BILL AND NATHAN, RECORD LECTURE!!!!

BILL RECORD LECTURE!!!
NPC SAT-type Problems

Exposition by William Gasarch—U of MD
Theory of P and NP: Paradigm Shift

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Computability and Complexity

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**Computability** The study of what problems can be solved in good time and which ones cannot be solved in good time. We think SAT cannot be solved in good time.
A Brief History of Theory of Complexity

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I will present two threads of history of Theory of Computing.

**Warning** I am not a historian so some of what I say here may be exaggerated or wrong. But the general gist is correct.
1. In 1805 Gauss invented the Fast Fourier Transform for his own use and never thought to tell anyone. A statement like FFT runs in $O(n \log n)$ time would probably be very strange for him. In 1965 FFT was (re)invented by Cooley and Tukey.

2. In 1936 Turing defined The Turing Machine (he didn't call it that) as a model of computation. He did not concern himself with how many steps it took.

3. During WW II Turing helped crack the German Enigma Code. This requires real computers solving problems quickly. Turing did not combine this with his other work. (Of course, he was busy winning WW II at the time.) Thread continued on next slide.
Thread One: From Gauss to Gowers

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1. In 1963 Hartmanis and Stearns define $\text{DTIME}(T(n))$ - a problem such that there is a Turing Machine that will, on inputs of length $n$, solve it in $\leq T(n)$ time.

2. In 1960's Knuth was a Math Ugrad by day and a programmer by night. He realized Maybe I can use Math to analyze these algorithms!

3. In 1965 Cobham defined polynomial time.

4. Matching was known to be (in todays terms) $NP \cap \text{co-NP}$. In 1965 Jack Edmonds showed (in todays terms) that it was in $P$ and defined $P$. He had ideas about $NP$ and (in todays terms) conjectured $P \neq NP$. Possibly Whiggist History.

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Thread Two: Eulerian and Hamiltonian Graphs

Def
1. A graph is **Eulerian** if there is a cycle that hits every **edge** once.
2. A graph is **Hamiltonian** if there is a cycle that hits every **vertex** once.
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A graph is EUL iff every vertex has even degree. So $\text{EUL} \in \text{P}$. 
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NPC enabled people to state what they wanted (HAM \( \in P \)) and hence it could be shown unlikely (HAM is NPC).

Not an Isolated Example Many other vague open problems in math can now be stated more rigorously and either solved or shown hard to solve.
Why Do We Believe
$P \neq NP$?

Exposition by William Gasarch—U of MD
Why Do We Believe $P \neq NP$?

1. There are 3 polls of what theorists think of $P$ vs $NP$. 88% of those polled said $P \neq NP$.

   Some $P = NP$-ers emailed me privately that it was a protest vote—They think $P \neq NP$ but people should be more open minded.

2. The $NPC$ problems have been worked on for a long time (many before $P$ and $NP$ were defined) and none are in $P$.

3. Intuition: Coming up with a proof seems harder than verifying a proof.

4. $P \neq NP$ has great explanatory power. See next slide.
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Approximating Set Cover

Set Cover Given $n$ and $S_1, \ldots, S_m \subseteq \{1, \ldots, n\}$ find the least number of sets $S_i$'s that cover $\{1, \ldots, n\}$.
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2. Dinur and Steurer in 2013 showed that, assuming $P \neq \text{NP}$, for all $\epsilon$ there is no $(1 - \epsilon) \ln n \times \text{OPTIMAL}$ approx alg for *Set Cover*

3. These two proofs have nothing to do with each other yet give matching upper and lower bounds.

4. There are many other approx problems which (1) we have been unable to improve, and (2) $P \neq \text{NP}$ implies they cannot be improved.
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NPC Problems on Boolean Formulas

Exposition by William Gasarch—U of MD
Bounding
(1) Literals Per Clause
(2) Occurrences of a Var

Exposition by William Gasarch—U of MD
Two Types of SAT

1. **kSAT-b**: Clauses have $\leq k$ literals, each var occurs $\leq b$ times.

2. **EU-kSAT-b**: Clauses have $k$ literals, each var occurs $\leq b$ times.
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SAT means no bound on number of literals-per-clause.

