BILL AND NATHAN, RECORD LECTURE!!!!

BILL RECORD LECTURE!!!

ETH and NPC Probs on Graphs and Planar Graphs

Exposition by William Gasarch—U of MD

Is Assuming $\mathrm{P} \neq \mathrm{NP}$ Enough?

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So... What should Nathan try to prove now?

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Nathan Go to it!

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We will revisit this point after the next reduction.

NPC Problems on Graphs

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The Clique Problem

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Def

$$\mathrm{CLIQ} = \{(G, k) : G \text{ has a clique of size } k \}.$$

We show that CLIQ is NPC.

3SAT $\leq$ CLIQ

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2) Graph G with $7k$ vertices as follows: For each clause we have 7 vertices. Label them with the 7 ways to set the 3 vars to make the clause satisfiable. For example, for the clause $x \vee y \vee \neg z$, we have 7 vertices: TTT, TTF, TFT, TFF, FTT, FTF, FFF,

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There are no edges between vertices associated to the same clause. We put an edge between vertices associated with different clauses if the assignments do not conflict. Example:
$(x = T, y = T, z = T)$ has edge to $(w = F, x = T, z = T)$ but not to $(w = F, x = F, z = T)$.

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3) Example on next slide

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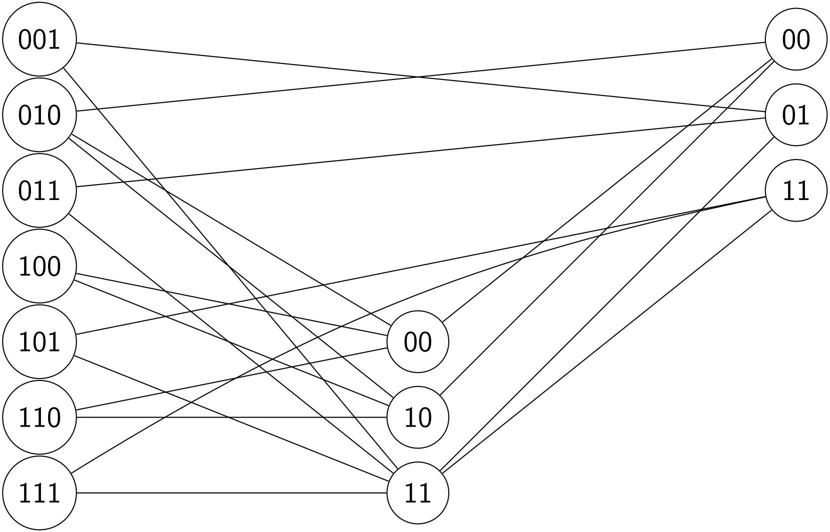
$(x \vee y \vee z) \quad \wedge \quad (w \vee \overline{z}) \quad \wedge \quad (\overline{x} \vee z)$



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Revisit ETH on next slide.

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This is equivalent to
$(\forall c)(\exists d)[A \in \mathrm{DTIME}(2^{dv}) \rightarrow \mathrm{3SAT} \in \mathrm{DTIME}(2^{cn})]$.

How to use ETH to Get $2^{\Omega(v)}$ Lower Bounds

Let A be a graph problem. We want to show that A takes $2^{\Omega(v)}$.
Thm (ETH) $\mathrm{3SAT} \le A$ by reduction f AND $(\exists a)[|f(\phi)| \le an]$ then A requires $2^{\Omega(v)}$.

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This takes time 2^{adn}. Take $d = \frac{c}{a}$ to get a 2^{cn} alg for $\mathrm{3SAT}$.
Note If the reduction took $2^{\epsilon n}$ time, would still work. If needed to do the reduction many times would still work.

Recap of Our Dilemma

We have $\mathrm{3SAT} \le \mathrm{CLIQ}$ where a formula on n variables and m clauses maps to a graph on $\le 7m$ vertices.
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Notation $\tilde{O}(f(n))$: ignore polys. Often used as $\tilde{O}(2^{cn})$.
ETH is equiv to: $(\exists c)(\forall p(n))$ $3\mathrm{SAT}$ requires $\ge p(n)2^{\Omega(cn)}$.

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(2) $\phi \in \mathrm{3SAT}$ iff $(\exists i)[\phi_i \in \mathrm{3SAT}]$.

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Need $(\forall c)(\exists d)[\mathrm{CLIQ} \in \mathrm{DTIME}(2^{dv}) \rightarrow \mathrm{3SAT} \in \mathrm{DTIME}(2^{cn})]$.
Erika What Lemma do we use? Right! SL!
Pick a c. We pick ϵ and d later.

