The Square Theorem

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These Slides Are Not the Complete Story

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During this talk I will go to Zoom White Board several times.
The Square Theorem

**Definition** Let $G \in \mathbb{N}$ and $c \in \mathbb{N}$. Let $\text{COL}: [G] \times [G] \rightarrow [c]$.

1. A *mono $L$* is 3 points

   $$(x, y), (x + d, y), (x, y + d)$$

   that are all the same color ($d \geq 1$). This is an isosceles $L$.

2. A *mono Square* is 4 points

   $$(x, y), (x + d, y), (x, y + d), (x + d, y + d)$$

   that are all the same color ($d \geq 1$). This is a square.
The Square Theorem

**Theorem** There exists $G$ such that for all $\text{COL} : [G] \times [G] \rightarrow [2]$ there exists a mono square.
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2. We will first prove *For all $c$ there exists $GG = GG(c)$ such that for all $\text{COL}: [GG] \times [GG] \rightarrow [c]$ there exists a mono $L$.*
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3. To prove **The Square Theorem** (about 2-coloring) we need to know that $GG(c)$ exists for a very large $c$. 
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3. To prove **The Square Theorem** (about 2-coloring) we need to know that $GG(c)$ exists for a very large $c$.

4. More Colors: *For all $c$ there exists $G = G(c)$ such that for all $\text{COL}: [G] \times [G] \rightarrow [c]$ there exists a mono square.* Proof needs a larger $c'$ for $GG(c')$. 
Theorem For all $c$ there exists $GG = GG(c)$ such that for all $COL: [GG] \times [GG] \rightarrow [c]$ there exists a mono $L$. 
The $L$ Theorem for $c = 2$

**Theorem** For all $c$ there exists $GG = GG(c)$ such that for all $COL: [GG] \times [GG] \rightarrow [c]$ there exists a mono $L$.

**Proof** We prove this for $c = 2$. We will set $H$ later. Let $COL: [H] \times [H] \rightarrow [c]$. 

Take the $[H] \times [H]$ grid and tile it with $3 \times 3$ tiles.

View a 2-coloring of $[H] \times [H]$ as a $2^9$-coloring of the tiles. This is very typical of VDW-Ramsey Theory: a 2-coloring of BLAH is viewed as a $X$-coloring of a different object where $X$ is quite large.
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This is very typical of VDW-Ramsey Theory: a 2-coloring of BLAH is viewed as a $\times$-coloring of a different object where $\times$ is quite large.
Why This Size Tile?

Any 2-coloring of the $3 \times 3$ tile will have two of the same color in the first column and hence an almost $L$
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Goto Zoom-White Board.
Make $H$ Big Enough To Get Two Tiles Same Color

Take $H = 3(2^9 + 1)$. 
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Take $H = 3(2^9 + 1)$.

View $[H] \times [H]$ grid of points as $[2^9 + 1] \times [2^9 + 1]$ grid of tiles. Look at the first column of tiles. Two are the same color.
Take $H = 3(2^9 + 1)$.


Look at the first column of tiles. Two are the same color.

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The $L$ Theorem for $c = 3$

First take $4 \times 4$-tiles.
The \( L \) Theorem for \( c = 3 \)

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Any 3-coloring of the \( 4 \times 4 \) tile will have two of the same color in the first column and hence an \textbf{almost} \( L \)
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First take $4 \times 4$-tiles.
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Make $H$ Big Enough To Get Two Tiles Same Color

Take $H = 4(3^{16} + 1)$. 

View $[H] \times [H]$ grid of points as $[3^{16} + 1] \times [3^{16} + 1]$ grid of tiles.
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View $[H] \times [H]$ grid of **points** as $[3^{16} + 1] \times [3^{16} + 1]$ grid of **tiles**.
Make $H$ Big Enough To Get Two Tiles Same Color

Take $H = 4(3^{16} + 1)$.

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Look at the first column of tiles. Two are the same color.
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Take Much Bigger Tiles

Take Tile so big that any 3-coloring of it has two different colored almost-L’s converging to the same point.
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Take Tile so big that any 3-coloring of it has two different colored almost-L’s converging to the same point.

Go to Zoom-White Board.
Theorem For all $c$ there exists $GG = GG(c)$ such that for all $COL: [GG] \times [GG] \rightarrow [c]$ there exists a mono $L$. 

We won’t prove this but I am sure any of you could prove it given what we have done so far. Would be messy. Easier to prove it from the Hales-Jewitt Theorem, which we won’t be doing.
**Full L Theorem**

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**Proof** $G$ will be $GG(2)GG(2^{GG(2)^2})$. 

Go to Zoom Whiteboard for rest of proof.
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For any 2-coloring of $[G] \times [G]$:

- Every tile has a mono $L$
- There is a mono $L$ of tiles.
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