_1, x_2$ such that $x_1$ connects to EVERY other vertex, and $x_2$ connects to NO other vertex. For all $n \geq 2$ there is $G$ with $n$ vertex that satisfies this sentence.
The Language of Graphs

Our logic has only one predicate: $E$ for edge. We will assume $E$ is symmetric and not reflexive.

**Example**

$$(\exists x) (\forall y) [x \neq y \implies E(x, y)]$$

There is a vertex $x$ that has an edge to EVERY other vertex.
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Conventions

1. The graphs are symmetric. So $E(x, y)$ really means $E(x, y) \land E(y, x)$.
2. No self loops, so $E(x, x)$ is always false.
3. $(\exists x_1) \cdots (\exists x_n)$ means they are DISTINCT.
4. $(\forall x_1) \cdots (\forall x_n)$ means they are DISTINCT.
Notation If $G$ is a graph and $\phi$ is a sentence then $G \models \phi$ means that $\phi$ is TRUE of $G$.

Definition If $\phi$ is a sentence in the language of graphs then $\text{spec}(\phi)$ is the set of all $n$ such that there is $G$ on $n$ vertices such that $G \models \phi$. 
Spectrum: Examples

\[ \phi = (\exists x_1, x_2, x_3)[E(x_1, x_2) \land E(x_1, x_3)] \]
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\[ \text{spec}(\phi) = \{3, 4, 5, \ldots\} \]
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\[ \phi = (\forall x)(\exists y \neq x)[E(x, y) \land (\forall z \neq y)[\neg E(x, z)]] \]  
Discuss

Is there a graph where \( \phi \) on 0 vertex?

YES.

Vacuously

Is there a graph where \( \phi \) on 1 vertex?

NO.

Discuss

Is there a graph where \( \phi \) on 2 vertex?

YES.

Discuss

Is there a graph where \( \phi \) on 3 vertex?

NO.
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Is there a graph where $\phi$ on 0 vertex? YES. Vacuously
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Is there a graph where \( \phi \) on 1 vertex?  NO.  Discuss
Is there a graph where \( \phi \) on 2 vertex?  YES.  Discuss
Is there a graph where \( \phi \) on 3 vertex?  NO.  Discuss
\[ \text{spec}(\phi) = \{0, 2, 4, 6, \ldots, \} \]
Spectrum: Another Example

\[(\forall x_1, x_2, x_3) \]
\[\neg (E(x_1, x_2) \land E(x_1, x_3) \land E(x_2, x_3))\]
\[\land\]
\[\neg (\neg E(x_1, x_2) \land \neg E(x_1, x_3) \land \neg E(x_2, x_3))\]

Discuss
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\[(\forall x_1, x_2, x_3)\]
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Discuss

This is asking for a graph without a 3-clique or 3-ind set. By Ramsey’s Theorem we know that all graphs of size \(\geq 6\) have a 3-clique or 3-ind set.

\(\text{spec}(\phi) = \{0, 1, 2, 3, 4, 5\}\).
Spectrum: Another Example

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\( \text{spec}(\phi) = \{0, 1, 2, 3, 4, 5\} \).

This is NOT the application of Ramsey Theory that this lecture is leading up to.
Spectrum: More Examples

\[ \phi = (\forall x)(\forall y)[E(x, y)]. \]
φ = (∀x)(∀y)[E(x, y)]. Discuss.
Spectrum: More Examples

\[ \phi = (\forall x)(\forall y)[E(x, y)]. \] Discuss.
For all \( n \), \( K_n \models \phi \). Hence, \( \text{spec}(\phi) = \mathbb{N} \).
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\[ \phi = (\forall x)(\forall y)[E(x, y)]. \text{ Discuss.} \]
For all \( n \), \( K_n \models \phi \). Hence, \( \text{spec}(\phi) = \mathbb{N} \).