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   Pick d, ϵ on next slide.

ETH implies CLIQ Requires $2^{\Omega(v)}$

Given c we want to pick d, ϵ such tha t

$$2^{(\epsilon+7de(\epsilon))n} < 2^{cn}$$

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Pick $\epsilon = \frac{c}{3}$.
Pick $d = \frac{c}{14e(\epsilon)}$.
Then

$$\epsilon + 7de(\epsilon) = \frac{c}{3} + \frac{c}{2} = \frac{5c}{6} < c.$$

Recap

From

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From

1. ETH

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From

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2. $3\mathrm{SAT} \leq \mathrm{CLIQ}$ is linear.

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A similar argument can be made for all of the graph problems in the rest of this talk.

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we get that CLIQ requires $2^{\Omega(v)}$.

A similar argument can be made for all of the graph problems in the rest of this talk.

So we won't bother.

The Ind. Set Problem

Def If G is a graph then an **ind. set** is a set of vertices such that no pair has an edge.

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Def

$$\mathrm{IS} = \{(G, k) : G \text{ has an ind. set of size } k\ \}.$$

We show that IS is NPC.

Two ways to prove IS is NPC

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Moral Once you have many NPC sets you can use them rather than SAT. In the future we won't bother with method one.
Bonus Reduction is linear, so assuming ETH and using SL we have IS requires $2^{\Omega(v)}$ time.

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Def If G is a graph then a **vertex cover** is a set of vertices such that every edge has at least one endpoint in that set.

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Def If G is a graph then a **vertex cover** is a set of vertices such that every edge has at least one endpoint in that set.

Def

$$\mathrm{VC} = \{(G,k) : G \text{ has a vertex cover of size } k\ \}.$$

We show that VC is NPC.

VC is NPC

We show $\mathrm{IS} \leq \mathrm{VC}$.

VC is NPC

We show $\mathrm{IS} \le \mathrm{VC}$.
GO TO BREAKOUT ROOMS TO TRY TO PROVE THIS.

VC is NPC

We show $\mathrm{IS} \le \mathrm{VC}$.
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$(G, k) \in \mathrm{IS}$ iff $(G, n-k) \in \mathrm{VC}$.
I leave the proof to you.

VC is NPC

We show $\mathrm{IS} \le \mathrm{VC}$.
GO TO BREAKOUT ROOMS TO TRY TO PROVE THIS.
$(G,k) \in \mathrm{IS}$ iff $(G, n-k) \in \mathrm{VC}$.
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RECAP

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2. IS is NPC.
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We now look at graph coloring.

Graph Coloring

Def A graph is k-colorable if can map vertices to $\{1, \ldots, k\}$ such that no adjacent vertices are the same color.

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Fractional Colorings One can define fractional colorings, so a graph can be $\frac{5}{2}$-colorable. For all $k > 2$, kCOL is NPC. We won't define or prove this.

3COL is NPC

Given $\phi = C_1 \vee \cdots \vee C_k$ in 3-CNF form we produce G such that

$$\phi \in \mathrm{3SAT} \text{ iff } G \in \mathrm{3COL}$$

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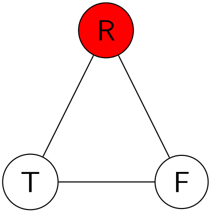
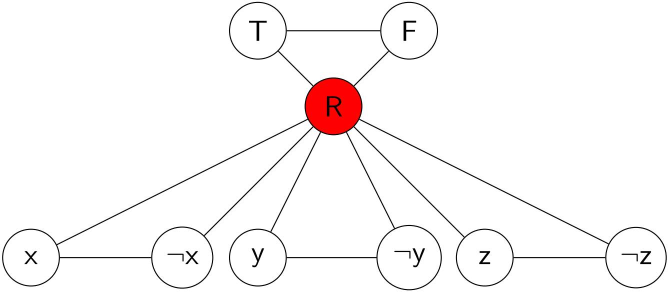
The third gadget involves the clauses.

