The Infinite Can Ramsey Thm

William Gasarch-U of MD

Hungarian Math Comp Problem

From the 1950 "Kürschák/Eötvös Math Competition":

There are 1950 cans of paint. Find an x *such that (1) there are either* x *cans of paint all the same color, or* x *cans of paint that are all different colors and (2) it is possible to have neither* $x + 1$ *cans that are all the same nor* $x + 1$ *cans that are all different.*

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From Homework, you know the answer is $\lfloor \sqrt{1949} \rfloor + 1 = 45$.

Can Ramsey Thm

The Can Ramsey Thm is for any number of colors.

It is named "Can Ramsey" in honor of the paint can problem on the 1950 Kürschák/Eötvös Math Competition

1-ary Ramsey's Thm

Thm: For every $COL : \mathbb{N} \rightarrow [c]$ there is an infinite homog set.

What if the number of colors was **infinite?**

Do not necessarily get a homog set since could color EVERY vertex differently. But then get infinite **rainbow set**.

One-Dim Can Ramsey Thm

Thm: Let V be a countable set. Let $COL : V \rightarrow \omega$. Then there exists either an infinite homog set (all the same color) or an infinite rainb set (all diff colors).

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Prove with your neighbor.

Ramsey's Thm For Graphs

Thm: For every $COL : \binom{\mathbb{N}}{2} \rightarrow [c]$ there is an infinite homog set.

What if the number of colors was **infinite?**

Do not necessarily get a homog set since could color EVERY edge differently. But then get infinite **rainbow set**.

Attempt

Thm: For every $COL : \binom{\mathbb{N}}{2} \rightarrow \omega$ there is an infinite homog set OR an infinite rainb set.

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FALSE:

- $COL(i,j) = \min\{i,j\}$.
- $COL(i,j) = \max\{i,j\}$.

Min-Homog, Max-Homog, Rainbow

Def: Let $COL : \binom{\mathbb{N}}{2} \rightarrow \omega$. Let $V \subseteq \mathbb{N}$. Assume $a < b$ and $c < d$.

- V is *homog* if $COL(a,b) = COL(c,d)$ iff $TRUE$.
- V is *min-homog* if $COL(a,b) = COL(c,d)$ iff $a = c$.
- V is *max-homog* if $COL(a,b) = COL(c,d)$ iff $b = d$.
- V is *rainb* if $COL(a,b) = COL(c,d)$ iff $a = c$ and $b = d$.

Can Ramsey Thm for $\binom{\mathbb{N}}{2}$: For all $COL : \binom{\mathbb{N}}{2} \rightarrow \omega$, there exists an infinite set V such that either V is homog, min-homog, max-homog, or rainb.

Proof of Can Ramsey Thm for $\binom{\mathbb{N}}{2}$, et

We are given $COL : \binom{\mathbb{N}}{2} \to \omega$.
Want to find infinite homog OR min-homog OR max-homog OR rainbow set.

We use COL to define $COL' : \binom{\mathbb{N}}{4} \to [16]$
We then apply **4-ary Ramsey Theorem.** (an **"Application!"**)

In the slides below $x_1 < x_2 < x_3 < x_4$.
All cases assume negation of prior cases.

Homog always means infinite Homog.

Pairs that begin the same way are same color

1. $COL(x_1, x_2) = COL(x_1, x_3) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 1.$
2. $COL(x_1, x_2) = COL(x_1, x_4) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 2.$
3. $COL(x_1, x_3) = COL(x_1, x_4) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 3.$
4. $COL(x_2, x_3) = COL(x_2, x_4) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 4.$

H is homog set,color 1 (rest similar)
$COL'' : H \rightarrow \omega$ is $COL''(x) =$ color of all (x, y) with $x < y \in H$.

Use **1-dim Can Ramsey!**:
Case 1: COL'' has homog set H' then H' homog for COL.
Case 2: COL'' has rainb set H' then H' min-homog for COL.

Pairs that End the same way are same color

1. $COL(x_1, x_3) = COL(x_2, x_3) \to COL'(x_1 < x_2 < x_3 < x_4) = 5.$
2. $COL(x_1, x_4) = COL(x_2, x_4) \to COL'(x_1 < x_2 < x_3 < x_4) = 6.$
3. $COL(x_1, x_4) = COL(x_3, x_4) \to COL'(x_1 < x_2 < x_3 < x_4) = 7.$
4. $COL(x_2, x_4) = COL(x_3, x_4) \to COL'(x_1 < x_2 < x_3 < x_4) = 8.$

H is homog set,color 5 (rest similar)
$COL'' : H \to \omega$ is $COL''(y) =$ color of all (x, y) with $x < y \in H$.

Use **1-dim Can Ramsey!**:
Case 1: COL'' has homog set H' then H' homog for COL.
Case 2: COL'' has rainb set H' then H' max-homog for COL.

