Mathematical Induction: Introduction and basic problems

CMSC 250

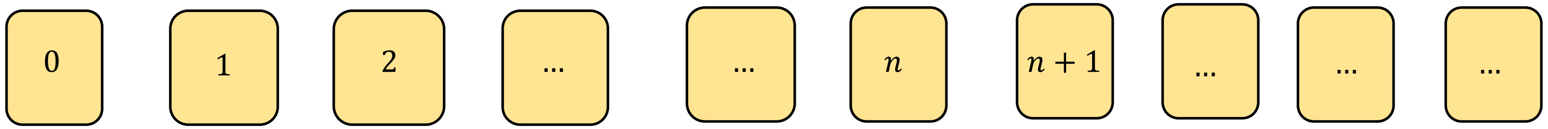
Lecture structure

- Considering campus closure, lectures will have to be given **remotely**.
- Every lecture will be broken down into 3-4 bite-sized Panopto videos.
- Power Point still projected, slides will be up by the time lecture is up.
- Can always tell Jason on CW or over e-mail if the picture or the audio aren't really good.
- Induction lectures delivered during second week of extended SpringBreak for practice with homework 6.
- We will see what we will do with lectures of week of Mar 30th – Apr 3rd.

FIRST VIDEO: INTRO AND BASIC SEQUENCE PROBLEMS

The idea behind induction

- Suppose that we want to prove that a proposition $P(n)$ is true for all natural numbers n.



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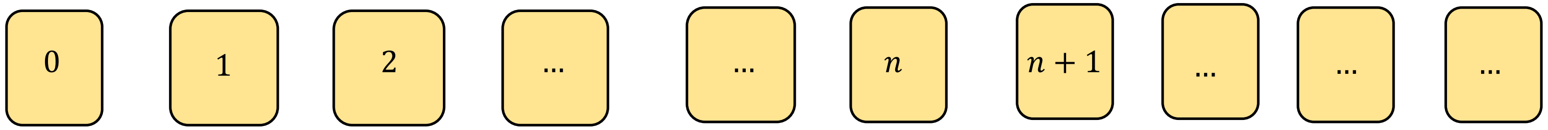

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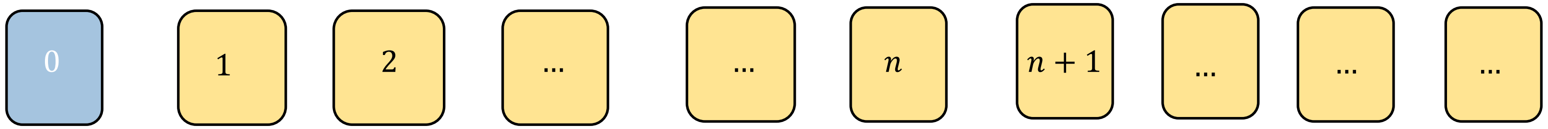

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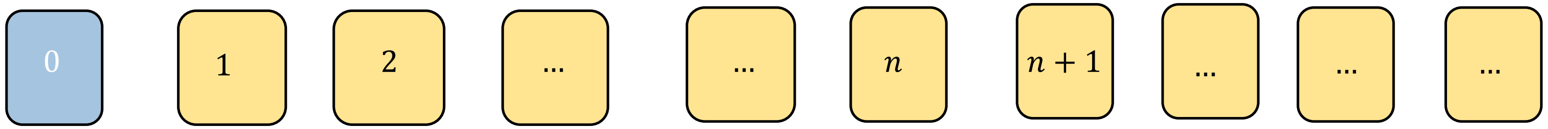

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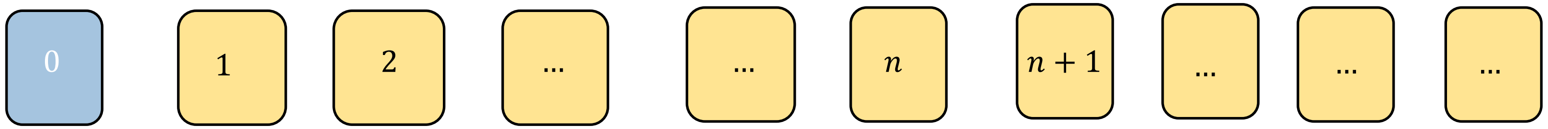

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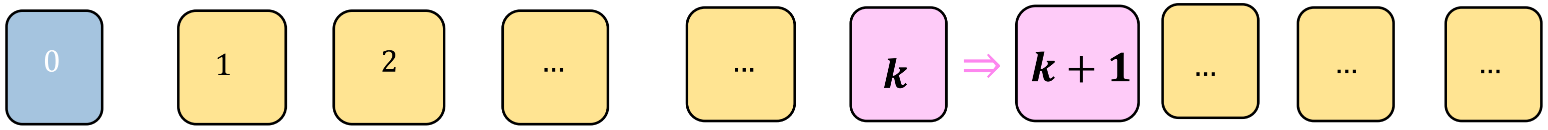

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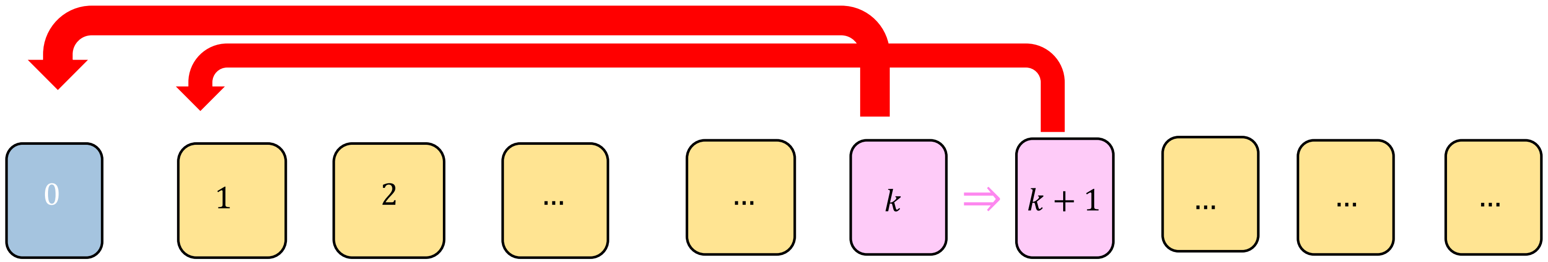

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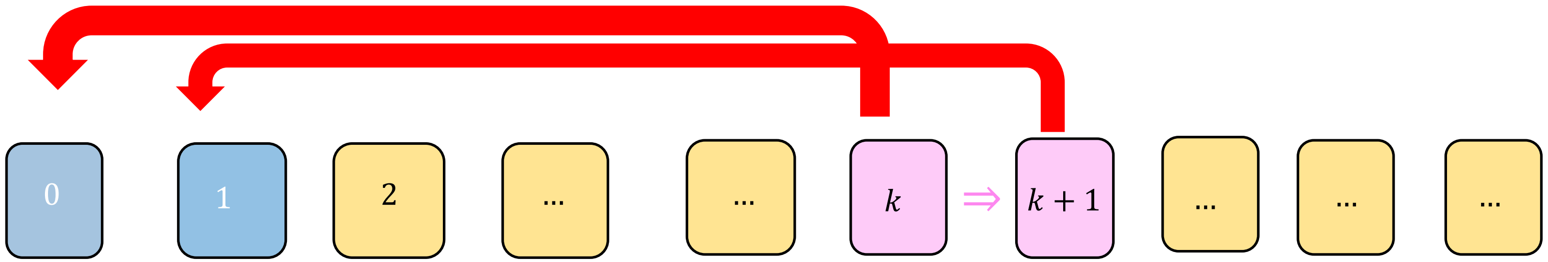