TRUE, FALSE, and RED

We have the following triangle. The colors are *not* part of the graph; however,
we will think of later when a variable is colored T (F) then it is set to T (F).
We will make sure that no variable is colored R



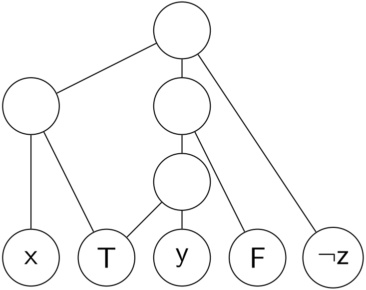
No var is R. $(x, \overline{x})$ is (T,F) or (F,T)



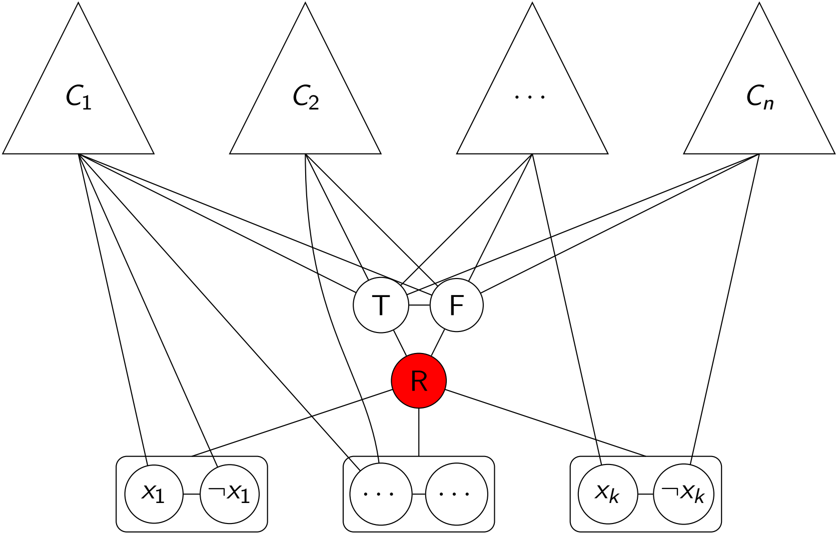
The Clause Gadget for $x \vee y \vee \neg z$

Recall that $x, y, \neg z$ are colored T or F.

1. If $x, y, \neg z$ are all colored F then NOT 3-colorable.
2. If $x, y, \neg z$ are anything else, then IS 3-colorable.



Putting it All Together



ETH and 3COL

You can check the reduction gives G of size $O(n+m)$.

ETH and $3\mathrm{COL}$

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ETH Using ETH and SL $3\mathrm{COL}$ requires $2^{\Omega(v)}$.

4-COL, 5-COL, ... all NPC

Erika How to show that, for $k \geq 3$, kCOL is NPC?

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Anyone Give me the reduction $\mathrm{3COL} \leq \mathrm{4COL}$.

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YES but by an insane reduction:

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Is there a sane reduction? Yes. Tell story about it.

Sanity

We give the Sane Reduction by first giving two Gadgets.

Gadget 1: x, y both $c \to z$ is c

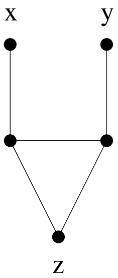
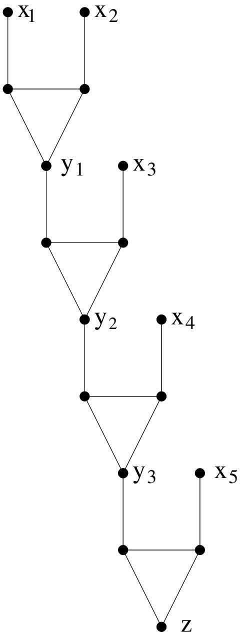


Figure: $\mathrm{GAD}(x, y, z)$

Gadget 2: x_1, x_2, x_3, x_4, x_5 all $c \rightarrow z$ is c



$\mathbf{4COL} \le \mathbf{3SAT}$

Given $G = (V, E)$ we want to construct $G' = (V', E')$ such that G is 4-Colorable iff G' is 3-Colorable

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Create a graph G' as follows:

SET UP This will not involve G at all except for the number of vertices.

For every $i \in V$ and $j \in \{1, 2, 3, 4\}$ have node v_{ij}.

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If v_{ij} is colored T in G' then i is colored j in G.

If v_{ij} is colored F in G' then i is NOT colored j in G.

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We use the tri-gadget to make sure that v_{ij} is not R.

Need to make sure that, for all $i \in V$:

1. At least one of $v_{i1}, v_{i2}, v_{i3}, v_{i4}$ is T
2. At most one of $v_{i1}, v_{i2}, v_{i3}, v_{i4}$ is T

Will do this on next slide.

4COL $\leq$ 3SAT SET UP

Fix i.

$4\mathrm{COL} \le 3\mathrm{SAT}$ SET UP

Fix i.