Easy Homog Cases

1. $COL(x_1, x_2) = COL(x_2, x_3) \Rightarrow COL(x_1, x_2, x_3, x_4) = 9.$
2. $COL(x_1, x_2) = COL(x_2, x_4) \Rightarrow COL(x_1, x_2, x_3, x_4) = 10.$
3. $COL(x_1, x_2) = COL(x_3, x_4) \Rightarrow COL(x_1, x_2, x_3, x_4) = 11.$
4. $COL(x_1, x_3) = COL(x_2, x_4) \Rightarrow COL(x_1, x_2, x_3, x_4) = 12.$
5. $COL(x_1, x_3) = COL(x_3, x_4) \Rightarrow COL(x_1, x_2, x_3, x_4) = 13.$
6. $COL(x_2, x_3) = COL(x_1, x_4) \Rightarrow COL(x_1, x_2, x_3, x_4) = 14.$
7. $COL(x_2, x_3) = COL(x_3, x_4) \Rightarrow COL(x_1, x_2, x_3, x_4) = 15.$

H is homog set,color 9 (rest similar)
For all $w < x < y < z \in H$.

$$COL(w, x) = COL(x, y) = COL(y, z).$$

Other cases, like $COL(w, y) = COL(x, z)$, are similar

Rainbow Case

If **NONE** of the above cases hold then $COL(x_1, x_2, x_3, x_4) = 16$.

Let H be homog set.

All edges from H diff colors, so Rainbow Set.

PROS and CONS of Proof

PRO: Each Case easy. Note that Rainbow case was easy.

CON: Lots of Cases. Use of 4-ary hypergraph Ramsey makes finite version have large bounds.
Let $\mathrm{CR}_2(k)$ = least n s.t. $\forall \mathrm{COL}\colon \binom{[n]}{2} \to \omega$, $\exists H$ of size k such that either H is homog, min-homog, max-homog, or rainb. If finitized, this proof obtains

$$\mathrm{CR}_2(k) \le R_4(k,16) \le 16^{16^{16^{O(k)}}}$$

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We will give anther proof which only uses 3-ary hypergraph Ramsey.

Def that Will Help Us

Def Let $COL : \binom{\mathbb{N}}{2} \rightarrow \omega$. If c is a color and $v \in \mathbb{N}$ then $\deg_c(v)$ is the number of c-colored edges with v as an endpoint.

Note: $\deg_c(v)$ could be infinite.

Needed Lemma

Lemma Let X be infinite. Let $COL : \binom{X}{2} \rightarrow \omega$. If for every $x \in X$ and $c \in \omega$, $\deg_c(x) \le 1$ then there is an infinite rainb set.
TRY TO PROVE WITH YOUR NEIGHBOR. I WILL THEN GIVE PROOF.

Proof

Let R be a MAXIMAL rainb set of X.

$$(\forall y \in X - R)[R \cup \{y\} \text{ is not a rainb set}].$$

We prove R is infinite.

Proof that R is infinite

Let $y \in X - R$. Why is $y \notin R$?

1. $(\exists u \in R, \exists \{a, b\} \in \binom{R}{2})[COL(y, u) = COL(a, b)]$.
2. $(\exists \{a, b\} \in \binom{R}{2})[COL(y, a) = COL(y, b)]$.
   If $c = COL(y, a)$ then $\deg_c(y) \geq 2$, so **Can't Happen!**

Map $X - R$ to $R \times \binom{R}{2}$: map $y \in X - R$ to $(u, \{a, b\})$ (item 1).
Map is injective: if y_1 and y_2 both map to $(u, \{a, b\})$ then $COL(y_1, u) = COL(y_2, u)$ but $\deg_c(u) \leq 1$.
Injection from $X - R$ to $R \times \binom{R}{2}$. If R finite then injection from an infinite set to a finite set Impossible! Hence R is infinite.

Can Ramsey Thm for $\mathbb{N}$

Thm: For all $COL : \binom{\mathbb{N}}{2} \rightarrow \omega$ there is either

- an infinite homog set,
- an infinite min-homog set,
- an infinite max-homog set, or
- an infinite rainb set.

Proof of Can Ramsey Thm for Graphs

Given $COL : \binom{\mathbb{N}}{2} \to \omega$. We use COL to obtain $COL' : \binom{\mathbb{N}}{3} \to [4]$
We will use the 3-ary Ramsey Theorem. In all of the below $x_1 < x_2 < x_3$.

1. If $COL(x_1, x_2) = COL(x_1, x_3)$ then $COL'(x_1 < x_2 < x_3) = 1$.
2. If $COL(x_1, x_3) = COL(x_2, x_3)$ then $COL'(x_1 < x_2 < x_3) = 2$.
3. If $COL(x_1, x_2) = COL(x_2, x_3)$ then $COL'(x_1 < x_2 < x_3) = 3$.
4. If none of the above occur then $COL'(x_1 < x_2 < x_3) = 4$.

Cases 1,2,3 are just like in the prior proof.
Case 4: For all x, for all c, $\deg_c(x) \le 1$ so have Rainbow by Lemma.

Better Bounds on Can Ramsey

Using 4-ary proof, 16 colors, bound was:

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$$\mathrm{CR}_2(k) \le 16^{16^{16^{O(k)}}}$$

Using new proof, 3-ary with 4 colors, bound is:

Better Bounds on Can Ramsey

Using 4-ary proof, 16 colors, bound was:

$$\mathrm{CR}_2(k) \le 16^{16^{16^{O(k)}}}$$

Using new proof, 3-ary with 4 colors, bound is:

Not obvious! Cases 1, 2, and 3 easy, but case 4 uses maximal sets.

Good news: Will be on homework.