1 REPLACE COMMENT ON APPROX NASH EQ WITH THIS

DONE

The NE problem really asks for an approximation to the NE. This means that if (x, y) is the NE (where x and y are vectors of probabilities that add to 1) then the algorithm produces (x', y') where x' is close to x and y' is close to y. We briefly discuss a different kind of approximation.

Def 1.1 An *ϵ-Nash equilibrium (henceforth just ϵ-equilibruim)* is a pair of mixed strategies (x, y) such that the following holds.

1. If the row player deviates from x, and the column player stll uses y, then the row player benefits by at most ϵ.

2. If the column player deviates from y, and the row player stll uses x, then the column player benefits by at most ϵ.

3. For each player, the payoff at (x, y) is at most ϵ less than the optimal.

There are essentially matching upper and lower bounds for the time needed to find an ϵ-equilibrium:

1. Lipton et al. [4] showed that, for all $\epsilon > 0$, there is an algorithm that finds an ϵ-equilibrium that runs in time $O(n^{\epsilon^{-2}\log n})$

2. Braverman et al. [1] showed that, assuming ETH, there exists ϵ^* such that any algorithm that finds an ϵ^*-equilibrium and requires time $O(n^{\log n})$

2 PUT IN THE PPAD PART

DONE

Def 2.1

1. Let C be a cake. Let $P_1, \ldots, P_n$ be n people. They each have a utilily function that maps areas of the cake to values. The entire cake maps to 1 and a single point maps to 0. If A and B are disjoint parts of the cake then, for any utility function U, $U(A \cup B) = U(A) + U(B)$.

2. A *allocation* of C is a partition $C = C_1 \cup \cdots \cup C_n$ of C where, for all $1 \le i \le n$, P_i gets piece C_i.

3. An allocation is *Proporitional* if every person, using their own utility function, gets $\ge \frac{1}{n}$.

4. An allocation is *Envy-Free* if every person, using their own utility function, think that nobody has a strictly larger piece than they have.

Stromquist [5] showed that, given any set of n utility functions there exists an envy-free allocation that only uses n cuts. The cuts could be at irraional points. His proof also yielded an algorithm that, given ϵ, found the cuts to within ϵ, in time $O(\log \frac{1}{\epsilon})$. This kind of problem falls neatly into the PPAD paradigm: we have a proof that something exists but we wonder if we can really find it. Deng et al. [2] showed that the problem of finding an approximate envy-free allocation for n people with $n-1$ cuts is PPAD-complete.

3 PUT IN THE PPA PART

DONE

1. Goos et al. [3] show that a variant of CHEVALLEY is PPA_q-complete (you will define PPA_q in Exercise 3.1). However, they do not think the original CHEVALLEY is PPA-complete (see there note on page 6).

Exercise 3.1

1. Let $q \in \mathsf{N}$ and let G be a bipartite graph. Show that if there is some vertex of degree $\not\equiv 0 \pmod{q}$ then there must be another one .

2. Define PPA_q and PPA_q-complete using Part 1 as motivation.

3. Read Goos et al. [3] which shows several problems are PPA_q-complete. Rewrite their proofs in your own words.

4 How do he Classes Relate?

DONE

We summarize what is know about how the classes relate, and what is open.

Exercise 4.1

1. Show that PF $\subseteq$ PPAD $\subseteq$ PPA $\subseteq$ FNP.

2. Show that PF $\subseteq$ PPAD $\subseteq$ PPP $\subseteq$ FNP.

3. (Open problem) For each subset inclusions in Part 1 and 2 resolve if the inclusion is equal or proper. (It is widely believed that all of the inclusions are proper.)

4. (Open problem) For each subset inclusions in Part 1 and 2 determine if an equality implies P = NP or some other unlikely conclusion.

5. (Open Problem) Resolve how PPA and PPP compare.

References

[1] M. Braverman, Y. Kun-Ko, and O. Weinstein. Approximating the best nash equilibrium in $n^{o(\log n)}$-time breaks the exponential time hypothesis. In P. Indyk, editor, *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2015, San Diego, CA, USA, January 4-6, 2015*, pages 970–982. SIAM, 2015. `https://doi.org/10.1137/1.9781611973730.66`.

[2] X. Deng, Q. Qi, and A. Saberi. Algorithmic solutions for envy-free cake cutting. *Oper. Res.*, 60(6):1461–1476, 2012. `https://doi.org/10.1287/opre.1120.1116`.

[3] M. Göös, P. Kamath, K. Sotiraki, and M. Zampetakis. On the complexity of modulo-q arguments and the chevalley - warning theorem. In S. Saraf, editor, *35th Computational Complexity Conference, CCC 2020, July 28-31, 2020, Saarbrücken, Germany (Virtual Conference)*, volume 169 of *LIPIcs*, pages 19:1–19:42. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020. `https://doi.org/10.4230/LIPIcs.CCC.2020.19`.

[4] R. J. Lipton, E. Markakis, and A. Mehta. Playing large games using simple strategies. In D. A. Menascé and N. Nisan, editors, *Proceedings 4th ACM Conference on Electronic Commerce (EC-2003), San Diego, California, USA, June 9-12, 2003*, pages 36–41. ACM, 2003. `https://doi.org/10.1145/779928.779933`.

[5] W. Stromquist. How to cut a cake fairly. *American Mathematics Monthly*, 87(8):640–644, 1980.