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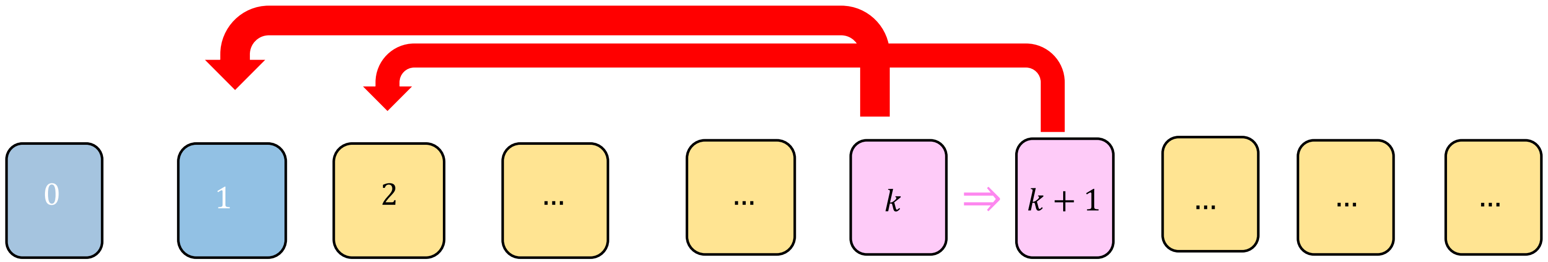

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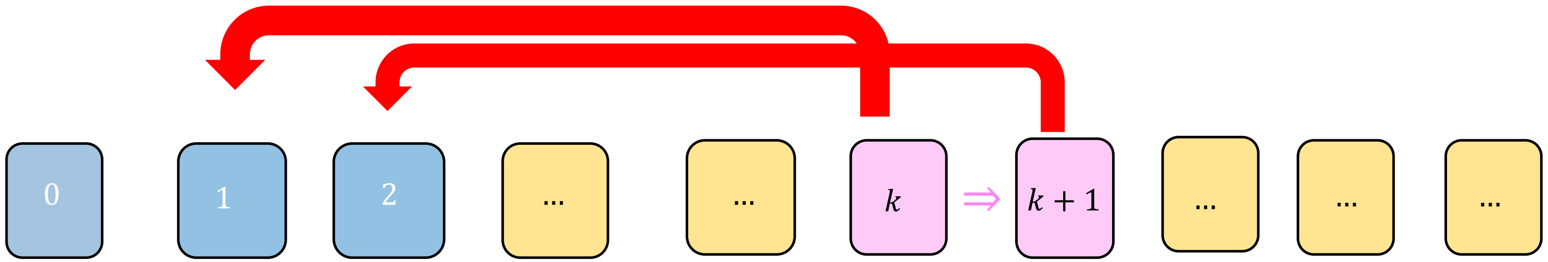

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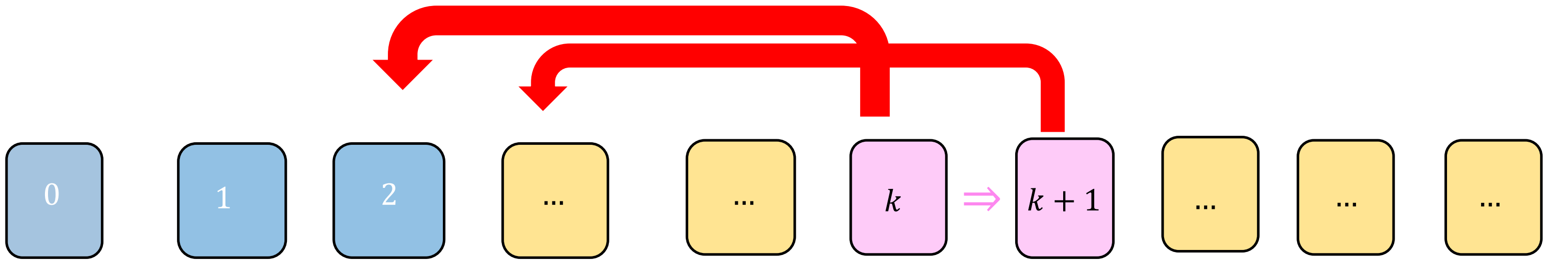

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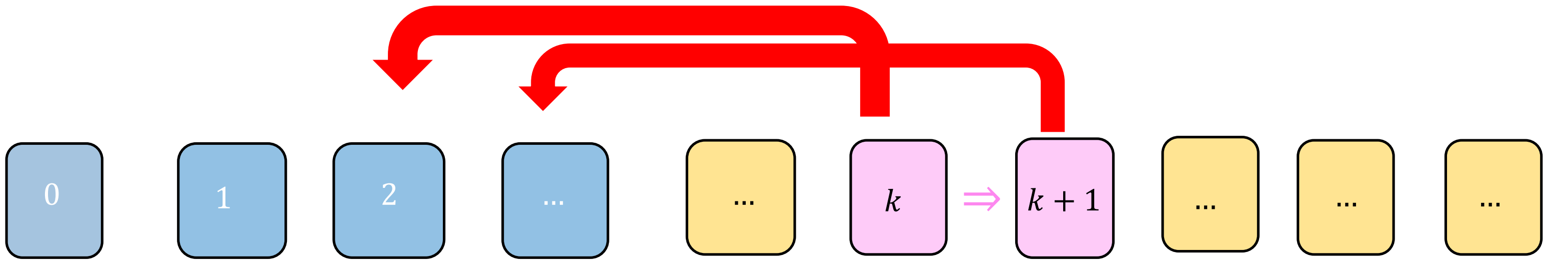

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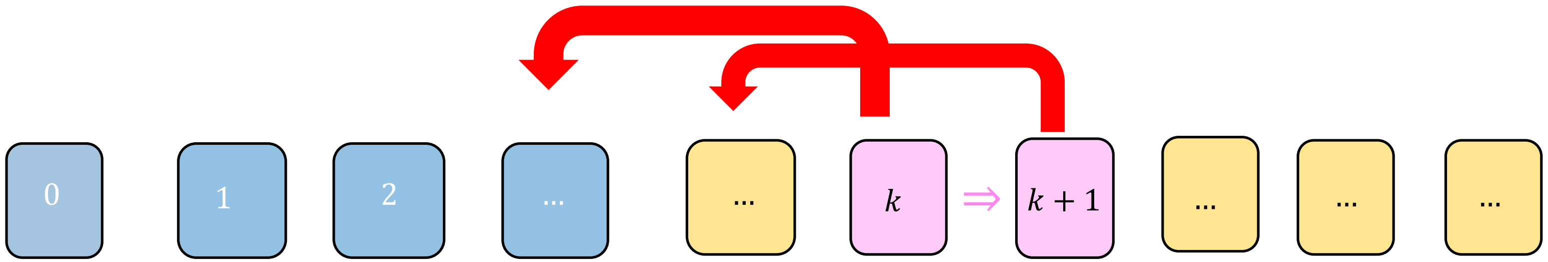

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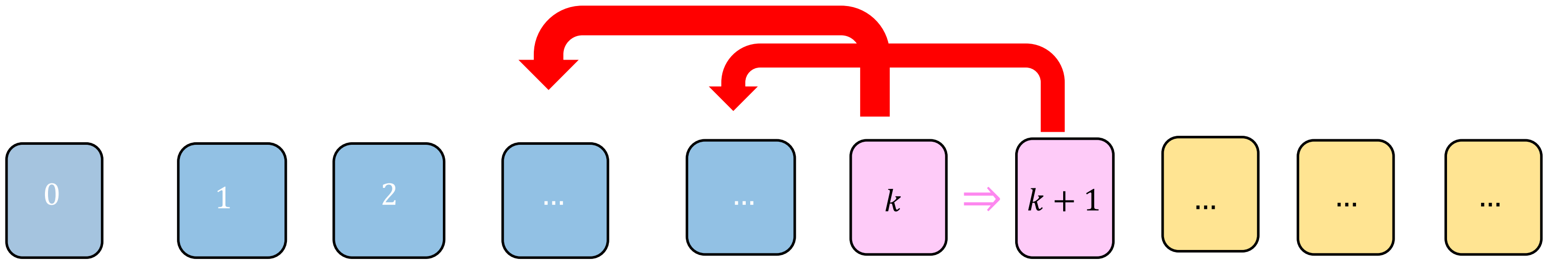

