An Application of Ramsey's Theorem to Logic

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There are x_1, x_2 such that x_1 connects to EVERY other vertex, and x_2 connects to NO other vertex.
For all $n \geq 2$ there is G with n vertex that satisfies this sentence.

Conventions

1. The graphs are symmetric. So $E(x, y)$ really means $E(x, y) \wedge E(y, x)$.
2. No self loops, so $E(x, x)$ is always false.
3. $(\exists x_1) \cdots (\exists x_n)$ means they are DISTINCT.
4. $(\forall x_1) \cdots (\forall x_n)$ means they are DISTINCT.

Spectrum: Examples

Notation If G is a graph and ϕ is a sentence then $G \models \phi$ means that ϕ is TRUE of G.

Definition If ϕ is a sentence in the language of graphs then $\mathrm{spec}(\phi)$ is the set of all n such that there is G on n vertices such that $G \models \phi$.

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$\mathrm{spec}(\phi) = \{0, 2, 4, 6, \ldots,\}$

Spectrum: Another Example

$(\forall x_1, x_2, x_3)$

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$\neg(E(x_1, x_2) \wedge E(x_1, x_3) \wedge E(x_2, x_3))$

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This is asking for a graph without a 3-clique or 3-ind set. By Ramsey's Theorem we know that all graphs of size ≥ 6 have a 3-clique or 3-ind set.
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This is NOT the application of Ramsey Theory that this lecture is leading up to.

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If G has 2 vertices then $G \models \phi$.
If $n \in \{0,1,3,4,5,\ldots\}$ and G has n vertices then $G \not\models \phi$.
$\mathrm{spec}(\phi) = \{2\}$.

Note how Simple Those Spectrum's Were

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Not going to bother with a vote—Always bet on Erika.

Lemma About Decidable We Need

Lemma

1. The following is decidable: Given a sentence ϕ and a graph G, determine if $G \models \phi$.
2. The following is decidable: Given a sentence ϕ and a number n, determine if $n \in \mathrm{spec}(\phi)$.

Proof Use brute force.

We will use Lemma without comment.

Note For many (ϕ, G) can do much better than brute force.

Main Theorem

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We will take ϕ to be

$$(\exists x_1)\cdots(\exists x_n)(\forall y_1)\cdots(\forall y_m)[\psi(x_1,\ldots,x_n,y_1,\ldots,y_m)]$$

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Proof of Claim 1 Let $G = (V, E)$ and $H = (V', E')$ where $V' \subseteq V$. Since $G \models \phi$

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H is just G with less vertices, and the vertices that remain have the same edges. And $v_1, \ldots, v_n$ are in H. Hence we DO have $(\forall y_1 \in V') \cdots (\forall y_m \in V')[\psi(v_1, \ldots, v_n, y_1, \ldots, y_m)]$, SO
$H \models (\forall y_1) \cdots (\forall y_m)[\psi(v_1, \ldots, v_n, y_1, \ldots, y_m)]$
End of Proof of Claim 1

Claim 2, The Main Claim

If $(\exists N \ge n + 2^n R(m))[N \in \mathrm{spec}(\phi)]$ then

$$\{n+m, \ldots, n+2^n R(m), \ldots\} \subseteq \mathrm{spec}(\phi).$$

Proof of Claim 2

Since $N \in \mathrm{spec}(\phi)$ there exists $G = (V, E)$, a graph on N vertices such that $G \models \phi$. Let $v_1, \ldots, v_n$ be such that:

$$(\forall y_1) \cdots (\forall y_m)[\psi(v_1, \ldots, v_n, y_1, \ldots, y_m)].$$

(Proof continued on next slide)

Proof of Claim 2 Continued

$$(\forall y_1)\cdots(\forall y_m)[\psi(v_1,\ldots,v_n,y_1,\ldots,y_m)].$$

Let $X = \{v_1,\ldots,v_n\}$ and $U = V - X$. Note that $|U| \geq 2^n R(m)$.
We define a 2^n-Coloring of U. $u \in U$ maps to $(b_1,\ldots,b_n)$:

$$b_i = \begin{cases} 0 \text{ if } (u,v_i) \notin E \\ 1 \text{ if } (u,v_i) \in E \end{cases} \tag{1}$$