We will look at all four of these for various values of \( k \), \( b \).
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We will look at all four of these for various values of \( k, b \).
1. 1SAT:

P, \phi \in 1\text{SAT} \iff \text{there is no } x \text{ such that both } x \text{ and } \neg x \text{ occur.}

2. 2SAT:
P. Known result. Sketch: Convert every clause \( L_1 \lor L_2 \) into \( (\neg L_1 \rightarrow L_2) \land (\neg L_2 \rightarrow L_1) \). Make a directed graph with literals as vertices and the \( \rightarrow \) as edges. \( \phi \in 2\text{SAT} \iff \text{there is no path from an } x \text{ to a } \neg x \).

3. 3SAT:

NPC by Cook. The \( k = 1 \) and \( k = 2 \) cases are of course still in \( P \) if you bound \( b \).

Hence we look at \( k = 3 \) and bound on \( b \).
No Bound on $b$

1. 1SAT: P,
   $\phi \in \text{1SAT}$ iff there is no $x$ such that both $x$ and $\neg x$ occur.

2. 2SAT:

   - Known result. Sketch: Convert every clause $L_1 \lor L_2$ into $(\neg L_1 \rightarrow L_2) \land (\neg L_2 \rightarrow L_1)$.
   - Make a directed graph with literals as vertices and the $\rightarrow$ as edges.
   - $\phi \in \text{2SAT}$ iff there is no path from an $x$ to a $\neg x$.
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$k = 3$ and $b = 1, 2$

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\( k = 3 \) and \( b = 1, 2 \)

3SAT-1: \( P \). Always satisfiable, just set all literals that appear to \( T \). EU version would still be in \( P \).

3SAT-2: \( P \)? NPC? Work on in Breakout Rooms.
3SAT, all vars occur $\leq 2$. P

1) Input $\phi$ in 3CNF, all vars occurs $\leq 2$. 

2) If a literal is only pos, set T, if only neg, set F. If clause has 1 literal, set true. These operations may solve problem.

3) Every clause has 2 or 3 literals, every literal occurs as pos and neg. We show SAT.

4) A clause with all NEG literals we call a NEG-clause. If no NEG-clauses then SAT easily. IF there is a NEG-clause then set a var in it to F. (Numb NEG-clauses) + (Numb of clauses) DECREASES. Eventually satisfy all clauses.

Moral
This was a clever trick! To prove P $\neq$ NP would need to show that no clever trick will get SAT into P. Hard!
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   These operations may solve problem.
3) Every clause has 2 or 3 literals, every literal occurs as pos and neg. We show SAT.
4) A clause with all NEG literals we call a NEG-clause. If no NEG-clauses then SAT easily.
   IF there is a NEG-clause then set a var in it to F.
3SAT, all vars occur $\leq 2$. P

1) Input $\phi$ in 3CNF, all vars occurs $\leq 2$.  
2) If a literal is only pos, set T, if only neg, set F. If clause has 1 literal, set true.  
These operations may solve problem.  
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IF there is a NEG-clause then set a var in it to F.  
(Numb NEG-clauses) $+$ (Numb of clauses) DECREASES.
3SAT, all vars occur $\leq 2$. P

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If no NEG-clauses then SAT easily.
IF there is a NEG-clause then set a var in it to F.
(Numb NEG-clauses) + (Numb of clauses) DECREASES. Eventually satisfy all clauses.
3SAT, all vars occur $\leq 2$. P

1) Input $\phi$ in 3CNF, all vars occurs $\leq 2$.
2) If a literal is only pos, set $T$, if only neg, set $F$. If clause has 1 literal, set true.
   These operations may solve problem.
3) Every clause has 2 or 3 literals, every literal occurs as pos and neg. We show SAT.
4) A clause with all NEG literals we call a NEG-clause.
   If no NEG-clauses then SAT easily.
   IF there is a NEG-clause then set a var in it to $F$.
   (Numb NEG-clauses) + (Numb of clauses) DECREASES.
   Eventually satisfy all clauses.

**Moral** This was a clever trick! To prove $P \neq NP$ would need to show that no clever trick will get SAT into $P$. Hard!
3SAT, all vars occur $\leq 3$

3SAT-3: There are $\leq 3$ clauses per literal and every var occurs $\leq 3$ times.
3SAT, all vars occur $\leq 3$

3SAT-3: There are $\leq 3$ clauses per literal and every var occurs $\leq 3$ times.
In P? NPC? Breakout Rooms!
We will prove this **NPC**. Erika- how will we do it?
3SAT, all vars occur $\leq 3$. NPC