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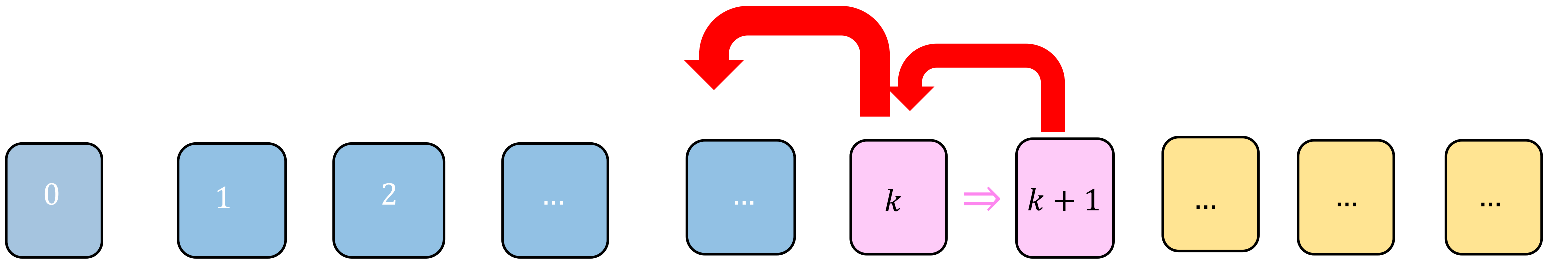

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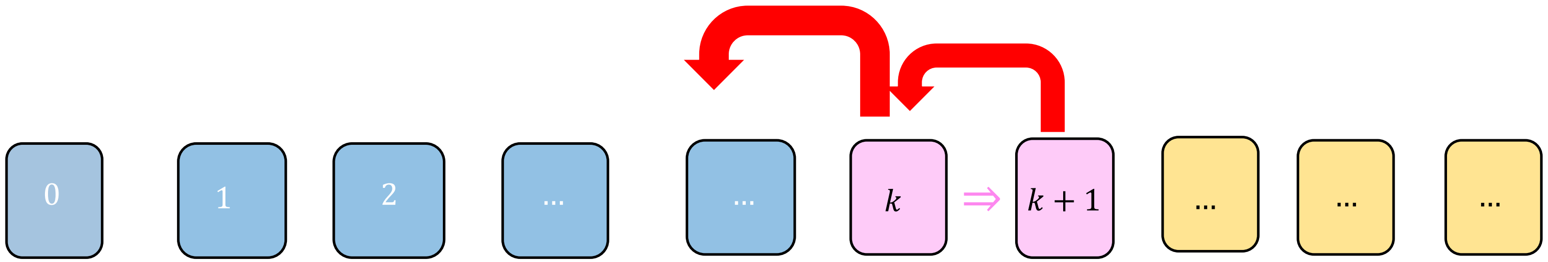

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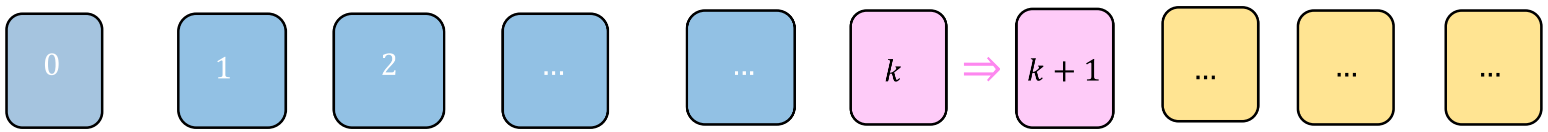

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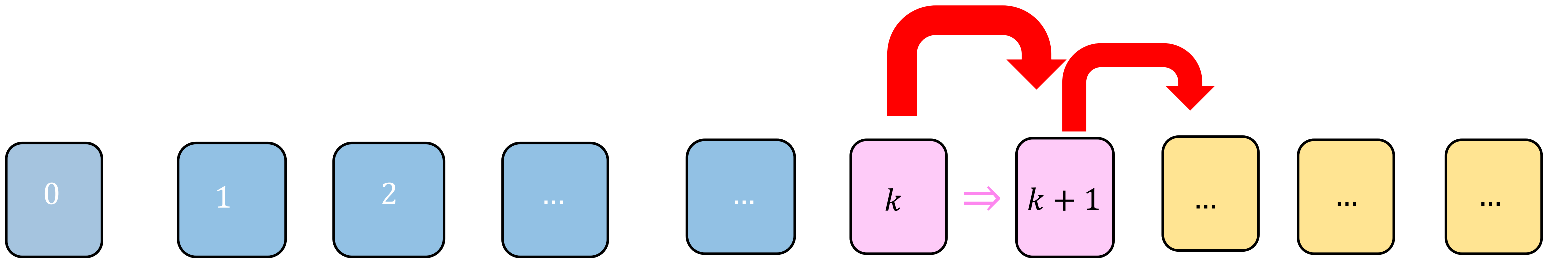

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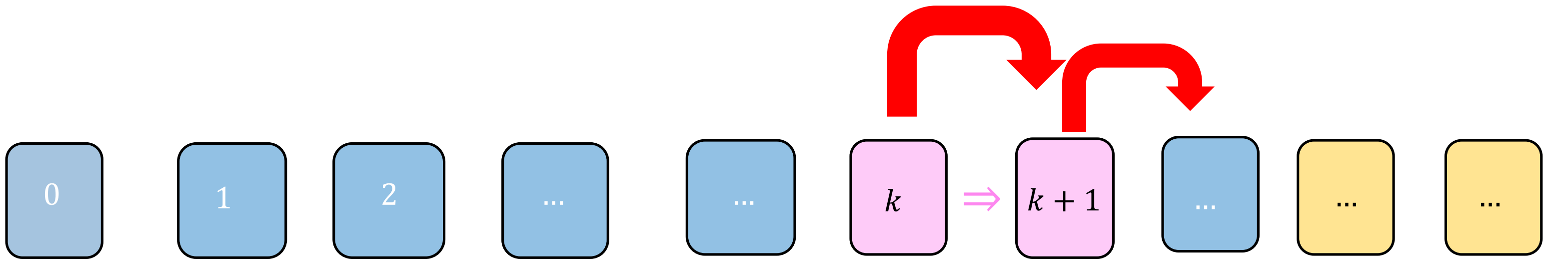

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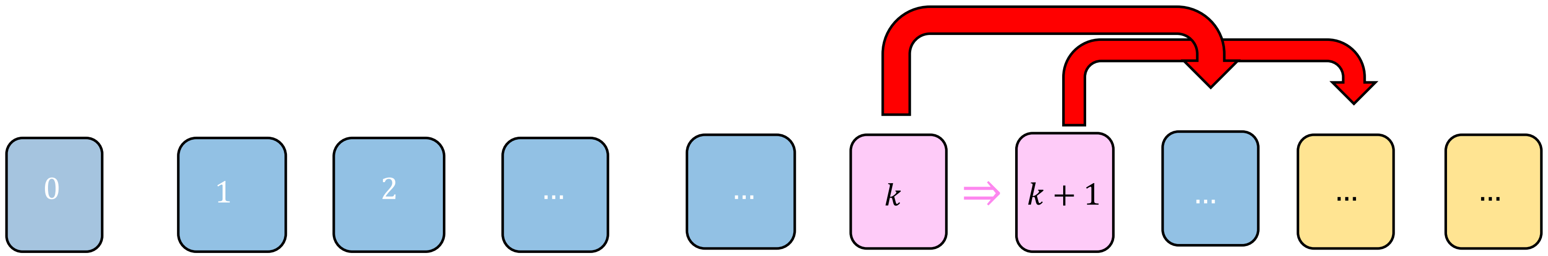

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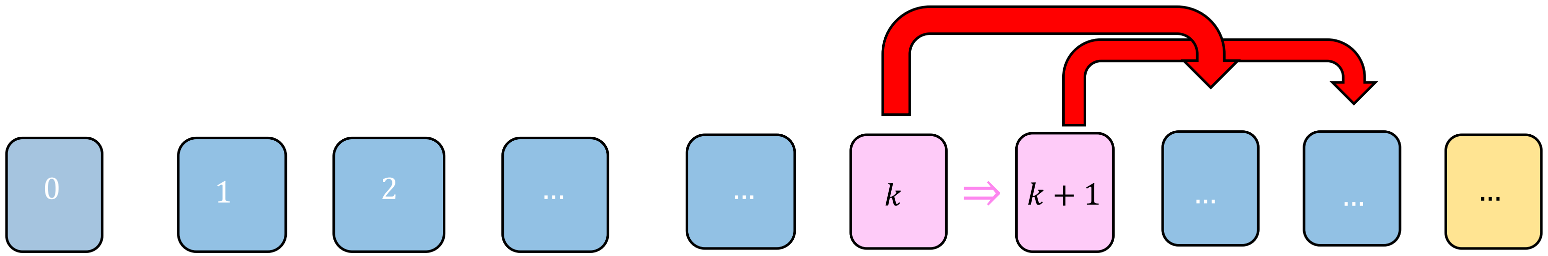