Hence every $u \in U$ is mapped to a description of how it relates to every element in X. Since $|U| \geq 2^n R(m)$ there exists $R(m)$ vertices that map to the same vector. Apply Ramsey's Theorem to these $R(m)$ vertices to obtain homog set $u_1,\ldots,u_m$.
(Proof continued on next slide)

Proof of Claim 2 Continued

- Either the u_i's form a clique or the u_i's form an ind. set. We will assume the u_i's form a clique (the other case is similar).
- All of the u_i's map to the same vector. Hence they all look the same to $v_1, \ldots, v_n$.

Example All u_i have edge to $\{v_1, v_3, v_{17}\}$ but no other v_j.

Let H_0 be G restricted to $X \cup \{u_1, \ldots, u_m\}$. By Claim 1 $H_0 \models \phi$.
For every $p \geq 1$ we form a graph $H_p = (V_p, E_p)$ on $n + m + p$ vertices such that $H_p \models \phi$:
Informally add $m + p$ vertices that are **just like the u_i's**.
Formally Next Slide.

Proof of Claim 2 Continued, Formal $H_p = (V_p, E_p)$

- $V_p = X \cup \{u_1, \ldots, u_m, u_{m+1}, \ldots, u_{m+p}\}$ where $u_{m+1}, \ldots, u_{m+p}$ are new vertices.
- E_p is the union of the following edges.
  - The edges in H_0,
  - Make $\{u_1, \ldots, u_{m+p}\}$ form a clique.
  - Let $(b_1, \ldots, b_n)$ be the vector that all of the elements of $\{u_1, \ldots, u_m\}$ mapped to. For $m+1 \le j \le m+p$, for $1 \le i \le m$ such that $b_i = 1$, put an edge between u_j and v_i. **Example** All of the u_j's have a edge to $\{v_1, v_3, v_{17}\}$ but nothing else.

X sees all of the $u_1, \ldots, u_{m+p}$ as the same. Hence any subset of the $\{u_1, \ldots, u_{m+p}\}$ of size m looks the same to X and to the other u_i's. Hence $H_p \models \phi$, so $n+m+p \in \mathrm{spec}(\phi)$.

End of Proof of Claim 2

Claim 3

$\phi = (\exists x_1)\cdots(\exists x_n)(\forall y_1)\cdots(\forall y_m)[\psi(x_1,\ldots,x_n,y_1,\ldots,y_m)]$.
$N_0 = n + 2^n R(m)$.
$N_0 \notin \mathrm{spec}(\phi) \implies \mathrm{spec}(\phi) \subseteq \{0,\ldots,N_0-1\}$.

Proof of Claim 3

By Claim 2
$\{N_0,\ldots\} \cap \mathrm{spec}(\phi) \neq \emptyset \implies \{n+m,\ldots,N_0,\ldots\} \subseteq \mathrm{spec}(\phi)$.
We take the contrapositive with a weaker premise.

$$N_0 \notin \mathrm{spec}(\phi) \implies \{N_0,\ldots\} \cap \mathrm{spec}(\phi) = \emptyset$$

$$\implies \mathrm{spec}(\phi) \subseteq \{0,\ldots,N_0-1\}.$$

End of Proof of Claim 3

Recap Both Claims

We put a subcase of Claim 2, and Claim 3, next to each other to recap what we know.
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Algorithm for Finding $\mathrm{spec}(\phi)$

1. Input

$$\phi = (\exists x_1) \cdots (\exists x_n)(\forall y_1) \cdots (\forall y_m)[\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)].$$

2. Let $N_0 = n + 2^n R(m)$. Determine if $N_0 \in \mathrm{spec}(\phi)$.

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   2.1 If YES then by Claim 2 $\{n + m, \ldots\} \subseteq \mathrm{spec}(\phi)$.
   For $0 \le i \le n + m - 1$ test if $i \in \mathrm{spec}(\phi)$. You now know $\mathrm{spec}(\phi)$ which is co-finite. Output it.