We will prove this NPC. Erika- how will we do it? By a Reduction

1) Input $\phi$ in 3CNF. Want $\phi'$ 3CNF with all vars occurring $\leq 3$ times such that $\phi \in \text{SAT}$ iff $\phi' \in \text{SAT}$.  

\begin{itemize}
\item 2) If a var occurs $\leq 3$ times then leave it alone.
\item 3) If a var occurs $m \geq 4$ times then
\begin{itemize}
\item a) Add new vars $x_1, \ldots, x_m$. Replace $i$th occurrence of $x$ with $x_i$.
\item b) Add the clauses $x_1 \rightarrow x_2$, $x_2 \rightarrow x_3$, $\ldots$, $x_{m-1} \rightarrow x_m$, $x_m \rightarrow x_1$.
\end{itemize}
\end{itemize}

(Formally $x_1 \rightarrow x_2$ is $(\neg x_1 \lor x_2)$.)

Clearly $\phi \in \text{3CNF}$ and all variables occur $\leq 3$ times.

Clearly $\phi \in \text{SAT}$ iff $\phi' \in \text{SAT}$.

Moral

Going from $b \leq 2$ to $b \leq 3$ matters!
3SAT, all vars occur $\leq 3$. NPC

We will prove this NPC. Erika- how will we do it? By a Reduction
1) Input $\phi$ in 3CNF. Want $\phi'$ 3CNF with all vars occurring $\leq 3$ times such that $\phi \in SAT$ iff $\phi' \in SAT$.
2) If a var occurs $\leq 3$ times then leave it alone.
3SAT, all vars occur \leq 3. NPC

We will prove this NPC. Erika- how will we do it? By a Reduction
1) Input \( \phi \) in 3CNF. Want \( \phi' \) 3CNF with all vars occurring \( \leq 3 \)
times such that \( \phi \in \text{SAT} \) iff \( \phi' \in \text{SAT} \).
2) If a var occurs \( \leq 3 \) times then leave it alone.
3) If a var occurs \( m \geq 4 \) times then
We will prove this NPC. Erika- how will we do it? By a Reduction

1) Input $\phi$ in 3CNF. Want $\phi'$ 3CNF with all vars occurring $\leq 3$ times such that $\phi \in \text{SAT}$ iff $\phi' \in \text{SAT}$.

2) If a var occurs $\leq 3$ times then leave it alone.

3) If a var occurs $m \geq 4$ times then

a) Add new vars $x_1, \ldots, x_m$. Replace $i$th occurrence of $x$ with $x_i$. 


3SAT, all vars occur $\leq 3$. NPC

We will prove this NPC. Erika- how will we do it? By a Reduction

1) Input $\phi$ in 3CNF. Want $\phi'$ 3CNF with all vars occurring $\leq 3$ times such that $\phi \in \text{SAT}$ iff $\phi' \in \text{SAT}$.

2) If a var occurs $\leq 3$ times then leave it alone.

3) If a var occurs $m \geq 4$ times then
   a) Add new vars $x_1, \ldots, x_m$. Replace $i$th occurrence of $x$ with $x_i$.
   b) Add the clauses $x_1 \rightarrow x_2$, $x_2 \rightarrow x_3$, $\ldots$, $x_{m-1} \rightarrow x_m$, $x_m \rightarrow x_1$.

(Formally $x_1 \rightarrow x_2$ is $(\neg x_1 \lor x_2)$.)
3SAT, all vars occur ≤ 3. NPC

We will prove this NPC. Erika- how will we do it? By a Reduction
1) Input $\phi$ in 3CNF. Want $\phi'$ 3CNF with all vars occurring ≤ 3 times such that $\phi \in \text{SAT}$ iff $\phi' \in \text{SAT}$.
2) If a var occurs ≤ 3 times then leave it alone.
3) If a var occurs $m \geq 4$ times then
   a) Add new vars $x_1, \ldots, x_m$. Replace $i$th occurrence of $x$ with $x_i$.
   b) Add the clauses $x_1 \rightarrow x_2$, $x_2 \rightarrow x_3$, $\ldots$, $x_{m-1} \rightarrow x_m$, $x_m \rightarrow x_1$.
   (Formally $x_1 \rightarrow x_2$ is $(\neg x_1 \lor x_2)$.)
Clearly $\phi \in 3\text{CNF}$ and all variables occur ≤ 3 times.
3SAT, all vars occur \(\leq 3\). NPC

We will prove this NPC. Erika- how will we do it? By a Reduction

1) Input \(\phi\) in 3CNF. Want \(\phi'\) 3CNF with all vars occurring \(\leq 3\) times such that \(\phi \in \text{SAT}\) iff \(\phi' \in \text{SAT}\).