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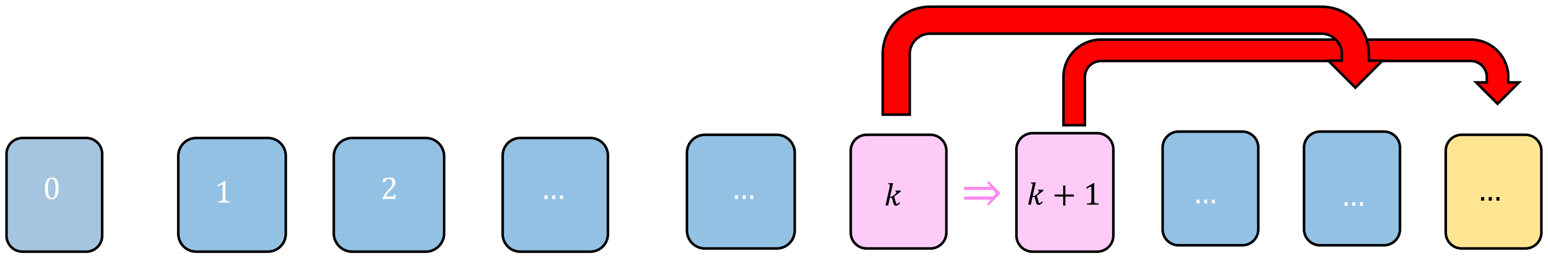

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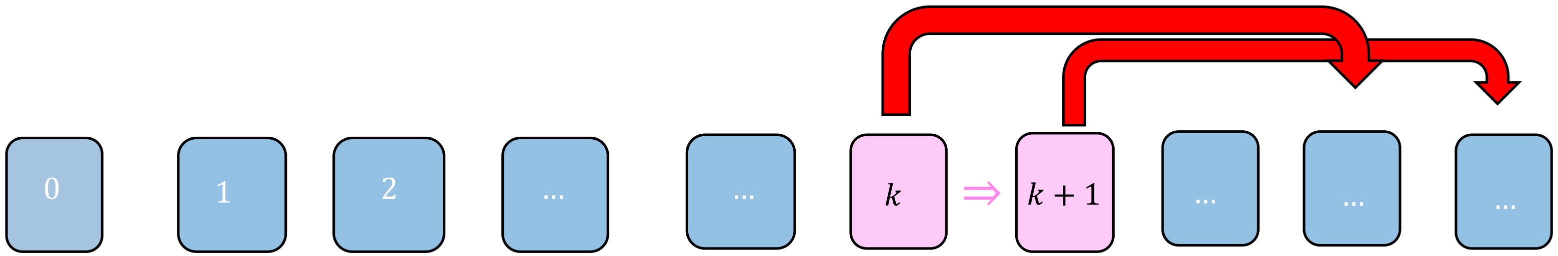

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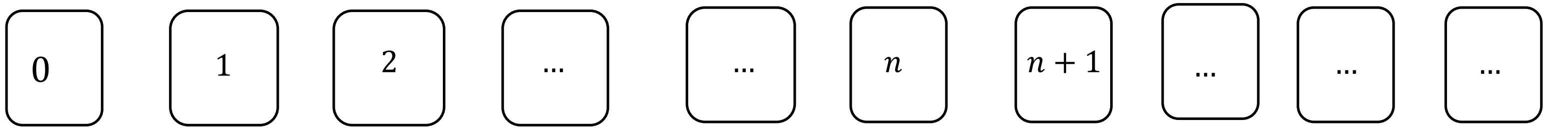

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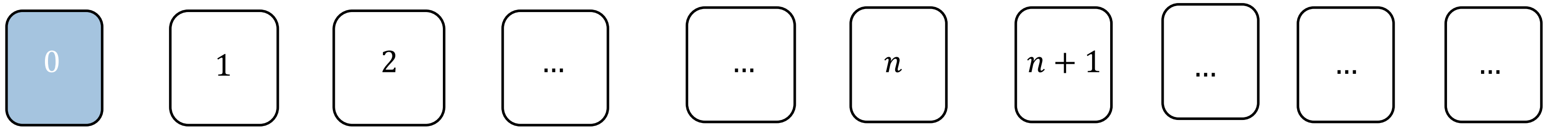

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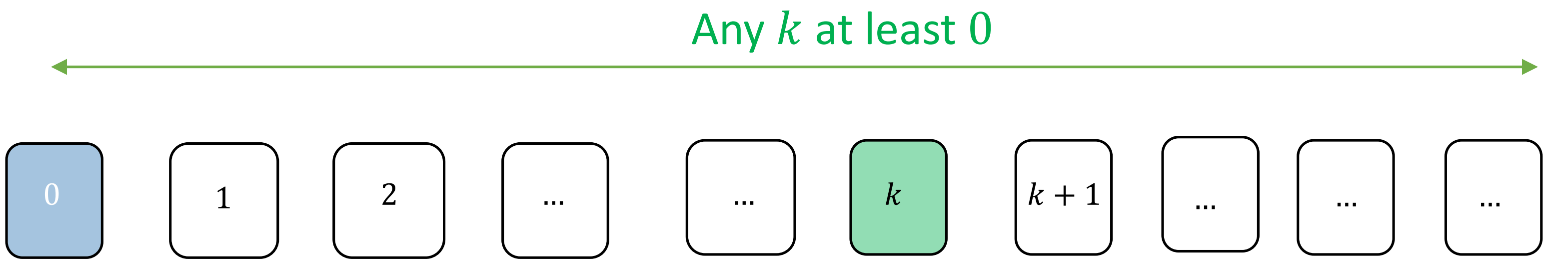

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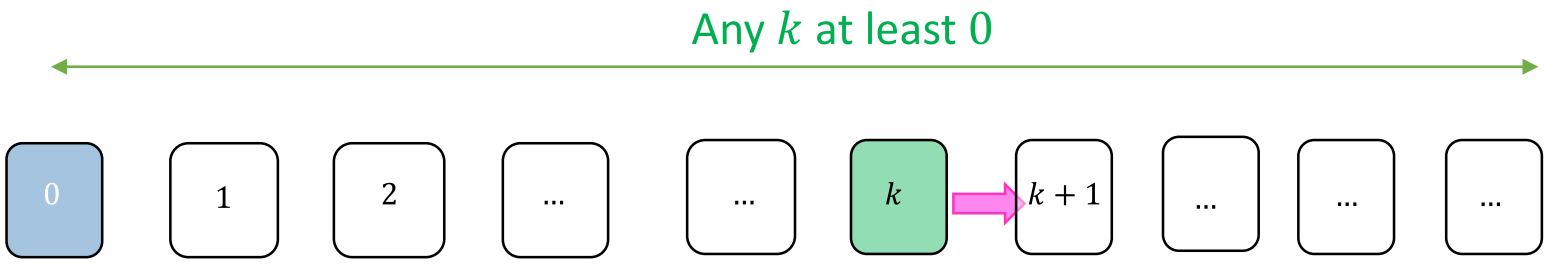

