Rado's Theorem

Exposition by William Gasarch

June 19, 2020

VDW and Extended VDW

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$A, A+D, \ldots, A+(k-1)D$ are color CCC. So $COL(D) \neq CCC$.
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$A, A+\frac{XD}{k-1}, A+\frac{2XD}{k-1}, \ldots, A+\frac{(k-1)XD}{k-1}$. So $COL(\frac{XD}{k-1}) \neq CCC$.

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Real EVDW

What I presented above is NOT the EVDW. This is:
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This is an exercise. It might be on a HW or the Final.

Notation

For this talk

$$\mathbb{N} = \{1, 2, 3, \ldots, \}$$

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Take $x = a$, $y = d$, $z = a + d$.

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$w = 5d$

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So $E = EVDW(3, 5, c)$.

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Thm For all c there exists S such that for all $\mathrm{COL}\colon [S] \to [c]$ $(\exists w, x, y, z$ mono and distinct) with $w + 2x + 3y = 5z$.

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So $E = EVDW(6, 1, c)$.

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$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$

Case $e_1 < e_2, e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod{5}$

$$a_1 + 2a_2 = 4a_3$$

$$5^{e_1} b_1 + 2 \times 5^{e_2} b_2 = 4 \times 5^{e_3} b_3$$

$$b_1 + 2 \times 5^{e_2 - e_1} b_2 = 4 \times 5^{e_3 - e_1} b_3$$

Take this mod 5

$$b \equiv 0 \pmod{5} \text{ contradiction}$$

Case $e_2 < e_1, e_3$, Case $e_3 < e_1, e_2$

Both cases similar to $e_1 < e_2, e_3$ case.

Case $e_2 < e_1, e_3$, Case $e_3 < e_1, e_2$

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$e_2 < e_1, e_3$: $5^{e_1 - e_2} b_1 + 2b_2 = 4 \times 5^{e_3 - e_2}$, so $2b_2 \equiv 0 \pmod 5$.

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$e_3 < e_1, e_2$: $5^{e_1-e_3}b_1 + 2 \times 5^{e_2=e_3}b_2 = 4b_3$, so $4b_3 \equiv 0 \pmod 5$.

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- 5 primes, so can go from $2b_2 \equiv 0 \pmod 5$ to $b_2 \equiv 0 \pmod 5$.
- For $e_1 < e_2, e_3$ used that coeff of b_1 was $1 \neq 0$.
- For $e_2 < e_1, e_3$ used that coeff of b_2 was $2 \neq 0$.
- For $e_3 < e_1, e_2$ used that coeff of b_3 was $4 \neq 0$.

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$$5^{e_1} b_1 + 2 \times 5^{e_2} b_2 = 4 \times 5^{e_3} b_3$$

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Then mod 5

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Then mod 5

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- Could not have used the prime 3 instead of 5.
- Used that sum of coeff of b_1 and b_2 was $3 \neq 0$.

Case $e_1 = e_3 < e_2, e_2 = e_3 < e_1$

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$b \equiv 4b \pmod 5$, $b \equiv 0$.

$e_2 = e_3 < e_1$: $5^{e_1 - e_2} b_1 + 2 b_2 = 4 b_3$
$2b \equiv 4b \equiv 5$, $b \equiv 0$.

Case $e_1 = e_2 = e_3$

$$5^{e_1} b_1 + 2 \times 5^{e_2} b_2 = 4 \times 5^{e_3} b_3$$

$$b_1 + 2b_2 = 4b_3$$

$$b + 2b \equiv 4b \pmod 5$$

$$b \equiv 0 \pmod 5$$

Rado's Theorem

Thm Let $a_1, \ldots, a_k \in \mathbb{Z}$. TFAE

- Some subset of the a_i's sums to 0.
- For all c, for all $\mathrm{COL}\colon \mathbb{N} \rightarrow [c]$ there exists mono solution to

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We will not prove this.

From what I did above:

- Given any particular $(a_1, \ldots, a_k) \in \mathbb{Z}$ with some subset summing to 0 you should be able to show that any finite coloring of $\mathbb{N}$ has a mono solution.
- Given any particular $(a_1, \ldots, a_k) \in \mathbb{Z}$ with NO subset sums to 0 you should be able to define a finite coloring of $\mathbb{N}$ with no mono solution.

Other Equations

1. There is a matrix form of Rado that I don't care about.
2. **Folkman's Thm** For all k, c there exists $N = N(k, c)$ such that for all $\mathrm{COL}\colon [N] \to [c]$ there exists $a_1, \ldots, a_k$ such that ALL non-empty sums of the a_i's are the same color.
3. For all c there exists $N = N(c)$ such that for any $COL\colon [N] \to [c]$ there is a mono solution to $16x^2 + 9y^2 = z^2$. (This equation has certain properties that make it work, so there is really a more general theorem here.) `http://fourier.math.uoc.gr/~ergodic/Slides/Host.pdf`

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Research Obtain a human-readable proof with perhaps a much bigger N, but which can be generalized to $c = 3$ and beyond.