Van Der Warden's (VDW) Theorem

Exposition by William Gasarch

May 12, 2020

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During this talk I will go to Zoom White Board several times.

VDW's Theorem

Definition Let $W, k, c \in \mathbb{N}$. Let $\mathrm{COL}\colon [W] \to [c]$. A **mono** ***k*-AP** is an arithmetic progression of length k where every elements has the same color. We often say

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We will determine W later.
Let $COL\colon [W] \to [2]$.

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Worst Case Scenario B_1 and B_{33} same color. So need B_{65} to exist.

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How many colorings of a block already have a mono 3-AP.

Side Note: Can Get By With Less Blocks (cont)

RRRXY with $X, Y \in \{R, B\}$. 4 colorings.
BBBXY with $X, Y \in \{R, B\}$. 4 colorings.
RBRRR
RBRBR
BRBBB
BRBRB
RBBBX with $X \in \{R, B\}$. 2 colorings.
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I have 16 blocks which already have a mono 3-AP. I might have missed some. but if not then can replace 32 with 18.

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I really do not care.

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- In every block there exists x, y same color and z such that x, y, z are 3-AP in same block. (This is why blocks-of-5.)

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One can work out by hand that

$$W(3,2) = 9.$$

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Note that we **do not** do
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How large? The bounds are not primitive recursive.

A False Prediction

In 1983 there were two thoughts in the air

1. $W(k, c)$ is not prim rec and a **logician** will prove this deep result. Perhaps like the Large Ramsey Numbers (1977) though not that big.
2. $W(k, c)$ is surely prim rec and a **combinatorist** will prove this perhaps with a clever elementary technique.

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- Proof is elementary. Can be in a this class but won't.
- Bounds still of Mae-type.

Deep Math From Search for Better Upper Bounds on VDW Numbers

Exposition by William Gasarch

May 12, 2020

A Man, A Plan, A Canal: Panama!

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We outline a plan for getting better upper bounds on $W(k, c)$.

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It DID succeed! (Oh! Thats a good thing!)

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Examples

1. For all k, $\{x : x \equiv 0 \pmod{k}\}$ has upper den $\frac{1}{k}$.
2. $\{x^2 : x \in \mathbb{N}\}$ has upper den 0.

A Conjecture, 1936

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Conjecture If $A \subseteq \mathbb{N}$ has positive upper density then, for all k, A has a k-AP.

Theorem Conj implies VDW's Theorem. HW or Final.

The hope was that the proof of Conj would require a new proof of VDW's Theorem that would lead to better bounds.

Roth's Theorem, 1952

Theorem If $A \subseteq \mathbb{N}$ has positive upper density then A has a 3-AP.

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Theorem If $A \subseteq \mathbb{N}$ has positive upper density then A has a 3-AP.

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- Roth won the Fields Medal in 1958 for his work on Diophantine approximation (so not for this work).

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- Even so, it introduced very deep methods.
- Proof is elementary but strains the use of the word **elementary**.

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- What is better financially: Fields Medal when you are 40 or Abel prize when you are 70? Fields Medal can lead to better jobs and pay while you are still young. I wish this was my dilemma.

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  - Combinatorics was less respected in 1975 then in 1998.
  - Causes of change: (1) combinatorics using deep math, (2) CS inspired new problems in combinatorics.

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None of these results used mathematics of interest.

Known Lower Bounds

1. Easy Use of Prob Method (was on HW) $W(k,2) \ge \sqrt{k}2^{k/2}$ (Easy extension to 3 colors)
2. Very sophisticated use yields $W(k,2) \ge \frac{2^k}{k^\epsilon}$ (Does not extend to 3 colors.)
3. If p is prime then $W(p,2) \ge p(2^p - 1)$. Constructive! (Does not extend to 3 colors.)

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- There is also a ConservaMedical Medal- an alternative to the Nobel Prize in Medicine. It went to Donald Trump for his Medical Advice on Covonavirus. I am kidding.