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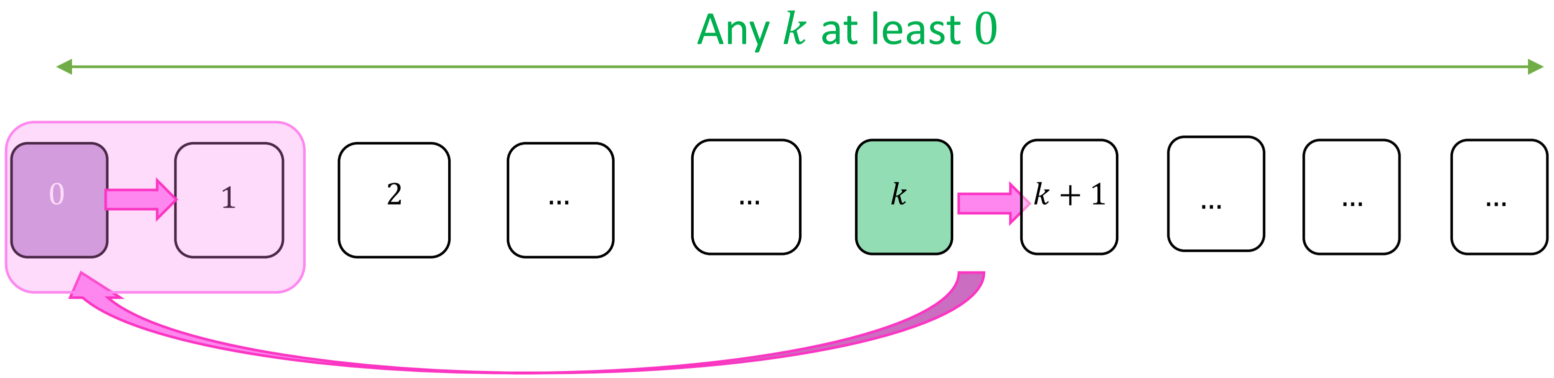

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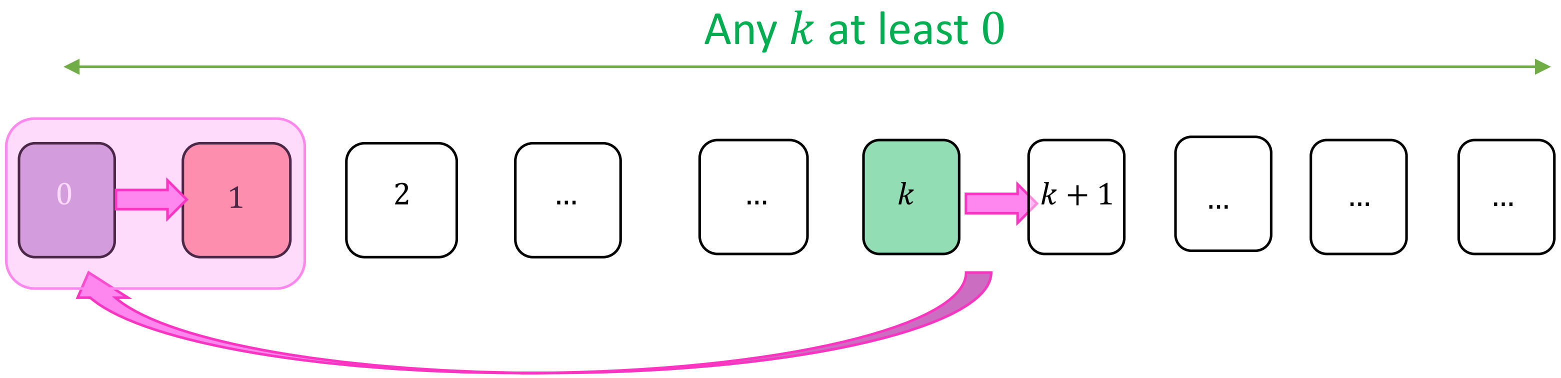

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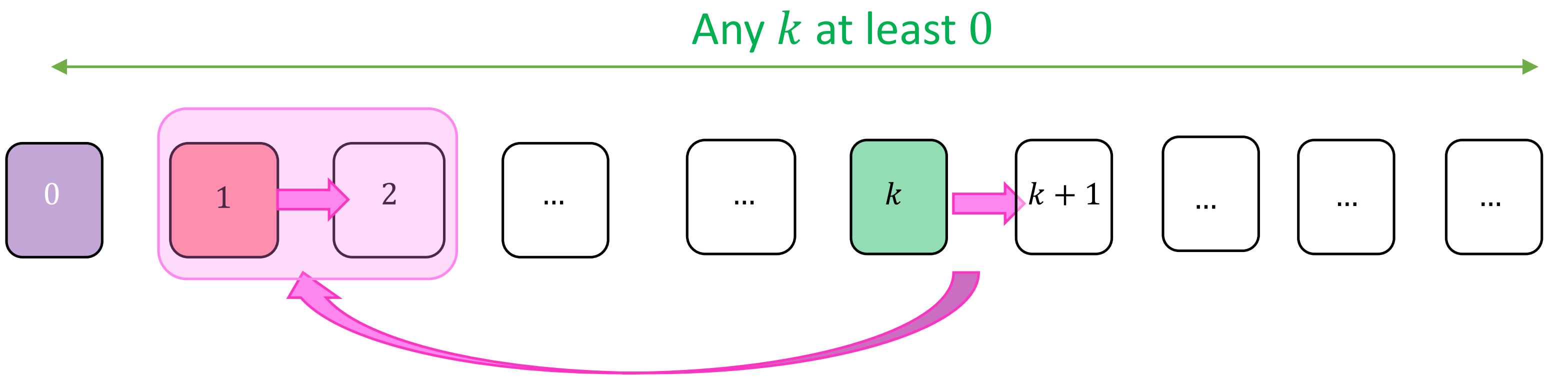

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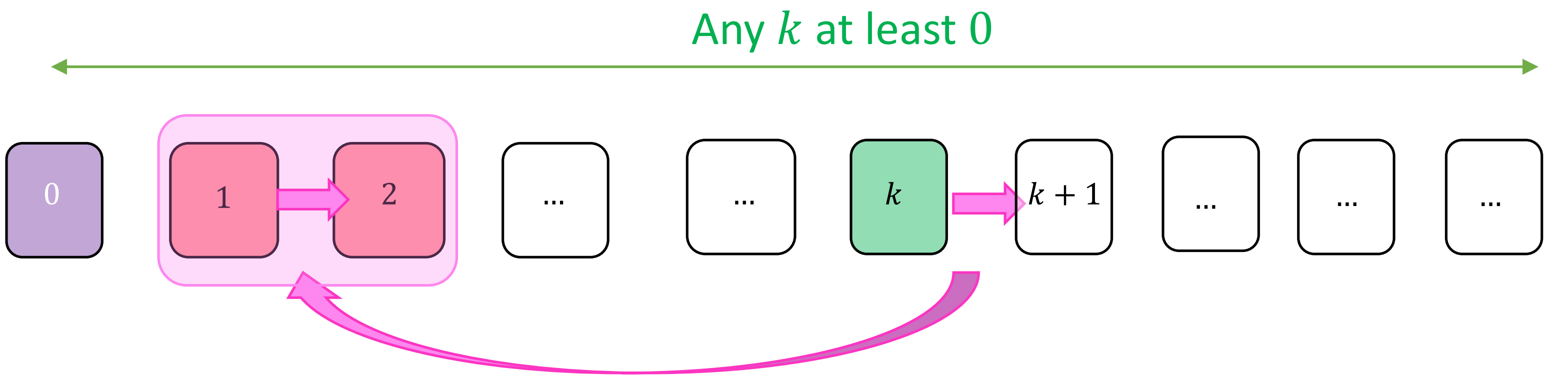

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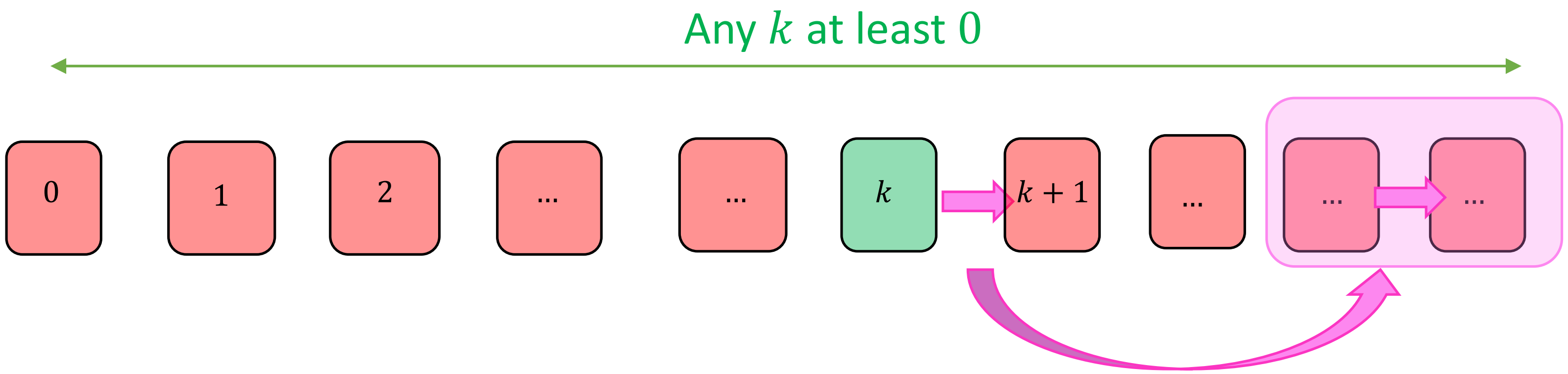

(We fast-forwarded here to save some time.)

