Mathematical Induction: Introduction and basic problems

CMSC 250
Lecture structure

• Considering campus closure, lectures will have to be given remotely.
• Every lecture will be broken down into 3-4 bite-sized Panopto videos.
• Power Point still projected, slides will be up by the time lecture is up.
• Can always tell Jason on CW or over e-mail if the picture or the audio aren’t really good.
• Induction lectures delivered during second week of extended SpringBreak for practice with homework 6.
• We will see what we will do with lectures of week of Mar 30th – Apr 3rd.
FIRST VIDEO: INTRO AND BASIC SEQUENCE PROBLEMS
The idea behind induction

• Suppose that we want to prove that a proposition $P(n)$ is true for all natural numbers $n$. 
The idea behind induction

• Suppose that we want to prove that a proposition \( P(n) \) is true for all natural numbers \( n \).
• We will prove two separate things:
The idea behind induction

• Suppose that we want to prove that a proposition $P(n)$ is true for all natural numbers $n$.

• We will prove two separate things:
  1. For $n = 0$, $P(n)$ is true
The idea behind induction

• Suppose that we want to prove that a proposition $P(n)$ is true for all natural numbers $n$.

• We will prove two separate things:
  1. For $n = 0$, $P(n)$ is true (simplifiable to “$P(0)$ is true”).
The idea behind induction

• Suppose that we want to prove that a proposition $P(n)$ is true for all natural numbers $n$.
• We will prove two separate things:
  1. $P(0)$ is true
The idea behind induction

• Suppose that we want to prove that a proposition $P(n)$ is true for all natural numbers $n$.

• We will prove two separate things:
  1. $P(0)$ is true.
  2. For $n = k \geq 0$, if $P$ is true for $k$ (symb. “$P(k)$”, or “$P(k)$ holds”), it then must be true for $k + 1$ (symb. “$P(k + 1)$”, or “$P(k + 1)$ holds”).
The idea behind induction

• Suppose that we want to prove that a proposition $P(n)$ is true for all natural numbers $n$.

• We will prove two separate things:
  1. $P(0)$ is true
  2. For $n = k \geq 0$, if $P$ is true for $k$, it then must be true for $k + 1$

• Since $k \geq 0...$
The idea behind induction

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• Since $k \geq 0$...

```
0  1  2  ...  ...  k  ⇒  k + 1  ...  ...
```
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  2. For $n = k \geq 0$, if $P$ is true for $k$, it then must be true for $k + 1$

• Since $k \geq 0$...

| 0 | 1 | 2 | ... | ... | k | k + 1 | ... | ... | ... |
The idea behind induction

• Suppose that we want to prove that a proposition $P(n)$ is true for all natural numbers $n$.

• We will prove two separate things:
  1. $P(0)$ is true
  2. For $n = k \geq 0$, if $P$ is true for $k$, it then must be true for $k + 1$

• Since $k \geq 0$...

\[ \begin{align*}
0 & \quad 1 & \quad 2 & \quad \ldots & \quad \ldots & \quad k & \quad \Rightarrow & \quad k + 1 & \quad \ldots & \quad \ldots & \quad \ldots
\end{align*} \]
How we’ll make it work

1. Inductive base: We will prove (explicitly, no matter how dumb it may sometimes seem) that $P(0)$ is true.
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1. **Inductive base**: We will **prove** (explicitly, no matter how dumb it may sometimes seem) that $P(0)$ is true.

2. **Inductive hypothesis**: We will **assume** that, for $n = k \geq 0$, $P(k)$ holds.
How we’ll make it work

1. Inductive **base**: We will *prove* (explicitly, no matter how dumb it may sometimes seem) that $P(0)$ is true.

2. Inductive **hypothesis**: We will *assume* that, for $n = k \geq 0$, $P(k)$ holds.

3. Inductive **step**: We will *prove* that if $P(k)$ holds, then $P(k + 1)$ holds.

Any $k$ at least 0
How we’ll make it work

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• So everything falls into place!

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1. Inductive base: We will **prove** (explicitly, no matter how dumb it may sometimes seem) that $P(0)$ is true.

2. Inductive hypothesis: We will **assume** that, for $n = k \geq 0$, $P(k)$ holds.

3. Inductive step: We will **prove** that if $P(k)$ holds, then $P(k + 1)$ holds.

• So everything falls into place!

(We fast-forwarded here to save some time.)
An introductory example

• Suppose that we have the sequence \( a \) such that:

\[
a_n = \begin{cases} 
1, & n = 0 \\
2a_{n-1}, & n \geq 1 
\end{cases}
\]

• First few terms:

\( 1, 2, 4, 8, 16, \ldots \)

• We will prove, via mathematical induction, that for all \( n \geq 0 \),

\( a_n = 2^n \)
Inductive Base

• For \( n = 0 \), we will prove that \( P(0) \) is true, where \( P(0) \) is the statement:

\[
a_0 = 2^0
\]

• This is trivial to prove, since by the base case of the sequence \( a \) we have \( a_0 = 1 = 2^0 \).

• So \( P(0) \) is true.
Inductive Hypothesis

• For $n = k \geq 0$, we assume that $P(k)$ is true:

$$a_k = 2^k$$
Inductive Step

- Given that $P(k)$ is true, we will prove that $P(k + 1)$ is true, where $P(k + 1)$ is the statement:

$$a_{k+1} = 2^{k+1}$$
Inductive Step

• Given that $P(k)$ is true, we will prove that $P(k + 1)$ is true, where $P(k + 1)$ is the statement:

$$a_{k+1} = 2^{k+1}$$

• Since $k \geq 0$, $k + 1 \geq 1$. 
Inductive Step

• Given that $P(k)$ is true, we will prove that $P(k + 1)$ is true, where $P(k + 1)$ is the statement:

$$a_{k+1} = 2^{k+1}$$

• Since $k \geq 0$, $k + 1 \geq 1$.