2) If a var occurs \(\leq 3\) times then leave it alone.

3) If a var occurs \(m \geq 4\) times then
   a) Add new vars \(x_1, \ldots, x_m\). Replace \(i\)th occurrence of \(x\) with \(x_i\).
   b) Add the clauses \(x_1 \rightarrow x_2, x_2 \rightarrow x_3, \ldots, x_{m-1} \rightarrow x_m, x_m \rightarrow x_1\).

(Formally \(x_1 \rightarrow x_2\) is \((\neg x_1 \lor x_2\)).

Clearly \(\phi \in 3\text{CNF}\) and all variables occur \(\leq 3\) times.

Clearly \(\phi \in \text{SAT}\) iff \(\phi' \in \text{SAT}\).
We will prove this NPC. Erika- how will we do it? By a Reduction
1) Input $\phi$ in 3CNF. Want $\phi'$ 3CNF with all vars occurring $\leq 3$
times such that $\phi \in \text{SAT}$ iff $\phi' \in \text{SAT}$.
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      (Formally $x_1 \rightarrow x_2$ is $\neg x_1 \vee x_2$.)
Clearly $\phi \in 3\text{CNF}$ and all variables occur $\leq 3$ times.
Clearly $\phi \in \text{SAT}$ iff $\phi' \in \text{SAT}$
**Moral** Going from $b \leq 2$ to $b \leq 3$ matters!
**EU-3SAT-3?**

**EU-3SAT-3:** Every clause has *exactly* 3 literals. Every variable occurs $\leq 3$ times. P? NPC?
EU-3SAT-3: Every clause has exactly 3 literals. Every variable occurs $\leq 3$ times. P? NPC?
Go to breakout rooms to work on this.
EU-3SAT-3 is in $P$

EU-3SAT-3 with $b \leq 3$ is in $P$. 

This needs a known Theorem and its Corollary.

For this slide $G = (A, B, E)$ is a bipartite graph.

A Matching of $A$ into $B$ is a set of disjoint edges so that every element of $A$ is an endpoint of some edge. View as an injection of $A$ into $B$.

$X \subseteq A$.

$E(X) = \{ y \in Y : (\exists x \in X) [(x, y) \in E] \}$.

Hall's Matching Theorem

If, for all $X \subseteq A$, $|E(X)| \geq |X|$, then there exists a matching from $A$ to $B$.

Corollary

If there exists $k$ such that (1) for every $x \in A$, $\deg(x) \geq k$, and (2) for every $y \in B$, $\deg(y) \leq k$, then there is a matching from $A$ to $B$.

We will use these on the next slide.
EU-3SAT-3 is in $P$

EU-3SAT-3 with $b \leq 3$ is in $P$.
This needs a known Theorem and its Corollary.
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EU-3SAT-3 is in $P$

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EU-3SAT-3 is in $P$

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Hall’s Matching Theorem If, for all $X \subseteq A$, $|E(X)| \geq |X|$ then there exists a matching from $A$ to $B$.

Corollary If there exists $k$ such that (1) for every $x \in A$, $\text{deg}(x) \geq k$, and (2) for every $y \in B$, $\text{deg}(y) \leq k$, then there is a matching from $A$ to $B$. We will use these on the next slide.
EU-3SAT-3 is in \( P \)

EU-3SAT-3 with \( b \leq 3 \) is in \( P \).
This needs a known Theorem and its Corollary.
For this slide \( G = (A, B, E) \) is a bipartite graph.
A **Matching of \( A \) into \( B \)** is a set of disjoint edges so that every element of \( A \) is an endpoint of some edge. View as an injection of \( A \) into \( B \).
\( X \subseteq A \). \( E(X) = \{ y \in Y : (\exists x \in X) [(x, y) \in E] \} \).

**Hall’s Matching Theorem** If, for all \( X \subseteq A \), \( |E(X)| \geq |X| \) then there exists a matching from \( A \) to \( B \).

**Corollary** If there exists \( k \) such that (1) for every \( x \in A \), \( \deg(x) \geq k \), and (2) for every \( y \in B \), \( \deg(y) \leq k \), then there is a matching from \( A \) to \( B \).

We will use these on the next slide.
Every EU-3CNF-3 fml is Satisfiable

Let $\phi$ be EU-3CNF-3. $\phi = C_1 \lor \cdots \lor C_m$.