An introductory example

- Suppose that we have the sequence a such that:

$$a_n = \begin{cases} 1, & n = 0 \\ 2a_{n-1}, & n \geq 1 \end{cases}$$

- First few terms:

$$1, 2, 4, 8, 16, \ldots$$

- We will prove, via mathematical induction, that for all $n \geq 0$,

$$a_n = 2^n$$

Inductive Base

- For $n = 0$, we will **prove** that $P(0)$ is true, where $P(0)$ is the statement:

$$a_0 = 2^0$$

- This is trivial to prove, since by the base case of the sequence a we have $a_0 = 1 = 2^0$.
- So $P(0)$ is true.

Inductive Hypothesis

- For $n = k \geq 0$, we **assume** that $P(k)$ **is true**:

$$a_k = 2^k$$

Inductive Step

- Given that $P(k)$ is true, we will **prove** that $P(k+1)$ is true,where $P(k+1)$ is the statement:

$$a_{k+1} = 2^{k+1}$$

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- Given that $P(k)$ is true, we will **prove** that $P(k+1)$ is true,where $P(k+1)$ is the statement:

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- Since $k \geq 0, k+1 \geq 1.$

Inductive Step

- Given that $P(k)$ is true, we will **prove** that $P(k+1)$ is true,where $P(k+1)$ is the statement:

$$a_{k+1} = 2^{k+1}$$

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- We can therefore use the **recursive rule** of the sequence's definition to derive $a_{k+1} = 2 \cdot a_k \ (I)$

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- Since $k \geq 0$, $k + 1 \geq 1$.
- We can therefore use the **recursive rule** of the sequence's definition to derive $a_{k+1} = 2 \cdot a_k \ (I)$
- From our assumption of $P(k)$, we know that $a_k = 2^k \ (II)$
- $(I) \overset{(II)}{\Longrightarrow} a_{k+1} = 2^{k+1}$
- So $P(k+1)$ is also true and we are done.

Here's another

- Suppose that we have the sequence s defined as follows:

$$s_n = \begin{cases} 0, & n = 0 \\ s_{n-1} + 10, & n \geq 1 \end{cases}$$

- Using weak induction, prove that $(\forall n \in \mathbb{N})[\, 5 \mid s_n \,]$

Inductive Base

- For $n = 0, s_0 = 0\ (I)$.
- Furthermore, it is the case that $5 \mid 0\ (II)$.
- $(I, II) \Rightarrow 5 \mid s_0 \Rightarrow P(0)\ \ holds$

Inductive Hypothesis

- Suppose that $n = k \geq 0$. We will assume that $P(k)$ holds, i.e:

$$(5 \mid s_k) \Leftrightarrow (\exists r \in \mathbb{Z})[s_k = 5r]$$

Could also use the mod definition!

Inductive Step

- Given $P(k)$, we will now attempt to prove $P(k+1)$, i.e:

$$(5 \mid s_{k+1}) \Leftrightarrow (\exists \ell \in \mathbb{Z})[s_{k+1} = 5\ell]$$

- Since $k \geq 0, k+1 \geq 1$ and we can use the recursive part of the definition of s:

$$s_{k+1} = s_{(k+1)-1} + 10 = s_k + 10 \overset{\text{(By I.H)}}{=} 5 \cdot r + 10 = 5r + 5 * 2 = 5(r+2) = 5\,\ell$$

You do this!

- The sequence b is defined as:

$$b_n = \begin{cases} 1, & n = 0 \\ 4 + b_{n-1}, & n \geq 1 \end{cases}$$

- Prove that for all $n \geq 0$, b_n is odd

END OF FIRST INDUCTION LECTURE VIDEO

Feel free to either take notes from the first video, discuss what you learned, or just skip to the second video altogether.

SECOND VIDEO: SUM PROBLEMS

$$\sum_{i=0}^{n} f(n)$$

The Gaussian Sum

- We will prove that the sum of the first n numbers is equal to $\frac{n(n+1)}{2}$.
- Symbolically:

$$\underbrace{1 + 2 + 3 + \cdots + (n - 1) + n}_{\sum_{i=1}^{n} i} = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Inductive base

- For $n = 0$, we will **prove** that $P(0)$ holds

$$\sum_{i=1}^{0} i \overset{?}{=} \frac{0(0+1)}{2}$$

Remember: $P(n)$ is

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

- LHS: $\sum_{i=1}^{0} i = 0$ (recall this fact from our sequences lecture)
- RHS: $\frac{0(0+1)}{2} = 0$
- Since LHS = RHS for $n = 0$, $P(0)$ has been proven true.

Inductive Hypothesis

- For $n = k \geq 0$, we **assume** that $P(k)$ **is true**:

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

So, we **assume** that

$$P(k) \Leftrightarrow \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

is true for an arbitrary $k \geq 0$

- Inductive Hypothesis done!

Inductive step

- Given that $P(k)$ is true, we will **prove** that $P(k+1)$ is true.

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2} \Rightarrow \sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

Inductive step

- Given that $P(k)$ is true, we will **prove** that $P(k+1)$ is true.

Just adding 1 to k

Just adding 1 to $k+1$

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2} \Rightarrow \sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

Inductive step

- Given that $P(k)$ is true, we will **prove** that $P(k+1)$ is true.

Just adding 1 to k

Just adding 1 to $k+1$

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2} \Rightarrow \sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

This is our goal!

Inductive step, contd.

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

- Starting from the LHS of the relation to prove, we have:

$$\sum_{i=1}^{k+1} i = 1 + 2 + \cdots + k + (k+1)$$

Inductive step, contd.

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

- Starting from the LHS of the relation to prove, we have:

$$\sum_{i=1}^{k+1} i = 1 + 2 + \cdots + k + (k+1) = \sum_{i=1}^{k} i + (k+1) \qquad (1)$$

Inductive step, contd.

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

- Starting from the LHS of the relation to prove, we have:

$$\sum_{i=1}^{k+1} i = 1 + 2 + \cdots + k + (k+1) = \sum_{i=1}^{k} i + (k+1) \quad (1)$$

- **From the Inductive Hypothesis**, we have that

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2} \quad (2)$$

Inductive step, contd.

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

- Starting from the LHS of the relation to prove, we have:

$$\sum_{i=1}^{k+1} i = 1 + 2 + \cdots + k + (k+1) = \sum_{i=1}^{k} i + (k+1) \quad (1)$$

- **From the Inductive Hypothesis**, we have that

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2} \quad (2)$$

Inductive step, contd.

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

- Starting from the LHS of the relation to prove, we have:

$$\sum_{i=1}^{k+1} i = 1 + 2 + \cdots + k + (k+1) = \sum_{i=1}^{k} i + (k+1) \quad (1)$$

- **From the Inductive Hypothesis**, we have that

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2} \qquad (2)$$

- Substituting (2) into (1) yields (next slide):

Inductive step, contd.

$$\sum_{i=1}^{k+1} i = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+2)(k+1)}{2}$$

$$= RHS$$

Inductive step, contd.

$$\sum_{i=1}^{k+1} i = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+2)(k+1)}{2}$$

$$= RHS$$

- So, when $P(k)$ is true, $P(k+1)$ was also proven true.
- We conclude that $P(n)$ is true $\forall n \geq 0$. $\square$

And one for you!