\[ \phi = (\exists x, y, z)\left(\forall w \notin \{x, y, z\}\right)\left[E(w, x) \land E(w, y) \land E(w, z)\right]. \]
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Discuss.
For all \( n \geq 0 \) \( K_{n,3} \models \phi. \) If \( G \) has 0,1, or 2 vertices, \( G \not\models \phi. \)
\( \text{spec}(\phi) = \{3, 4, 5, \ldots, \}. \)
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\[ \phi = (\exists x_1)(\exists x_2)(\forall y)[x_1 = y \lor x_2 = y]. \]
Spectrum: More Examples

$\phi = (\forall x)(\forall y)[E(x, y)]$. Discuss.
For all $n$, $K_n \models \phi$. Hence, $\text{spec}(\phi) = \mathbb{N}$.

$\phi = (\exists x, y, z)(\forall w \notin \{x, y, z\})[E(w, x) \land E(w, y) \land E(w, z)]$.
Discuss.
For all $n \geq 0$ $K_{n,3} \models \phi$. If $G$ has 0,1, or 2 vertices, $G \not\models \phi$.
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\[ \phi = (\exists x_1)(\exists x_2)(\forall y)[x_1 = y \lor x_2 = y]. \] Discuss.

If \( G \) has 2 vertices then \( G \models \phi \).

If \( n \in \{0, 1, 3, 4, 5, \ldots\} \) and \( G \) has \( n \) vertices then \( G \not\models \phi \).

\( \text{spec}(\phi) = \{2\} \).
Note how Simple Those Spectrum’s Were

\[ \phi = (\forall x)(\forall y)[E(x, y)]. \]
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All of these sentence were of the form \((E^*A^*\)-sentences).\)
\[ (\exists x_1) \cdots (\exists x_n)(\forall y_1) \cdots (\forall y_m)[\psi(x_1, \ldots, x_n, y_1, y \ldots, y_m)] \]
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All of these sentence \text{spec} was finite or cofinite.
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Or is there a Theorem? Will Erika say Use Ramsey Theory?
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(\exists x_1) \cdots (\exists x_n)(\forall y_1) \cdots (\forall y_m)[\psi(x_1, \ldots, x_n, y_1, y \ldots, y_m)]
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All of these sentence spec was finite or cofinite. Coincidence? Or is there a Theorem? Will Erika say Use Ramsey Theory? Not going to bother with a vote—Always bet on Erika.
Lemma About Decidable We Need

Lemma

1. The following is decidable: Given a sentence $\phi$ and a graph $G$, determine if $G \models \phi$.

2. The following is decidable: Given a sentence $\phi$ and a number $n$, determine if $n \in \text{spec}(\phi)$.

Proof Use brute force.

We will use Lemma without comment.

Note For many $(\phi, G)$ can do much better than brute force.
Main Theorem

**Theorem** The following function is computable: Given $\phi$, an $E^*A^*$ sentence in the theory of graphs, output $\text{spec}(\phi)$. ($\text{spec}(\phi)$ will be a finite or cofinite set; hence it will have an easy description.)
Main Theorem

**Theorem** The following function is computable: Given $\phi$, an $E^*A^*$ sentence in the theory of graphs, output $\text{spec}(\phi)$. ($\text{spec}(\phi)$ will be a finite or cofinite set; hence it will have an easy description.)

We will take $\phi$ to be

$$(\exists x_1) \cdots (\exists x_n)(\forall y_1) \cdots (\forall y_m)[\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)]$$
Claim 1

Let $G \models \phi$ with witnesses $v_1, \ldots, v_n$. Let $H$ be an induced subgraph of $G$ that contains $v_1, \ldots, v_n$. Then $H \models \phi$. 
Claim 1

Let $G \models \phi$ with witnesses $v_1, \ldots, v_n$. Let $H$ be an induced subgraph of $G$ that contains $v_1, \ldots, v_n$. Then $H \models \phi$.