Form a bipartite graph:

1. Clauses on the left, variables on the right.
2. Edge from $C$ to $x$ if either $x$ or $\neg x$ is in $C$.

Every clause has degree 3.
Every EU-3CNF-3 fml is Satisfiable

Let $\phi$ be EU-3CNF-3. $\phi = C_1 \lor \cdots \lor C_m$.

Form a bipartite graph:

1. Clauses on the left, variables on the right.
2. Edge from $C$ to $x$ if either $x$ or $\neg x$ is in $C$.

Every clause has degree 3. Every variable has degree $\leq 3$.
By Corollary there is a matching of $C$’s to $V$’s. This gives a satisfying assignment.
Every EU-3CNF-3 fml is Satisfiable

Let $\phi$ be EU-3CNF-3. $\phi = C_1 \lor \cdots \lor C_m$.

Form a bipartite graph:

1. Clauses on the left, variables on the right.
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Every clause has degree 3. Every variable has degree $\leq 3$.

By Corollary there is a matching of $C$’s to $V$’s. This gives a satisfying assignment.

**Moral** The algorithm used a THEOREM in math that perhaps you did not know. To prove $P \neq NP$ would need to say this can’t happen. Hard!
A Variant of SAT

Exposition by William Gasarch—U of MD
1-in-3-SAT

Def 1-in-3-SAT (1-in-3-SAT) is the problem of, given a formula $D_1 \land \cdots \land D_m$ find an assignment that satisfies exactly one literal-per-clause. We will show that 1-in-3-SAT is NPC.
1-in-3-SAT

**Def 1-in-3-SAT (1-in-3-SAT)** is the problem of, given a formula $D_1 \land \cdots \land D_m$ find an assignment that satisfies *exactly* one literal-per-clause. We will show that 1-in-3-SAT is NPC.

*Is this a Natural Question?* VOTE, though this is an opinion question.
**Def 1-in-3-SAT** (1-in-3-SAT) is the problem of, given a formula $D_1 \land \cdots \land D_m$ find an assignment that satisfies *exactly* one literal-per-clause. We will show that 1-in-3-SAT is NPC.

**Is this a Natural Question?** VOTE, though this is an opinion question.

**My Opinion** The problem is *not* natural.
**Def 1-in-3-SAT** (1-in-3-SAT) is the problem of, given a formula $D_1 \land \cdots \land D_m$ find an assignment that satisfies **exactly** one literal-per-clause. We will show that 1-in-3-SAT is NPC.

*Is this a Natural Question?* VOTE, though this is an opinion question.

*My Opinion* The problem is **not** natural.

*So why are we studying it* Discuss.
Def **1-in-3-SAT** (1-in-3-SAT) is the problem of, given a formula $D_1 \land \cdots \land D_m$ find an assignment that satisfies *exactly* one literal-per-clause. We will show that 1-in-3-SAT is NPC.

**Is this a Natural Question?** VOTE, though this is an opinion question.

**My Opinion** The problem is **not** natural.

**So why are we studying it** Discuss.

**Its a means to an end** We will show natural problems NPC by using reductions from 1-in-3-SAT. We will do a reduction from a variant of 1-in-3-SAT.
1-in-3-SAT is NPC

Given $\phi = C_1 \land \cdots \land C_m$ in 3CNF create the $\phi'$ as follows:
1-in-3-SAT is NPC

Given $\phi = C_1 \land \cdots \land C_m$ in 3CNF create the $\phi'$ as follows:
Replace clause $(L_1 \lor L_2 \lor L_3)$ with

$$(\neg L_1 \lor a \lor b) \land (b \lor L_2 \lor c) \land (c \lor d \lor \neg L_3).$$

where $a, b, c, d$ are new variables.
1-in-3-SAT is NPC

Given $\phi = C_1 \land \cdots \land C_m$ in 3CNF create the $\phi'$ as follows:
Replace clause $(L_1 \lor L_2 \lor L_3)$ with

$$(\neg L_1 \lor a \lor b) \land (b \lor L_2 \lor c) \land (c \lor d \lor \neg L_3).$$

where $a, b, c, d$ are new variables.
Leave it to the reader to prove

$$\phi \in \text{3SAT} \text{ iff } \phi' \in \text{1-in-3-SAT}.$$
Mono 1-in-3-SAT

Mono 1-in-3-SAT (mono-1-in-3-SAT): Given a formula \( E_1 \land \cdots \land E_p \) where all vars occur positively, is there an assignment that satisfies exactly one literal-per-clause.
Mono 1-in-3-SAT

**Mono 1-in-3-SAT (mono-1-in-3-SAT):** Given a formula $E_1 \land \cdots \land E_p$ where all vars occur positively, is there an assignment that satisfies **exactly** one literal-per-clause.