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

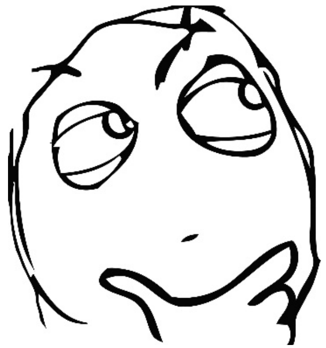
Inductive Base

- For $n = 0$, LHS = $\sum_{i=1}^{0} i^2 = 0$
- RHS=$\frac{0(0+1)(2*0+1)}{2} = 0$
- Since LHS = RHS, $P(0)$ holds and we are done.

Inductive Base

- For $n = 0$, LHS = $\sum_{i=1}^{0} i^2 = 0$
- RHS=$\frac{0(0+1)(2*0+1)}{2} = 0$
- Since LHS = RHS, $P(0)$ holds and we are done.

- You could also start from $n = 1$! LHS = RHS in both cases
  - $n = 0$ sometimes makes the math easier (RHS in this case)



Inductive Hypothesis

- Suppose that $n = k \geq 0$. *(Or 1 in the alternative scenario)*
- We will then assume $P(k)$, i.e:

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Inductive Step

- We will now attempt to prove $P(k+1)$, i.e

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

Careful with factoring please!!!

Inductive Step

- We will now attempt to prove $P(k+1)$, i.e

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

Careful with factoring please!!!

- By leveraging associativity of sum, the LHS can be written as follows:

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$$

Inductive Step

- We will now attempt to prove $P(k+1)$, i.e

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Careful with factoring please!!!

- By leveraging associativity of sum, the LHS can be written as follows:

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$$

We can apply the I.H here!

Inductive Step

- By I.H, we can now write:

$$\sum_{i=1}^{k+1} i^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

- Remember: we **want** this to be equal to

$$\frac{(k+1)(k+2)(2k+3)}{6}$$

- **We will fearlessly manipulate the algebra until it does!**

Inductive Step - Algebra

$$\frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} =$$

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} = \frac{(k+1)[2k^2 + 7k + 6]}{6}$$

- If only we could prove that $2k^2 + 7k + 6 = (k+2)(2k+3)$, we'd be done!
- But…. $(k+2)(2k+3) = 2k^2 + 3k + 4k + 6 = 2k^2 + 7k + 6$! ☺
- So we're done.

And one with more than 1 variable!

- Prove that the sum of the first n terms of a **geometric sequence** with $m \in (\mathbb{R} - \{1\})$ and $a_0 = 1$ is equal to $\frac{m^n - 1}{m - 1}$.
- Symbolically:

$$\sum_{i=0}^{n-1} m^i = \frac{m^n - 1}{m - 1}$$

And one with more than 1 variable!

- Prove that the sum of the first n terms of a **geometric sequence** with $m \in (\mathbb{R} - \{1\})$ and $a_0 = 1$ is equal to $\frac{m^n - 1}{m - 1}$.
- Symbolically:

$$\sum_{i=0}^{n-1} m^i = \frac{m^n - 1}{m - 1}$$

- In this instance, we have two variables, m and n, and it's spectacularly easy to confuse ourselves about which variable we will be focusing on.
  - So, we will **explicitly** say, at the beginning of our proof, that we will be performing a proof by induction on n.

Proof

- Proof : We attempt to prove $P(n),\ \forall n \in \mathbb{N}$. We proceed via induction on n.
- **Inductive base**: We attempt to prove $P(0)$.

$$P(0): \sum_{i=0}^{0-1} m^i = \frac{m^0 - 1}{m - 1} \Leftrightarrow \sum_{i=0}^{-1} m^i = \frac{m^0 - 1}{m - 1} \Leftrightarrow 0 = 0$$

So $P(0)$ is true.

- **Inductive hypothesis**: Suppose $n = k \geq 0$. We assume $P(k)$, i.e

$$\sum_{i=0}^{k-1} m^i = \frac{m^k - 1}{m - 1}$$

Proof (contd.)

- **Inductive step:** We will attempt to prove $P(k+1)$, i.e

$$\sum_{i=0}^{k} m^i = \frac{m^{k+1} - 1}{m - 1}$$

$(k+1)-1$

From the LHS to the RHS:

By the I.H

$$LHS = \sum_{i=0}^{k} m^i = \sum_{i=0}^{k-1} m^i + m^k = \frac{m^k - 1}{m-1} + m^k = \frac{m-1 + m^k(m-1)}{m-1} = \frac{m^{k+1}-1}{m-1} = RHS \;\square$$

END OF SECOND INDUCTION LECTURE VIDEO

Feel free to either take notes from the second video, discuss what you learned, or just skip to the third video altogether.

THIRD VIDEO: COIN PROBLEMS!

A coin problem

- We will prove that every dollar amount $\geq$ 4 cents can be **exclusively** paid for by 2 and/or 5 cent coins.

Theorem expressed in quantifiers

- All quantifiers implicitly assumed over $\mathbb{N}$.

$$(\forall n \geq 4)(\exists n_1, n_2)[n = 2n_1 + 5n_2]$$

Inductive base

- The least amount of money we are required to prove the statement for is 4¢, so we will attempt to prove $P(4)$.
- For $n = 4$, we have 4¢. Since $4¢ = 2 \times 2¢$, we are done (we have shown that the amount of 4¢ can be exclusively paid for by using only 2 and/or 5 cent coins)

Inductive hypothesis 

- Let $n = k \geq 4$.
- Assume $P(k) \Leftrightarrow (\exists k_1, k_2)[k = 2k_1 + 5k_2]$

Inductive step

- We will prove that $P(k) \Rightarrow P(k+1)$, i.e that we can pay an amount of money equal to $k+1$ cents using **only** 2¢ or 5¢ coins.
- In terms of algebra, what we want to prove is:

$$(\exists k_3, k_4 \in \mathbb{N})\ [k+1 = 2k_3 + 5k_4]$$

Inductive step

- We will prove that $P(k) \Rightarrow P(k+1)$, i.e that we can pay an amount of money equal to $k+1$ cents using **only** 2¢ or 5¢ coins.
- In terms of algebra, what we want to prove is:

$$(\exists k_3, k_4 \in \mathbb{N})\ [k+1 = 2k_3 + 5k_4]$$

Different variables from I.H!

Inductive Step (contd.)

- From the Inductive Hypothesis (I.H), we have that for some specific positive integers k_1 and k_2:

$$k = 2k_1 + 5k_2$$

Inductive Step (contd.)

- From the Inductive Hypothesis (I.H), we have that for some specific positive integers k_1 and k_2:

$$k = 2k_1 + 5k_2$$

1. **Case #1:** $k_1 \geq 2$

- I have at least 2 2¢ coins, so I can take away 2 2¢ coins and add one 5¢ coin

Inductive Step (contd.)

- From the Inductive Hypothesis (I.H), we have that for some specific positive integers k_1 and k_2:

$$k = 2k_1 + 5k_2$$

1. **Case #1:** $k_1 \geq 2$

- I have at least 2 2¢ coins, so I can take away two 2¢ coins and add one 5 ¢ coin
- By adding 1 on both sides of the I.H we obtain:

$$k + 1 = 2k_1 + 5k_2 + 1 = 2k_1 + 5k_2 + (5 - 2 * 2) =$$

$$= (2k_1 - 4) + (5k_2 + 5) = 2\underbrace{(k_1 - 2)}_{k_3} + 5\underbrace{(k_2 + 1)}_{k_4} = 2k_3 + 5k_4$$

Inductive Step (contd.)

