Exposition by William Gasarch (gasarch@cs.umd.edu)

Small Ramsey Numbers

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April 19, 2020

Lets Party Like Its January of 2019

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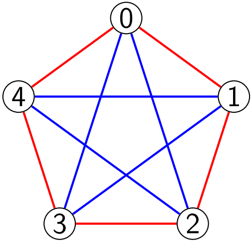
Question What if we color the edges of K_5?

Coloring of K_5 with no Mono K_3



This graph is not arbitrary.
$SQ_5 = \{x^2 \pmod 5) : 0 \le x \le 4\} = \{0, 1, 4\}$.

- If $i - j \in SQ_5$ then RED.
- If $i - j \notin SQ_5$ then BLUE.

Asymmetric Ramsey Numbers

Definition $R(a, b)$ is least n such that for all 2-colorings of K_n there is **either** a red K_a or a blue K_b.

1. $R(a, b) = R(b, a)$.
2. $R(2, b) = b$
3. $R(a, 2) = a$

$R(a,b) \le R(a-1,b) + R(a,b-1)$

Theorem $R(a,b) \le R(a-1,b) + R(a,b-1)$
Proof
Let $n = R(a-1,b) + R(a,b-1)$. $COL : \binom{[n]}{2} \rightarrow [2]$.
Case 1 $(\exists v)[\deg_R(v) \ge R(a-1,b)]$. Look at the $R(a-1,b)$ vertices that are RED to v. By Definition of $R(a-1,b)$ either

- There is a RED K_{a-1}. Combine with v to get RED K_a.
- There is a BLUE K_b.

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Case 3
$(\forall v)[\deg_R(v) \le R(a-1,b) - 1 \wedge \deg_B(v) \le R(a,b-1) - 1]$
$(\forall v)[\deg(v) \le R(a-1,b) + R(a,b-1) - 2 = n-2]$
Not possible since every vertex of K_n has degree $n-1$.

Lets Compute Bounds on $R(a, b)$

- $R(3,3) \leq R(2,3) + R(3,2) \leq 3 + 3 = 6$
- $R(3,4) \leq R(2,4) + R(3,3) \leq 4 + 6 = 10$
- $R(3,5) \leq R(2,5) + R(3,4) \leq 5 + 10 = 15$
- $R(3,6) \leq R(2,6) + R(3,5) \leq 6 + 15 = 21$
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Can we make some improvements to this? YES!

$R(3,4) \leq 9$

Theorem $R(3,4) \leq 9$.
Let COL be a 2-coloring of the edges of K_9.
Case 1 $(\exists v)[\deg_R(v) \geq 4]$. $v_1, \ldots, v_4$ are RED to v.

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Case 2 $(\exists v)[\deg_B(v) \geq 6]$. $v_1, \ldots, v_6$ are BLUE to v. Either:
(1) a RED K_3, or
(2) a BLUE K_3, which together with v is a BLUE K_4.
NOTE Can't have any $\deg_R(v) \leq 2$.

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Case 3 $(\forall v)[\deg_R(v) = 3]$. The RED subgraph has 9 nodes each of degree 3. Impossible!

Reminder of the Odd Vertex Things

Lemma Let $G = (V, E)$ be a graph.

$$V_{\mathrm{even}} = \{v : \deg(v) \equiv 0 \pmod 2\}$$
$$V_{\mathrm{odd}} = \{v : \deg(v) \equiv 1 \pmod 2\}$$

Then $|V_{\mathrm{odd}}| \equiv 0 \pmod 2$.

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Recall that for any graph $G = (V, E)$:

$$\sum_{v \in V_{\mathrm{even}}} \deg(v) + \sum_{v \in V_{\mathrm{odd}}} \deg(v) = \sum_{v \in V} \deg(v) = 2|E| \equiv 0 \pmod 2.$$

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$$\sum_{v \in V_{\mathrm{odd}}} \deg(v) \equiv 0 \pmod 2.$$

Sum of odds $\equiv 0 \pmod 2$. Must have even numb of them. So $|V_{\mathrm{odd}}| \equiv 0 \pmod 2$.