Proof of Claim 1 Let $G = (V, E)$ and $H = (V', E')$ where $V' \subseteq V$. Since $G \models \phi$

\[ G \models (\forall y_1 \in V) \cdots (\forall y_m \in V)[\psi(v_1, \ldots, v_n, y_1, \ldots, y_m)] \]
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$$(\forall y_1 \in V') \cdots (\forall y_m \in V')[\psi(v_1, \ldots, v_n, y_1, \ldots, y_m)], \text{ SO}$$

$H \models (\forall y_1) \cdots (\forall y_m)[\psi(v_1, \ldots, v_n, y_1, \ldots, y_m)]$

End of Proof of Claim 1
Claim 2, The Main Claim

If \( (\exists N \geq n + 2^n R(m))[N \in \text{spec}(\phi)] \) then

\[
\{n + m, \ldots, n + 2^n R(m), \ldots\} \subseteq \text{spec}(\phi).
\]

Proof of Claim 2
Since \( N \in \text{spec}(\phi) \) there exists \( G = (V, E) \), a graph on \( N \) vertices such that \( G \models \phi \). Let \( v_1, \ldots, v_n \) be such that:

\[
(\forall y_1) \cdots (\forall y_m)[\psi(v_1, \ldots, v_n, y_1, \ldots, y_m)].
\]

(Proof continued on next slide)
Proof of Claim 2 Continued

\[(∀y_1)\cdots(∀y_m)[ψ(v_1,\ldots,v_n,y_1,\ldots,y_m)].\]

Let \(X = \{v_1,\ldots,v_n\}\) and \(U = V - X\). Note that \(|U| \geq 2^n R(m)\).

We define a \(2^n\)-Coloring of \(U\). \(u \in U\) maps to \((b_1,\ldots,b_n)\):

\[b_i = \begin{cases} 
0 & \text{if } (u,v_i) \notin E \\
1 & \text{if } (u,v_i) \in E 
\end{cases} \quad (1)\]

Hence every \(u \in U\) is mapped to a description of how it relates to every element in \(X\). Since \(|U| \geq 2^n R(m)\) there exists \(R(m)\) vertices that map to the same vector. Apply Ramsey’s Theorem to these \(R(m)\) vertices to obtain homog set \(u_1,\ldots,u_m\).

(Proof continued on next slide)
Either the $u_i$’s form a clique or the $u_i$’s form an ind. set. We will assume the $u_i$’s form a clique (the other case is similar).

All of the $u_i$’s map to the same vector. Hence they all look the same to $v_1, \ldots, v_n$.

**Example** All $u_i$ have edge to $\{v_1, v_3, v_{17}\}$ but no other $v_j$.

Let $H_0$ be $G$ restricted to $X \cup \{u_1, \ldots, u_m\}$. By Claim 1 $H_0 \models \phi$.

For every $p \geq 1$ we form a graph $H_p = (V_p, E_p)$ on $n + m + p$ vertices such that $H_p \models \phi$:

**Informally** add $m + p$ vertices that are just like the $u_i$’s.

**Formally** Next Slide.
Proof of Claim 2 Continued, Formal $H_p = (V_p, E_p)$

- $V_p = X \cup \{u_1, \ldots, u_m, u_{m+1}, \ldots, u_{m+p}\}$ where $u_{m+1}, \ldots, u_{m+p}$ are new vertices.
- $E_p$ is the union of the following edges.
  - The edges in $H_0$,
  - Make $\{u_1, \ldots, u_{m+p}\}$ form a clique.
  - Let $(b_1, \ldots, b_n)$ be the vector that all of the elements of $\{u_1, \ldots, u_m\}$ mapped to. For $m + 1 \leq j \leq m + p$, for $1 \leq i \leq m$ such that $b_i = 1$, put an edge between $u_j$ and $v_i$.