**Thm** $1$-in-$3$-SAT $\leq$ mono-$1$-in-$3$-SAT

Given $3$CNF form $\phi(x_1, \ldots, x_n) = C_1 \lor \cdots \lor C_k$ want $\phi'$ such that $\phi \in 1$-in-$3$-SAT iff $\phi' \in$ mono-$1$-in-$3$-SAT.
Mono 1-in-3-SAT

**Mono 1-in-3-SAT (mono-1-in-3-SAT):** Given a formula $E_1 \land \cdots \land E_p$ where all vars occur positively, is there an assignment that satisfies exactly one literal-per-clause.

**Thm** 1-in-3-SAT $\leq$ mono-1-in-3-SAT

Given 3CNF form $\phi(x_1, \ldots, x_n) = C_1 \lor \cdots \lor C_k$ want $\phi'$ such that $\phi \in 1$-in-3-SAT iff $\phi' \in$ mono-1-in-3-SAT.

1) New Vars $t, f$ and new clause $E = (t \lor f \lor f)$. Any 1-in-3-SAT assignment of $\phi$ will set $t$ to $T$ and $f$ to $F$. 
Mono 1-in-3-SAT

**Mono 1-in-3-SAT (mono-1-in-3-SAT):** Given a formula $E_1 \land \cdots \land E_p$ where all vars occur positively, is there an assignment that satisfies **exactly** one literal-per-clause.

**Thm** 1-in-3-SAT $\leq$ mono-1-in-3-SAT

Given 3CNF form $\phi(x_1, \ldots, x_n) = C_1 \lor \cdots \lor C_k$ want $\phi'$ such that $\phi \in$ 1-in-3-SAT iff $\phi' \in$ mono-1-in-3-SAT.

1) New Vars $t$, $f$ and new clause $E = (t \lor f \lor f)$. Any 1-in-3-SAT assignment of $\phi$ will set $t$ to $T$ and $f$ to $F$.

2) For each $x_j$ have new var $x'_j$ and clause $D_j = (f \lor x_j \lor x'_j)$. Any 1-in-3-SAT assignment for $\phi$ will set $x_j$, $x'_j$ to opposites.
Mono 1-in-3-SAT

**Mono 1-in-3-SAT (mono-1-in-3-SAT):** Given a formula $E_1 \land \cdots \land E_p$ where all vars occur positively, is there an assignment that satisfies exactly one literal-per-clause.

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Given 3CNF form $\phi(x_1, \ldots, x_n) = C_1 \lor \cdots \lor C_k$ want $\phi'$ such that $\phi \in$ 1-in-3-SAT iff $\phi' \in$ mono-1-in-3-SAT.

1) New Vars $t, f$ and new clause $E = (t \lor f \lor f)$. Any 1-in-3-SAT assignment of $\phi$ will set $t$ to $T$ and $f$ to $F$.

2) For each $x_j$ have new var $x'_j$ and clause $D_j = (f \lor x_j \lor x'_j)$. Any 1-in-3-SAT assignment for $\phi$ will set $x_j, x'_j$ to opposites.

3) For each $C_i$ let $C'_i$ be obtained by replacing every $\overline{x_j}$ with $x'_j$. 
Mono 1-in-3-SAT

Mono 1-in-3-SAT (mono-1-in-3-SAT): Given a formula $E_1 \land \cdots \land E_p$ where all vars occur positively, is there an assignment that satisfies exactly one literal-per-clause.

Thm 1-in-3-SAT $\leq$ mono-1-in-3-SAT

Given 3CNF form $\phi(x_1, \ldots, x_n) = C_1 \lor \cdots \lor C_k$ want $\phi'$ such that $\phi \in 1$-in-3-SAT iff $\phi' \in$ mono-1-in-3-SAT.

1) New Vars $t$, $f$ and new clause $E = (t \lor f \lor f)$. Any 1-in-3-SAT assignment of $\phi$ will set $t$ to $T$ and $f$ to $F$.

