# CMSC 858M: Algorithmic Lower Bounds: Fun With Hardness Proofs
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## 1 Overview

This chapter discusses graph problems that are NP-Complete even when restricted to planar graphs. Rather than reducing the planar version to the non-planar version, which is frequently done using a crossover gadget, this chapter will introduce PL-3SAT (Planar 3SAT) which is also NP-hard, and then reduce it directly to various graph problems.

It also defines the *degree* of a graph as the maximum degree of (number of edges connecting to) any vertex, and a regular graph as a graph in which all of the vertices have the same degree.

## 2 Clique, Independent Set, and Vertex Cover

Clique (Independent Set) is the question of whether a given graph has a subset of $k$ vertices such that every pair (no pair) of vertices in the subset has an edge between them.

Vertex Cover is the question of whether a given graph has a subset of $k$ vertices such that every edge has a vertex in the subset as at least one of its endpoints.

All three of those graph problems are NP-Complete.

Clique restricted to planar graphs is in P.

IS and VC are both NP-hard when restricted to planar graphs of degree 3.

### 2.1 Planar 3SAT

A CNF formula can be represented by a bipartite graph with variables on one side and clauses on the other. Put a red edge between variable $x$ and clause $C$ if $x$ is in $C$, a blue line between them if $\neg x$ is in $C$, and no edge between them if neither is in $C$. A SAT formula can be called planar if the associated bipartite graph is planar. It can also be checked in polynomial time whether a formula is planar.

PL-SAT is NP-hard even with (or without) the further restrictions that all variables are connected in a cycle, and that $x$ is a left-endpoint of any clause containing $x$, and a right endpoint of any clause containing $\neg x$. This is proven using a crossover gadget to replace any crossings in the non-planar version of the problem.

## 2.2 Graph Coloring and Variants

2-coloring a graph means assigning one of two different colors to each vertex in a graph such that no adjacent vertices are the same color. This problem can be extended to any number of colors.

2-coloring a graph can be done in polynomial time.

3-coloring a graph in NP-hard, even when restricted to a planar graph of degree 4.

Graphs restricted to degree 3 are always 3-colorable.

Planar graphs are always 4-colorable.

## 2.3 Vertex Cover and Variants

PL-VC-DEG3 (vertex cover on a planar graph of degree 3) is NP-complete. This is shown by a reduction from 3SAT that when restricted to planar formulas results in planar graphs.

## 2.4 Dominating Set

Dominating Set is the question of whether a graph has a subset of $k$ vertices such that every vertex in the graph is either in the subset or adjacent to a member of the subset.

PL-DOM is NP-complete. This can be shown by a reduction from either PL-VC or from 3SAT.

## 2.5 Planar Directed Hamiltonian Graphs and Variants

A directed graph is a graph where the edges have a direction. The prefix D- in front of a graph problem will indicate the graph is directed.

A Hamiltonian cycle (path) is a cycle (path) that visits every vertex exactly once. PL-D-HAMC (the problem of whether there exists a planar directed Hamiltonian cycle) is NP-hard.

## 2.6 Shakashaka

The Nikoli game Shakshaka is NP-Complete.

## 2.7 Planar Rectilinear 3SAT and Planar Monotone Rectilinear 3SAT

A rectilinear formula is one where every variable is on the x axis, and every clause is above (or below) the x axis and every connection is vertical.

PL-RECT-3SAT is then a 3SAT problem that can be represented by a planar rectilinear graph.

PL-RECT-MONO-3SAT is the same as above except every clause is either all positive or all negativel, with positive clauses above the x axis and negative clauses below.

PL-RECT-3SAT and PL-RECT-MONO-3SAT are bot NP-complete. However, for PL-RECT-MONO-3SAT if all clauses are positive or all are negative, or they can all be connected in a path, then it is polynomial time solvable.

## 2.8 Planar 1-in-3SAT

Planar 1-in-3SAT is NP-complete. This can be used to show that Planar Exact Covering by 3-sets and Planar 3-Dimensional Matching are NP-complete.

## 2.9 Exact Covering by 3-Sets

X3C (Exact Covering by 3-Sets) is given a set $\mathsf{X}$ with a multiple of 3 elements, and a collection of subsets of $\mathsf{X}$ each containing three elements, is there a set of those subsets containing every element of $\mathsf{X}$ exactly once?

This can be represented as a bipartite graph with the elements of $\mathsf{X}$ on the left, and the subsets on the right, with each subset having an edge connecting it to the three elements of $\mathsf{X}$ it contains. PL-X3C is when this graph can be made planar.
X3C and PL-X3C are both NP-complete by a reduction from Planar-1-in-3SAT.

## 2.10 Matching

A 2D Matching (usually but not necessarily on a bipartite graph) is the question of whether there is a disjoint set of edges covering every vertex exactly once. This problem is polynomial

A 3D matching is given three groups of people (males, females, and zans), the goal is to assign them into triples in which everyone is happy. This can be represented as a 3D hyper-graph, or as a bipartite graph with people on one side and triples on the other (with edges connecting each person to the triples they are in. The question is whether there is a disjoint set of edges covering all people. If the bipartite graph representation is planar then it is a PL-3DM.

3DM and PL-3DM turn out to be basically the same as X3C and PL-X3C so they are therefore NP-complete.

### 2.11 Triangulation

Rectilinear Positive Planar 1-in-3SAT is NP-complete.

Let $S$ be a set of points in a plane. A *triangulation* of $S$ is a graph of the vertices in $S$ in which all edges are straight lines, the graph is planar, and if any more edges were added to the graph it would no longer be planar. The Minimum Weight Triangulation is the triangulation that minimizes the sum of weights of the edges.

### 2.12 Planar NAE SAT

Planar NAE 3SAT is can be solved in polynomial time by a reduction to planar Max Cut.

### 2.13 Flattening Fixed-Angle Chains

The question of which molecules can be drawn in 2D is NP-hard.

## 3 Other Interesting Problems

The set packing problem is given a set and some subsets of the set, are there $k$ subsets that are pairwise disjoint. This can be shown NP-hard by reducing from clique.

The problem of Path With Forbidden Pairs is given a graph, a start vertex, an end vertex, and a list of pairs, is there a path from the start to the end that visits at most one of each pair?

[2] The game Mastermind framed as a decision problem for whether there is a valid solution given a series of guesses and scores is NP-Complete. This was shown by a reduction from Vertex Cover.

[3] The game Corral is a puzzle in which the player is given a grid of squares, some of which contain numbers. The player's goal is to form a closed loop containing all of the numbers, such that the numbers represent how many squares in the loop can be seen (vertically or horizontally in contiguous segments contained within the loop) from that square. This is NP-complete by a reduction from Planar 3coloring.

[4] Completing partial Latin Squares (nxn squares which contain the numbers 1-n exactly once per row and once per column) is NP-hard by a reduction from the triangulation of tripartite graphs.

Sudoku can (unsurprisingly) be proven NP-complete by a reduction from Latin Squares.

## References

[1] Erik Demaine, William Gasarch, and Mohammad Hajiaghayi *Fun With Hardness: Algorithmic Lower Bounds* .

[2] Jeff Stuckman and Guo-Qiang Zhang *Mastermind is NP-Complete* December 2005

[3] Erich Friedman *Corral Puzzles are NP-complete*

[4] Charles J. COLBOURN *THE COMPLEXITY OF COMPLETING PARTIAL LATIN SQUARES* April 1982