1. Have $\mathrm{GAD}(v_{i1}, v_{i2}, v_{i3}, v_{i4}, T)$.

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Have: any 3-coloring (proper or not) of G' will induce a 4-coloring (proper or not) on G.

$4\mathrm{COL} \le 3\mathrm{SAT}$ SET UP

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Have: any 3-coloring (proper or not) of G' will induce a 4-coloring (proper or not) on G.

Need: if G' has a proper 3-coloring then the induced 4-coloring is proper.

$4COL \leq 3SAT$ The Heart of the Construction

Need to make sure that two adjacent vertices of G have different colors.

$4\mathrm{COL} \le 3\mathrm{SAT}$ The Heart of the Construction

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Similar proof shows $\mathrm{kCOL} \le 3\mathrm{COL}$.

NPC Problems on Planar Graphs

Exposition by William Gasarch—U of MD

Restrict to Planar Graphs

We look at the graph problems we just proved NPC and see what happens when restricted to planar graphs.

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Testing Planarity is in P so we can assume the graph given IS Planar.

Clique on Planar Graphs

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If $k \leq 4$ can do brute force in $O(n^k) \leq O(n^4)$ time.

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NPC Coloring of Graphs and Planar Graphs

Exposition by William Gasarch—U of MD

Coloring of Graphs and Planar Graphs

$$\mathrm{kCOL} = \{G : G \text{ is } k\text{-colorable}\}$$

$$\mathrm{PL\text{-}kCOL} = \{G : G \text{ is planar and } k\text{-colorable}\}$$

In breakout rooms discuss planar $3\mathrm{COL}$, $4\mathrm{COL}$, $5\mathrm{COL}$, $\ldots$.

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Hence for all $k \geq 4$, $\mathrm{PL\text{-}kCOL}$ is in P.

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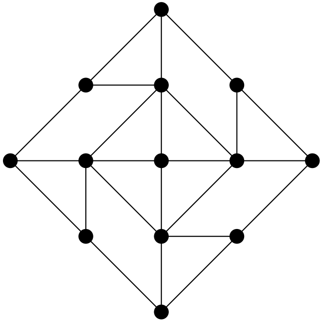
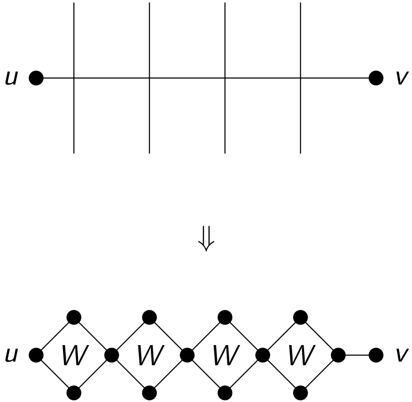
Gadget is on next slide and is all we need for the proof.

Crossover Gadget

$W =$ 

How to Use Crossover Gadget



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4. **Good News** We are not dealing with graphs anywhere near as complicated as K_n.

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No There **is** a $2^{O(\sqrt{v})}$ algorithm for $\mathrm{PL\text{-}3COL}$. Comes from work on graphs of bounded treewidth.

Do We Really Want to Devise a New Crossover Gadgets For Every Problem?

Exposition by William Gasarch—U of MD

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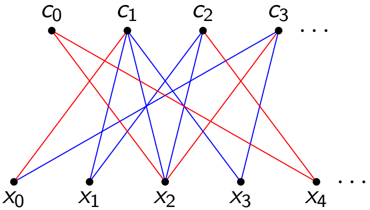
Rather than do all of these proofs and devise different crossover gadgets, we define a planar variant of **SAT**.

But SAT is not a graph problem!

Graph Associated to a CNF Formula



$$c_0 = (\neg x_2 \wedge \neg x_4)$$
$$c_1 = (\neg x_0 \wedge x_1 \wedge x_2 \wedge x_3)$$
$$c_2 = (x_1 \wedge x_2 \wedge \neg x_4)$$
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Figure: Bipartite Graph Associated to a CNF Formula

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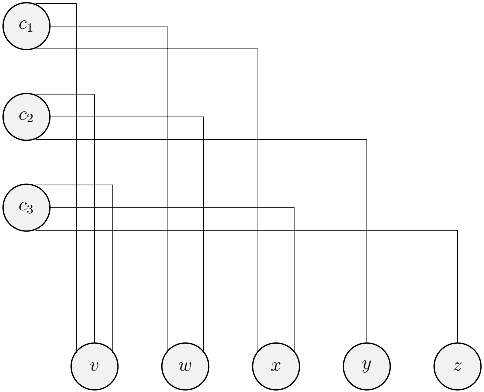
   $\mathrm{3SAT} \le X$ that
   map ϕ to G, and
   if ϕ is planar then G is planar.
3. Hence we will show many planar graph problems NPC without having to construct a gadget for each one.

