1 The Unique Games Conjecture

Recall that in our definitions of GAP-MAXR the promise is that either (1) there is a label covering with one vertex per A_i and B_j which covers all superedges, or (2) every such label covering covers at most an ϵ fraction of the superedges. What if we relaxed the promise of part (1)? Consider the following gap problem.

Def 1.1 ϵ-2-sided-GAP-MAXR.

INSTANCE: A bipartite $G = (A, B, E)$ that has the vertices partitioned as in Definition **??**.

QUESTION: We only look at label cover which takes exactly one element from each A_i and each B_j. We are promised that one of the following occurs.

- There is such a label covering which covers fraction $(1 - \epsilon)$ of the superedges.
- Every such label covering covers at most an ϵ fraction of the superedges.

The question is to determine which case happens.

Khot [16] made the following conjecture.

Conjecture 1.2 *The* Unique Games Conjecture (UGC) *is that, for all* $\epsilon > 0$*,* ϵ*-2-sided-*GAP-MAXR *is* NP*-hard. (The name* Unique Games Conjecture *comes from another formulation of it.)*

For more on UGC see Khot's survey [17] and Klarreich exposition [20]. Is the conjecture true? We argue both sides.

Argument for UGC

1. UGC has great explanatory power. There are many examples of this. We give one. Consider the Vertex Cover Problem (VC).
   - There is a poly time 2-approximation for VC (so returns twice the min number of vertices needed).
   - The 2-approximation result is very old. Despite many attempts to improve it it stays stubbornly at 2-approx.

- Dinur and Safra [7] showed that, assuming P $\neq$ NP, or all $\epsilon > 0$, VC has no poly time $1.360 - \epsilon$-approximation.
- Khot and Regev [19] showed that, assuming UGC, or all $\epsilon > 0$, VC has no poly time $2 - \epsilon$-approximation.

We note that the proof of the upper bound of 2, and the proof of the lower bound of $2 - \epsilon$, have nothing to do with each other.

2. Khot et al. [18] proved a weaker version, called the 2-2 games conjectures. See also the exposition by Klarreich [21].

Argument for UGC

1. It is possible we will obtain that explanatory power from the assumption P $\neq$ NP.
2. Arora et al. [3] obtained a subexponential algorithm for ϵ-2-sided-GAP-MAXR is NP-hard. Note that the algorithm is not polynomial and has not been improved on since 2010.

Unlike P vs NP and many other conjectures, the community is truly split on this conjecture.

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