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   For $0 \le i \le n+m-1$ test if $i \in \mathrm{spec}(\phi)$. You now know $\mathrm{spec}(\phi)$ which is co-finite. Output it.

   2.2 If NO then, by Claim 3 $\mathrm{spec}(\phi) \subseteq \{0,\ldots,N_0-1\}$.
   For $0 \le i \le N_0 - 1$ test if $i \in \mathrm{spec}(\phi)$. You now know $\mathrm{spec}(\phi)$ which is finite set. Output it.

End of Proof of Main Theorem

Other Sentences. Part I

What other Sentences could we look at?
E^*A^* sentences with more complicated objects than graphs.

1. **Colored Graphs** c kinds of edges.
2. **a-ary Hypergraphs** a-ary Hyperedges.
3. **Colored a-ary Hypergraphs** c kinds of a-ary Hyperedges.
4. **$\le a$-ary Hypergraphs** all arities $\le a$ allowed.
5. **Colored $\le a$-ary Hypergraphs** c_i colors for the i-arity sets.

Discuss for which of these is spec decidable.

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Is spec for colored $\le a$-hypergraphs decidable? Vote.
YES-and will be on HW, YES-but hard, NO, Unknown to Science.

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YES-and will be on HW or Final.
Key ingredient already was on the midterm: Ramsey theory on $\le a$-hypergraphs.

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$(E^*A^*)^*$-sentences, only predicate $E(x, y)$. **Nick** sentences.

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$(E^*A^*)^*$-sentences, only predicate $E(x, y)$. **Nick** sentences.

Is spec for Nick Sentences decidable? **Vote**
YES, NO, Unknown to Science. YES
Known If ϕ is a Nick sentence then $\mathrm{spec}(\phi)$ is a union of AP's (called a semi-linear ser) OR the complement of such (proof is hard). So for example

$$\{4, 7, 10, \ldots\} \cup \{11, 22, 33, \ldots\} \text{ is a Semi-linear Set}$$

Known If A is semi-linear then there exists ϕ with $\mathrm{spec}(\phi) = A$. Will be on HW or final.

Other Sentences. Part III

$(E^*A^*)^*$-sentences, predicates of arity $\leq a$-ary. **McKenzie** sentences.
Is spec for McKenzie Sentences decidable? **Vote**.
YES, NO, Unknown to Science.

Other Sentences. Part III

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Is spec for McKenzie Sentences decidable? **Vote**.

YES, NO, Unknown to Science.YES.

Other Sentences. Part III

$(E^*A^*)^*$-sentences, predicates of arity $\leq a$-ary. **McKenzie** sentences.

Is spec for McKenzie Sentences decidable? **Vote**.

YES, NO, Unknown to Science. YES.

Known If ϕ is a Mackenzie sentence then $\mathrm{spec}(\phi) \in \mathit{EXPTIME}$.

Other Sentences. Part III

$(E^*A^*)^*$-sentences, predicates of arity $\leq a$-ary. **McKenzie** sentences.
Is spec for McKenzie Sentences decidable? **Vote**.
YES, NO, Unknown to Science.YES.
Known If ϕ is a Mackenzie sentence then $\mathrm{spec}(\phi) \in \mathit{EXPTIME}$.
Also Known If $A \in \mathit{EXPTIME}$ then there exists Mackenzie ϕ such that $\mathrm{spec}(\phi) = A$.

App, “App”, or ““App””

App This was **not** a problem people came up with to find an app of Ramsey’s Theorem. Ramsey was working on this problem in logic and proved Ramsey’s Theorem to help him solve it. So the question in Logic is legit.

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"App" While origin is legit, do we care now? I do, and my advisor Harry Lewis does (I have been in email contact with him about this lecture and he gave me several pointers and facts) but do YOU care?

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""App"" This would be unfair. I reserve the 4-quotes if either NOBODY cares or ONLY I care. (When I prove primes are infinite FROM van Der Waerden's Theorem, feel free to use 4 quotes. I am not kidding.)

Vote App OR "App" OR ""App""