PL-3SAT is NPC

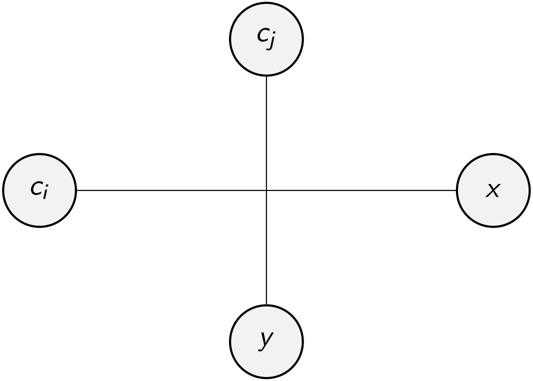
Given ϕ in 3CNF form we will

1. Draw the graph of ϕ as a grid (figure 1)
2. Note what crossings look like (figure 2)
3. Have a crossover gadget (figure 3).
4. Say what we add to the formula to get that crossover gadget.

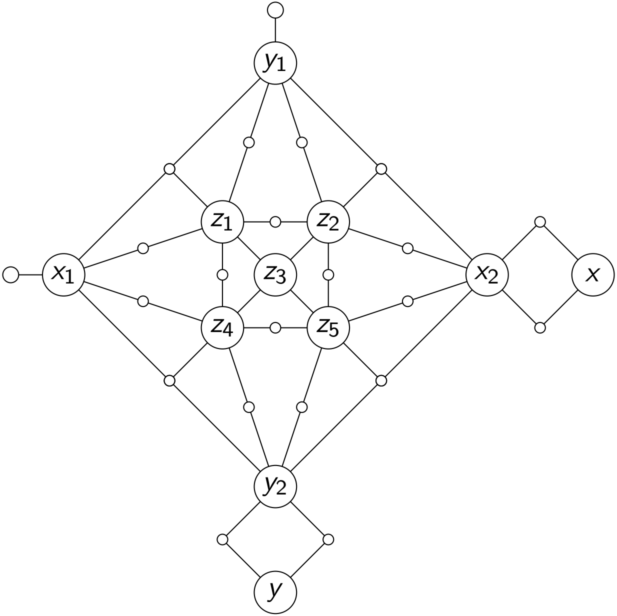
$$(v \vee \neg w \vee x) \wedge (v \vee \neg w \vee \neg y) \wedge (v \vee \neg x \vee z)$$



The Kinds of Crossings We Will Deal With



Crossover Gadget For Planar SAT



Add to the Boolean Fml

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Add to the Boolean Fml

The crossover gadget does not tell us what to add to the formula since if C and x are connected then either x or $\overline{x}$ could be in C. We add the following to the formula

$(\overline{x_2} \vee \overline{y_2} \vee z_5) \wedge (\overline{x_2} \vee \overline{z_5}) \wedge (y_2 \vee \overline{z_5})$ (which is $(x_2 \wedge y_2) \leftrightarrow z_5$)

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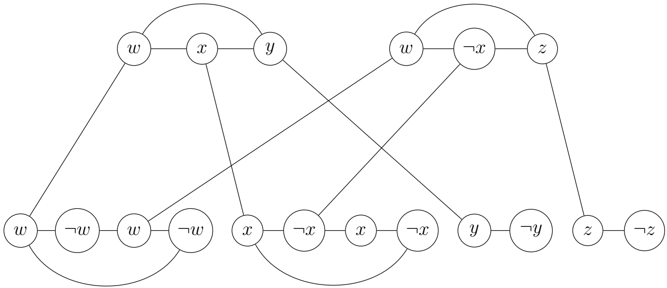
We give an example of the key gadget on the next slide.