**Example** All of the $u_j$’s have a edge to $\{v_1, v_3, v_{17}\}$ but nothing else.

$X$ sees all of the $u_1, \ldots, u_{m+p}$ as the same. Hence any subset of the $\{u_1, \ldots, u_{m+p}\}$ of size $m$ looks the same to $X$ and to the other $u_i$’s. Hence $H_p \models \phi$, so $n + m + p \in \text{spec}(\phi)$.

**End of Proof of Claim 2**
Claim 3

\[ \phi = (\exists x_1) \cdots (\exists x_n)(\forall y_1) \cdots (\forall y_m)[\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)]. \]

\[ N_0 = n + 2^n R(m). \]

\[ N_0 \notin \text{spec}(\phi) \implies \text{spec}(\phi) \subseteq \{0, \ldots, N_0 - 1\}. \]

**Proof of Claim 3**

By Claim 2

\[ \{N_0, \ldots\} \cap \text{spec}(\phi) \neq \emptyset \implies \{n + m, \ldots, N_0, \ldots\} \subseteq \text{spec}(\phi). \]

We take the contrapositive with a weaker premise.

\[ N_0 \notin \text{spec}(\phi) \implies \{N_0, \ldots\} \cap \text{spec}(\phi) = \emptyset \]

\[ \implies \text{spec}(\phi) \subseteq \{0, \ldots, N_0 - 1\}. \]

**End of Proof of Claim 3**
Recap Both Claims

We put a subcase of Claim 2, and Claim 3, next to each other to recap what we know.
Let $N_0 = n + 2^n R(m)$. 
Recap Both Claims

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Let $N_0 = n + 2^n R(m)$.

Claim 2

If $N_0 \in \text{spec}(\phi)$ then $\{n + m, \ldots,\} \subseteq \text{spec}(\phi)$. 
Recap Both Claims

We put a subcase of Claim 2, and Claim 3, next to each other to recap what we know.
Let $N_0 = n + 2^n R(m)$.

Claim 2
If $N_0 \in \text{spec}(\phi)$ then $\{n + m, \ldots, \} \subseteq \text{spec}(\phi)$.

Claim 3
If $N_0 \notin \text{spec}(\phi)$ then $\text{spec}(\phi) \subseteq \{0, \ldots, N_0 - 1\}$.
Algorithm for Finding $\text{spec}(\phi)$

1. Input

$$\phi = (\exists x_1) \cdots (\exists x_n)(\forall y_1) \cdots (\forall y_m)[\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)].$$

2. Let $N_0 = n + 2^n R(m)$. Determine if $N_0 \in \text{spec}(\phi)$.
**Algorithm for Finding** $\text{spec}(\phi)$

1. **Input**

   \[
   \phi = (\exists x_1) \cdots (\exists x_n)(\forall y_1) \cdots (\forall y_m)[\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)].
   \]

2. **Let** $N_0 = n + 2^n R(m)$. **Determine if** $N_0 \in \text{spec}(\phi)$.

   2.1 **If YES** then by Claim 2 $\{n + m, \ldots\} \subseteq \text{spec}(\phi)$.
   
   For $0 \leq i \leq n + m - 1$ test if $i \in \text{spec}(\phi)$. You now know $\text{spec}(\phi)$ which is co-finite. Output it.

   2.2 **If NO** then, by Claim 3 $\text{spec}(\phi) \subseteq \{0, \ldots, N_0 - 1\}$.
   
   For $0 \leq i \leq N_0 - 1$ test if $i \in \text{spec}(\phi)$. You now know $\text{spec}(\phi)$ which is finite set. Output it.

**End of Proof of Main Theorem**
Algorithm for Finding $\text{spec}(\phi)$

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$$\phi = (\exists x_1) \cdots (\exists x_n)(\forall y_1) \cdots (\forall y_m)[\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)].$$

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End of Proof of Main Theorem
What other Sentences could we look at? $E^* A^*$ sentences with more complicated objects than graphs.