- From the Inductive Hypothesis (I.H), we have that for some specific positive integers k_1 and k_2:

$$k = 2k_1 + 5k_2$$

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- I have at least 2 2¢ coins, so I can take away two 2¢ coins and add one 5 ¢ coin
- By adding 1 on both sides of the I.H we obtain:

$$k + 1 = 2k_1 + 5k_2 + 1 = 2k_1 + 5k_2 + (5 - 2 * 2) =$$
$$= (2k_1 - 4) + (5k_2 + 5) = 2\underbrace{(k_1 - 2)}_{k_1 - 2 \geq 0 \text{ because } k_1 \geq 2} + 5\underbrace{(k_2 + 1)}_{\text{In } \mathbb{N} \text{ by closure}} = 2k_3 + 5k_4$$

Inductive step

2. Case #2: $k_2 \geq 1$

- I have at least one 5¢ coin so I can take away one 5¢ coin and add three 2¢ coins
- By adding 1 on both sides of the I.H we obtain:

$$k + 1 = 2k_1 + 5k_2 + 1 = 2k_1 + 5k_2 + (3 * 2 - 5) =$$

$$= 2\underbrace{(k_1 + 3)}_{k_3} + 5\underbrace{(k_2 - 1)}_{k_4} = 2k_3 + 5k_4$$

Inductive step

2. Case #2: $k_2 \geq 1$

- I have at least one 5¢ coin so I can take away one 5¢ coin and add three 2¢ coins
- By adding 1 on both sides of the I.H we obtain:

$$k + 1 = 2k_1 + 5k_2 + 1 = 2k_1 + 5k_2 + (3 * 2 - 5) =$$

$$= 2\underbrace{(k_1 + 3)}_{\substack{(k_1+3) \in \mathbb{N} \\ \text{by closure}}} + 5\underbrace{(k_2 - 1)}_{\substack{k_2 - 1 \geq 0 \\ \text{because} \\ k_2 \geq 1}} = 2k_3 + 5k_4$$

Inductive step

3. Case #3: $(k_1 \leq 1) \wedge (k_2 = 0)$

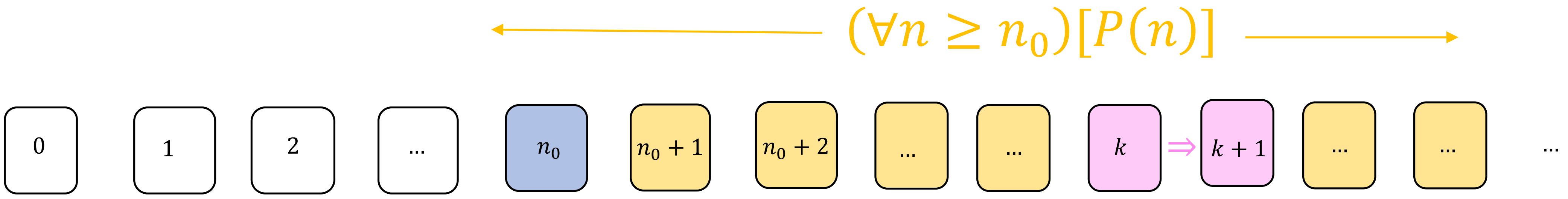
- This case means that we have either 0 or 2¢ at our disposal.
- But this is not possible, since we want to prove the theorem only for values ≥ 4¢
- So we're done. □

A note about the penny problem

- Note that we proved the theorem for $n \geq 4$
- Generally speaking, we can use induction to prove statements $P(n)\ \forall n \geq n_0$, where $n_0 \in \mathbb{N}$.
- Most of the time n_0 will be small (0, 1, 2, …)

A note about the penny problem

- Note that we proved the theorem for $n \geq 4$
- Generally speaking, we can use induction to prove statements $P(n)\ \forall n \geq n_0$, where $n_0 \in \mathbb{N}$.
- Most of the time n_0 will be small (0, 1, 2, …)
- If $P(n_0) \wedge (\forall k \geq n_0)[P(k) \Rightarrow P(k+1)]$ is true, then the inductive principle holds and we have the desired statement



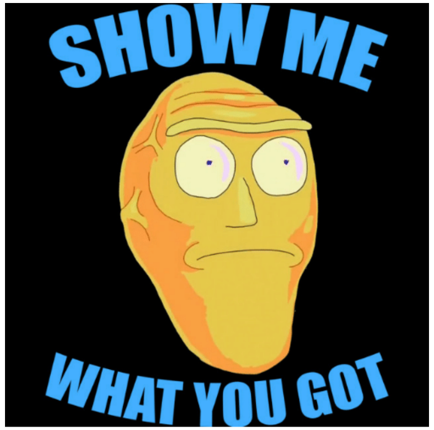
Another!

- Prove that every dollar amount equal to at least 112 cents can be paid for exclusively by 5 and 6 cent coins.

Another!

- Prove that every dollar amount equal to at least 112 cents can be paid for exclusively by 5 and 6 cent coins.
- Let’s do this one together.

A coin problem for **you**!



Prove to me that every dollar amount $\geq$ 20 cents can be **exclusively** paid for through combinations of **5**-cent coins and **6**-cent coins!

FOURTH VIDEO:
TREATING INEQUALITIES

Here's one with an inequality!

- Prove that for all integers n at least 4, $2^n < n!$

1. **I.B:** We will prove $P(4) \Leftrightarrow 2^4 < 4!\ \text{Done.}$
2. **I.H:** For $n = k \geq 4$, we assume $P(k)$, i.e $2^k < k!$
3. **I.S:** We will prove $P(k) \Rightarrow P(k+1)$, i.e

$$(2^k < k!) \Rightarrow (2^{k+1} < (k+1)!)$$

Inductive Step…

- Prove that for all integers n at least 4, $2^n < n!$

1. **I.B:** We will prove $P(4) \Leftrightarrow 2^4 < 4!$ Done.
2. **I.H:** For $n = k \geq 4$, we assume $P(k)$, i.e $2^k < k!$
3. **I.S:** We will prove $2^{k+1} < (k+1)!$

Inductive Step…

- Prove that for all integers n at least 4, $2^n < n!$

1. **I.B:** We will prove $P(4) \Leftrightarrow 2^4 < 4!$ Done.
2. **I.H:** For $n = k \geq 4$, we assume $P(k)$, i.e $2^k < k!$
3. **I.S:** We will prove $2^{k+1} < (k+1)!$
   - From algebra, we have that $2^{k+1} = 2^k \cdot 2$ (1)

Inductive Step...

- Prove that for all integers n at least 4, $2^n < n!$

1. **I.B:** We will prove $P(4) \Leftrightarrow 2^4 < 4!$ Done.
2. **I.H:** For $n = k \geq 4$, we assume $P(k)$, i.e $2^k < k!$
3. **I.S:** We will prove $2^{k+1} < (k+1)!$
   - From algebra, we have that $2^{k+1} = 2^k \cdot 2$ (1)
   - From the I.H, we have that $2^k < k! \overset{2>0}{\Longleftrightarrow} 2^k \cdot 2 < k! \cdot 2$ (2)

Inductive Step…

- Prove that for all integers n at least 4, $2^n < n!$

1. **I.B:** We will prove $P(4) \Leftrightarrow 2^4 < 4!$ Done.
2. **I.H:** For $n = k \geq 4$, we assume $P(k)$, i.e $2^k < k!$
3. **I.S:** We will prove $2^{k+1} < (k+1)!$
   - From algebra, we have that $2^{k+1} = 2^k \cdot 2$ (1)
   - From the I.H, we have that $2^k < k! \overset{2>0}{\Longleftrightarrow} 2^k \cdot 2 < k! \cdot 2$ (2)
   - Since $k \geq 4$, we have that $2 < k+1 \overset{k!>0}{\Longleftrightarrow} k! \cdot 2 < k!\,(k+1)$ (3)

Inductive Step…

- Prove that for all integers n at least 4, $2^n < n!$

1. **I.B:** We will prove $P(4) \Leftrightarrow 2^4 < 4!$ Done.
2. **I.H:** For $n = k \geq 4$, we assume $P(k)$, i.e $2^k < k!$
3. **I.S:** We will prove $2^{k+1} < (k+1)!$
   - From algebra, we have that $2^{k+1} = 2^k \cdot 2$ (1)
   - From the I.H, we have that $2^k < k! \overset{2>0}{\Longleftrightarrow} 2^k \cdot 2 < k! \cdot 2$ (2)
   - Since $k \geq 4$, we have that $2 < k+1 \overset{k!>0}{\Longleftrightarrow} k! \cdot 2 < k!(k+1)$ (3)
   - $(2) \overset{(3)}{\Rightarrow} 2^k \cdot 2 < (k+1)! \overset{(1)}{\Leftrightarrow} 2^{k+1} < (k+1)!$

An inequality problem for you!

- Using mathematical induction, prove that, for all naturals $n \geq 3$,

$$2n + 1 < 2^n$$