$(w \vee x \vee y) \wedge (w \vee \neg x \vee z)$



General Construction for VC and Planar VC

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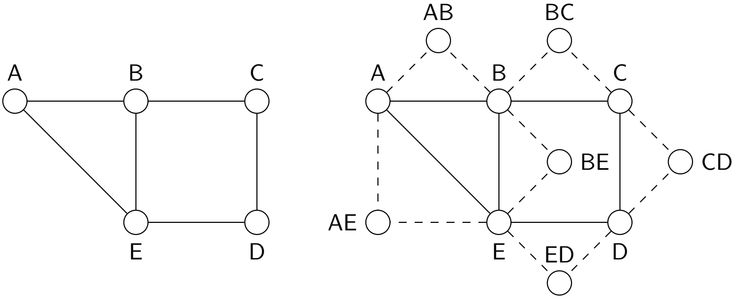
4) Seek a VC of size $5k$.

DOM and Planar DOM

Example of Gadget for DOM and Planar DOM



G =
Vertex Cover {B,C,E}

G' =
Dominating Set {B,E,CD}

Figure: Proof that Dominating Set is NP-Complete

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3) G' has a DOM of size $k \rightarrow G$ has a VC of size k: take the DOM set but if one of the vertices of form ab just take a.

Planar 1-in-3 SAT

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Given ϕ, a planar 3CNF formula, we form ϕ':
Replace every clause $L_1 \vee L_2 \vee L_3$ in ϕ (the L_i are literals) with

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Left to the reader to prove the ϕ' is planar and

$$\phi \in \mathrm{PL\text{-}3SAT} \text{ iff } \phi \in \mathrm{PL\text{-}1\text{-}in\text{-}3SAT}.$$

Exact Cover and Planar Exact Cover

Def $[n]$ is the set $\{1, \ldots, n\}$.
Def Exact Cover (X3C): Given $n \equiv 0 \pmod 3$ and a set $E_1, \ldots, E_m$ of 3-subsets of $[n]$ is does some set of $n/3$ of the E_i's cover $[n]$. Note that they cannot overlap.

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Def Planar Exact Cover ($\mathrm{PL\text{-}X3C}$): Input is an instance of $\mathrm{X3C}$ where the graph is planar.

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We do 1-in-3-SAT $\le$ X3C. Modifying to get PL-1-in-3SAT $\le$ PL-X3C is not automatic but not that hard.

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Example is on next page.
Small circles ae elements.
Large circles labeled E are 3-sets. They contain what they have an edge to.

Example of Reduction

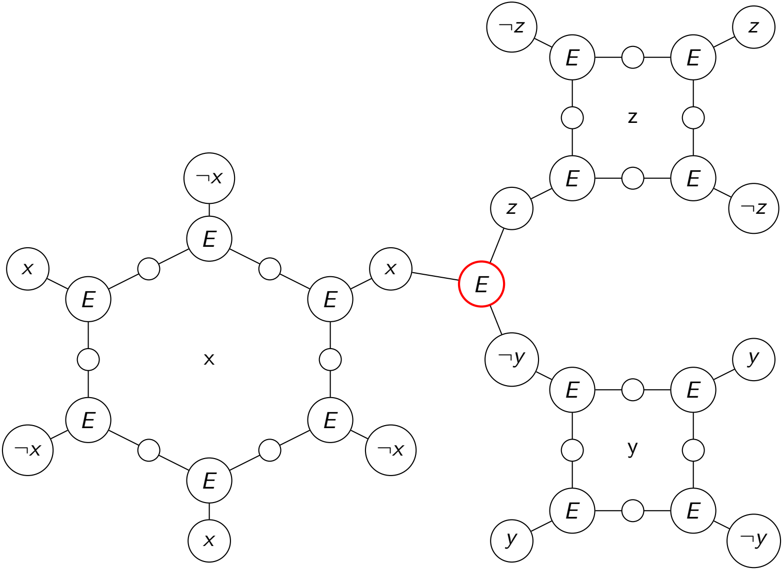

Figure: Gadget for PL-1-in-3SAT $\leq$ PL-X3C reduction.

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3) Intuition: Assume there is a 1-in-3 SAT assignment. Let x be a var set T. Then all of the x's in the var-gadget will be covered by the clauses they appear in. So half of the E's in the var-gadget are used. (This is not quite right since we are assuming that x appears in exactly m clauses.)

Planar Bipartite DOM set (PL-bi-DOM)

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We show that $\mathrm{PL\text{-}X3C} \leq \mathrm{PL\text{-}bi\text{-}DOM}$.

Given n and $E_1, \ldots, E_m$ we already have a planar bipartite graph.
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Easy to show that there is a covering of size $n/3$ iff there is a DOM set of size $\frac{n}{3} + m$.

BILL AND NATHAN, RECORD LECTURE!!!!

BILL: EITHER GO TO GRIPNP PACKET OR STOP RECORDING LECTURE!!!