1. **Colored Graphs** $c$ kinds of edges.
2. **$a$-ary Hypergraphs** $a$-ary Hyperedges.
3. **Colored $a$-ary Hypergraphs** $c$ kinds of $a$-ary Hyperedges.
4. **$\leq a$-ary Hypergraphs** all arities $\leq a$ allowed.
5. **Colored $\leq a$-ary Hypergraphs** $c_i$ colors for the $i$-arity sets.

Discuss for which of these is $\text{spec}$ decidable.

Is $\text{spec}$ for colored $\leq a$-hypergraphs decidable? Vote. YES-and will be on HW, YES-but hard, NO, Unknown to Science.

Key ingredient already was on the midterm: Ramsey theory on $\leq a$-hypergraphs.
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Other Sentences. Part I

What other Sentences could we look at? $E^* A^*$ sentences with more complicated objects than graphs.

1. **Colored Graphs** $c$ kinds of edges.
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Is spec for colored $\leq a$-hypergraphs decidable? Vote.
YES-and will be on HW, YES-but hard, NO, Unknown to Science.

YES-and will be on HW or Final.
**Key ingredient** already was on the midterm: Ramsey theory on $\leq a$-hypergraphs.
(E* A*)*-sentences, only predicate E(x, y). Nick sentences.
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**Is spec** for Nick Sentences decidable? **Vote**
YES, NO, Unknown to Science.
(\(E^*A^*\))^*-sentences, only predicate \(E(x, y)\). Nick sentences.

Is \(spec\) for Nick Sentences decidable? **Vote**
YES, NO, Unknown to Science. YES

**Known** If \(\phi\) is a Nick sentence then \(spec(\phi)\) is a union of AP’s (called a semi-linear set) OR the complement of such (proof is hard). So for example

\[
\{4, 7, 10, \ldots\} \cup \{11, 22, 33, \ldots\}\]

is a Semi-linear Set

**Known** If \(A\) is semi-linear then there exists \(\phi\) with \(spec(\phi) = A\). Will be on HW or final.
Other Sentences. Part III

\((E^* A^*)^*-\)sentences, predicates of arity \(\leq a\)-ary. McKenzie sentences.

Is \textit{spec} for McKenzie Sentences decidable? \textbf{Vote}.

\textbf{YES, NO, Unknown to Science.}
(E*A*)*-sentences, predicates of arity ≤ a-ary. McKenzie sentences.

Is \text{spec} for McKenzie Sentences decidable? \textbf{Vote}.
YES, NO, Unknown to Science. YES.
(E^* A^*)^*-sentences, predicates of arity $\leq a$-ary. McKenzie sentences.

Is $\text{spec}$ for McKenzie Sentences decidable? **Vote.**

YES, NO, Unknown to Science. **YES.**

**Known** If $\phi$ is a Mackenzie sentence then $\text{spec}(\phi) \in \text{EXPTIME}$. 
(\(E^*A^*\))^*-sentences, predicates of arity \(\leq a\)-ary. McKenzie sentences.

Is \text{spec} for McKenzie Sentences decidable? \textbf{Vote}. YES, NO, Unknown to Science. YES.

\textbf{Known} If \(\phi\) is a Mackenzie sentence then \(\text{spec}(\phi) \in \text{EXPTIME}\).

\textbf{Also Known} If \(A \in \text{EXPTIME}\) then there exists Mackenzie \(\phi\) such that \(\text{spec}(\phi) = A\).
App, “App”, or ““App””

App This was not a problem people came up with to find an app of Ramsey’s Theorem. Ramsey was working on this problem in logic and proved Ramsey’s Theorem to help him solve it. So the question in Logic is legit.
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**“App”** While origin is legit, do we care now? I do, and my advisor Harry Lewis does (I have been in email contact with him about this lecture and he gave me several pointers and facts) but do YOU care?
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““App”” This would be unfair. I reserve the 4-quotes if either NOBODY cares or ONLY I care. (When I prove primes are infinite FROM van Der Waerden’s Theorem, feel free to use 4 quotes. I am not kidding.)

Vote App OR “App” OR ““App””