2) For each $x_j$ have new var $x'_j$ and clause $D_j = (f \lor x_j \lor x'_j)$. Any 1-in-3-SAT assignment for $\phi$ will set $x_j$, $x'_j$ to opposites.

3) For each $C_i$ let $C'_i$ be obtained by replacing every $\overline{x}_j$ with $x'_j$.

$$\phi' = C'_1 \land \cdots \land C'_k \land D_1 \land \cdots \land D_n \land E.$$  

Leave it to the reader to show $\phi \in 1$-in-3-SAT iff $\phi' \in$ mono-1-in-3-SAT.
A Puzzle we Prove Hard Using mono-1-in-3-SAT

Exposition by William Gasarch—U of MD
Why is mono-1-in-3-SAT Important?

We care about the mono-1-in-3-SAT problem!
Why is \texttt{mono-1-in-3-SAT} Important?

We care about the \texttt{mono-1-in-3-SAT} problem! \textbf{NOT!}
Why is mono-1-in-3-SAT Important?

We care about the mono-1-in-3-SAT problem! **NOT!**
We will use it to show that a puzzle we DO care about is NPC.
Why is mono-1-in-3-SAT Important?

We care about the mono-1-in-3-SAT problem! NOT!
We will use it to show that a puzzle we DO care about is NPC

\[
\begin{array}{cccc}
S & E & N & D \\
+ & M & O & R & E \\
\hline
M & O & N & E & Y
\end{array}
\]

The SEND MORE MONEY Cryptarithms
Why is mono-1-in-3-SAT Important?

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\[
\begin{array}{ccccccc}
S & E & N & D \\
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\hline
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\end{array}
\]

The SEND MORE MONEY Cryptarithms

1) A carry can be at most 1. Hence \( M = 1 \).
Why is mono-1-in-3-SAT Important?

We care about the mono-1-in-3-SAT problem! NOT!
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\[
\begin{array}{cccc}
S & E & N & D \\
+ & M & O & R & E \\
\hline \\
M & O & N & E & Y \\
\end{array}
\]

The SEND MORE MONEY Cryptarithms
1) A carry can be at most 1. Hence \( M = 1 \).
2) Since \( M = 1 \), \( S + M + \text{poss carry} \leq 10 \). Since there is a carry, \( S + M + \text{poss carry} = 10 \) so \( O = 0 \).
Why is mono-1-in-3-SAT Important?

We care about the mono-1-in-3-SAT problem! **NOT**!
We will use it to show that a puzzle we DO care about is NPC

\[
\begin{array}{c}
S \\
E \\
N \\
D \\
\end{array}
\begin{array}{c}
+ \\
M \\
O \\
R \\
E \\
\end{array}
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3) Can keep on reasoning like this and we find:
Why is mono-1-in-3-SAT Important?

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\[
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3) Can keep on reasoning like this and we find:

\[
\begin{array}{ccccccc}
9 & 5 & 6 & 7 \\
+ & 1 & 0 & 8 & 5 \\
\hline 
1 & 0 & 6 & 5 & 2 \\
\end{array}
\]

The Solution to The SEND MORE MONEY Cryptarithms
How Did We Solve SEND+MORE=MONEY?

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Spoiler Alert:
How Did We Solve SEND + MORE = MONEY?

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Is the general problem NPC?
Spoiler Alert: Yes
We want to show that Cryptarithmetic is NPC. We need a definition.
Definition of Cryptarithmetic Problem

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**CRYPTARITHM**

**Input** \( B, m \in \mathbb{N} \). \( \Sigma \) is alphabet of \( B \) letters.

- \( x_0, \ldots, x_{m-1} \). Each \( x_i \in \Sigma \).
- \( y_0, \ldots, y_{m-1} \). Each \( y_i \in \Sigma \).
- \( z_0, \ldots, z_m \). Each \( z_i \in \Sigma \). The symbol \( z_m \) is optional.
Definition of Cryptarithms Problem

We want to show that Cryptarithms is \textit{NPC}. We need a definition.

**CRYPTARITHM**

**Input** $B, m \in \mathbb{N}$. $\Sigma$ is alphabet of $B$ letters.

$x_0, \ldots, x_{m-1}$. Each $x_i \in \Sigma$.

$y_0, \ldots, y_{m-1}$. Each $y_i \in \Sigma$.

$z_0, \ldots, z_m$. Each $z_i \in \Sigma$. The symbol $z_m$ is optional.