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Theorem $R(a,b) \leq$

1. $R(a,b-1) + R(a-1,b)$ always.
2. $R(a,b-1) + R(a-1,b) - 1$ if $R(a,b-1) \equiv R(a-1,b) \equiv 0 \pmod 2$

Some Better Upper Bounds

- $R(3,3) \leq R(2,3) + R(3,2) \leq 3 + 3 = 6.$
- $R(3,4) \leq R(2,4) + R(3,3) \leq 4 + 6 - 1 = 9.$
- $R(3,5) \leq R(2,5) + R(3,4) \leq 5 + 9 = 14.$
- $R(3,6) \leq R(2,6) + R(3,5) \leq 6 + 14 - 1 = 19.$
- $R(3,7) \leq R(2,7) + R(3,6) \leq 7 + 19 = 26$
- $R(4,4) \leq R(3,4) + R(4,3) \leq 9 + 9 = 18.$
- $R(4,5) \leq R(3,5) + R(4,4) \leq 14 + 18 - 1 = 31.$
- $R(5,5) \leq R(4,5) + R(5,4) = 62.$

Are these tight?

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Note $-1 = 2^2 \pmod 5$. Hence $a - b \in SQ$ iff $b - a \in SQ$. So the coloring is well defined.

$R(3,3) \geq 6$

$COL(a,b) =$ RED if $a - b \equiv SQ \pmod 5$, BLUE OW.

- Squares mod 5: 1,4.
- If there is a RED triangle then $a-b$, $b-c$, $c-a$ all SQ's. SUM is 0. So

$$x^2 + y^2 + z^2 \equiv 0 \pmod 5 \text{ Can show impossible}$$

- If there is a BLUE triangle then $a-b$, $b-c$, $c-a$ all non-SQ's. Product of nonsq's is a sq. So $2(a-b)$, $2(b-c)$, $2(c-a)$ all squares. SUM to zero- same proof.

UPSHOT $R(3,3) = 6$ and the coloring used math of interest!

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Use
$COL(a,b) =$ RED if $a - b \equiv SQ \pmod{17}$, BLUE OW.

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Same idea as above for K_5, but more cases.
UPSHOT $R(4,4) = 18$ and the coloring used math of interest!

$R(3,5) = 14$

$R(3,5) \geq 14$: Need coloring of K_{13} w/o RED K_3 or BLUE K_5.

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Can we extend these Patterns?

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$R(5,5)$—More on that later.

Revisit those Numbers

Int means Interesting Math. Bor means Boring Math.

- $R(3,3) \le 6$. TIGHT. Int
- $R(3,4) \le 9$. TIGHT. Int
- $R(3,5) \le 14$. TIGHT. Int
- $R(3,6) \le 19$. KNOWN: 18. Upper Bd Bor, Lower Bd Int
- $R(3,7) \le 26$. KNOWN: 23. Upper Bd Bor, Lower Bd Int
- $R(4,4) \le 18$. TIGHT. Int
- $R(4,5) \le 31$. KNOWN: 25. Both bd Bor
- $R(5,5) \le 62$. KNOWN: Between 43 and 49. Both Bor.

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Moral of the Story (Due Tuesday?)

1. At first there seemed to be **interesting mathematics** with mods and primes leading to nice graphs.
   (Joel Spencer) The Law of Small Numbers: Patterns that persist for small numbers will vanish when the calculations get to hard.
2. Seemed like a nice **Math** problem that would involve interesting and perhaps deep mathematics. No. The work on it is interesting and clever, but (1) the math is not deep, and (2) progress is slow.

When Will We Know $R(5,5)$

1. (Quote from Joel Spencer): *Erdos asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of* $R(5,5)$ *or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for* $R(6,6)$*. In that case, he believes, we should attempt to destroy the aliens.*

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2. I asked Stanislaw Radziszowski, the worlds leading authority on Small Ramsey Numbers, what $R(5,5)$ is and when we would know it. He said its 43 and we will **never** know it.

We Already Know $R(5,5)$!

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A Nice Case of Interdisciplinary Research

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5. Some people have fallen for it. Will tell stories in class.