**Question** Does there exists injection $\Sigma \rightarrow \{0, \ldots, B-1\}$ so that the arithmetic below is correct in base $B$?

\[
\begin{array}{cccc}
  x_{m-1} & \cdots & x_0 \\
  + & y_{m-1} & \cdots & y_0 \\
  \hline \\
  z_m & z_{m-1} & \cdots & z_0
\end{array}
\]
We Show CRYPTARITHM is NPC

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**Input** \( \phi(x_1, \ldots, x_n) = C_1 \land \cdots \land C_m \) where all vars occur positive.
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**Output** An instance \( J \) of CRYPTARITHM such that TFAE
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1. Exists assignment that satisfies exactly one var per clause.
2. Exists solution to CRYPTOARITHM \( J \).
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1. Exists assignment that satisfies exactly one var per clause.
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We do the reduction in three parts, so three more slides! We call the parts **gadgets**.
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\begin{array}{c}
0p0 \\
0p0 \\
1q0 \\
\end{array}
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$$
\begin{array}{c}
0 \ p \ 0 \\
0 \ p \ 0 \\
1 \ q \ 0
\end{array}
$$

We leave it to the reader to show that this ensures $0$ maps to $0$ and $1$ maps to $1$. 

For every variable $v$ we have a symbol $v \in \Sigma$. Our intent is
Vars ≡ 0, 1 (mod 4)

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If $v$ is false then $v \equiv 0 \pmod{4}$.
The following gadget ensures that $v \equiv 0, 1 \pmod{4}$.

\[
\begin{array}{cccccc}
0 & b & c & 0 & a & 0 \\
0 & b & c & 0 & a & 0 \\
0 & v & d & 0 & b & 0 \\
\end{array}
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Since $c + c = d$ the carry is $C \in \{0, 1\}$. 
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Since $c + c = d$ the carry is $C \in \{0, 1\}$.
Since $b + b = v$, $v = 2b + C$, so $v \equiv 0, 1 \pmod{4}$.
For every variable \( v \) we have a symbol \( v \in \Sigma \). Our intent is
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Since \( b + b = v \), \( v = 2b + C \), so \( v \equiv 0, 1 \pmod{4} \).

**Note** Do this for all vars \( v \), using a different \( a, b, c \) for each one.
Clauses Need to have Exactly One Var True

Clause is \((x \lor y \lor z)\).
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Clause is \((x \lor y \lor z)\). Gadget is:

\[
\begin{array}{cccccccccc}
0 & l & 0 & x & 0 & 1 & 0 & b & 0 & a & 0 \\
0 & z & 0 & y & 0 & c & 0 & b & 0 & a & 0 \\
0 & d & 0 & l & 0 & d & 0 & c & 0 & b & 0 \\
\end{array}
\]

\(b \equiv 0 \pmod{2}\), so \(b = \overline{a}\).

\(c \equiv 0 \pmod{4}\), so \(c = \overline{a}\).

\(d = c + 1\) so \(d \equiv 1 \pmod{4}\).

\(I + z = d\) so \(x + y + z \equiv 1 \pmod{4}\).

Note: For each clause use a different \(a\), \(b\), \(c\), \(I\).

So if \(J\) has a solution then \(\phi\) has a 1-in-3 assignment.

Need if \(\phi\) has a 1-in-3 assignment then \(J\) has sol. Left to reader.
Clauses Need to have Exactly One Var True

Clause is \((x \lor y \lor z)\). Gadget is:

\[
\begin{array}{cccccccc}
0 & 1 & 0 & x & 0 & 1 & 0 & b & 0 & a & 0 \\
0 & z & 0 & y & 0 & c & 0 & b & 0 & a & 0 \\
0 & d & 0 & l & 0 & d & 0 & c & 0 & b & 0 \\
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\(a + a = b\), so \(b \equiv 0 \pmod{2}\).
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\(x + y = l\) so \(x + y \equiv l \pmod{4}\).
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\(x + y = l\) so \(x + y \equiv l \pmod{4}\).
\(l + z = d\) so \(x + y + z \equiv 1 \pmod{4}\).
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Clause is \((x \lor y \lor z)\). Gadget is:

\[
\begin{array}{cccccccccccc}
0 & I & 0 & x & 0 & 1 & 0 & b & 0 & a & 0 \\
0 & z & 0 & y & 0 & c & 0 & b & 0 & a & 0 \\
0 & d & 0 & I & 0 & d & 0 & c & 0 & b & 0 \\
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