• We can therefore use the recursive rule of the sequence’s definition to derive $a_{k+1} = 2 \cdot a_k$ (I)
Inductive Step

• Given that $P(k)$ is true, we will prove that $P(k + 1)$ is true, where $P(k + 1)$ is the statement:

\[ a_{k+1} = 2^{k+1} \]

• Since $k \geq 0$, $k + 1 \geq 1$.

• We can therefore use the recursive rule of the sequence’s definition to derive $a_{k+1} = 2 \cdot a_k$ (I)

• From our assumption of $P(k)$, we know that $a_k = 2^k$ (II)
Inductive Step

• Given that \( P(k) \) is true, we will prove that \( P(k + 1) \) is true, where \( P(k + 1) \) is the statement:

\[
a_{k+1} = 2^{k+1}
\]

• Since \( k \geq 0 \), \( k + 1 \geq 1 \).

• We can therefore use the recursive rule of the sequence’s definition to derive

\[
a_{k+1} = 2 \cdot a_k \quad (I)
\]

• From our assumption of \( P(k) \), we know that \( a_k = 2^k \) \( (II) \)

\[
(I) \implies a_{k+1} = 2^{k+1}
\]
Inductive Step

• Given that \( P(k) \) is true, we will prove that \( P(k + 1) \) is true, where \( P(k + 1) \) is the statement:

\[
a_{k+1} = 2^{k+1}
\]

• Since \( k \geq 0 \), \( k + 1 \geq 1 \).
• We can therefore use the recursive rule of the sequence’s definition to derive

\[
a_{k+1} = 2 \cdot a_k \quad (I)
\]

• From our assumption of \( P(k) \), we know that \( a_k = 2^k \quad (II) \)

• \( (I) \Rightarrow a_{k+1} = 2^{k+1} \)
• So \( P(k + 1) \) is also true and we are done.
Here’s another

• Suppose that we have the sequence $s$ defined as follows:

$$s_n = \begin{cases} 
0, & n = 0 \\
 s_{n-1} + 10, & n \geq 1
\end{cases}$$

• Using weak induction, prove that $(\forall n \in \mathbb{N})[5 \mid s_n]$
Inductive Base

• For \( n = 0, s_0 = 0 \) (I).
• Furthermore, it is the case that \( 5 \mid 0 \) (II).
• \((I, II) \Rightarrow 5 \mid s_0 \Rightarrow P(0)\) holds
Inductive Hypothesis

• Suppose that $n = k \geq 0$. We will assume that $P(k)$ holds, i.e:

\[(5 \mid s_k) \iff (\exists r \in \mathbb{Z})[s_k = 5r]\]

Could also use the mod definition!
Inductive Step

• **Given** \( P(k) \), we will now attempt to prove \( P(k + 1) \), i.e:

\[
(5 \mid s_{k+1}) \iff (\exists \ell \in \mathbb{Z})[s_{k+1} = 5\ell]
\]

• Since \( k \geq 0, k + 1 \geq 1 \) and we can use the recursive part of the definition of \( s \):

\[
s_{k+1} = s_{(k+1)-1} + 10 = s_k + 10 = 5 \cdot r + 10 = 5r + 5 \cdot 2 = 5(r + 2) = 5 \ell \quad \text{(By I.H)}
\]
You do this!

• The sequence $b$ is defined as:

$$b_n = \begin{cases} 
1, & n = 0 \\
4 + b_{n-1}, & n \geq 1
\end{cases}$$

• Prove that for all $n \geq 0$, $b_n$ is odd
Feel free to either take notes from the first video, discuss what you learned, or just skip to the second video altogether.
SECOND VIDEO: SUM PROBLEMS

\[ \sum_{i=0}^{n} f(n) \]
The Gaussian Sum

- We will prove that the sum of the first $n$ numbers is equal to $\frac{n(n+1)}{2}$.
- Symbolically:

$$1 + 2 + 3 + \cdots + (n - 1) + n = \frac{n(n + 1)}{2}$$

$$\sum_{i=1}^{n} i = \frac{n(n + 1)}{2}$$
Inductive base

• For \( n = 0 \), we will prove that \( P(0) \) holds

\[
\sum_{i=1}^{0} i = 0(0 + 1) \quad \text{(recall this fact from our sequences lecture)}
\]

• LHS: \( \sum_{i=1}^{0} i = 0 \) (recall this fact from our sequences lecture)

• RHS: \( \frac{0(0+1)}{2} = 0 \)

• Since LHS = RHS for \( n = 0 \), \( P(0) \) has been proven true.

Remember: \( P(n) \) is

\[
\sum_{i=1}^{n} i = \frac{n(n + 1)}{2}
\]
Inductive Hypothesis

• For \( n = k \geq 0 \), we assume that \( P(k) \) is true:

\[
\sum_{i=1}^{k} i = \frac{k(k + 1)}{2}
\]

• Inductive Hypothesis done!
Inductive step

- Given that $P(k)$ is true, we will prove that $P(k + 1)$ is true.

$$\sum_{i=1}^{k} i = \frac{k(k + 1)}{2} \Rightarrow \sum_{i=1}^{k+1} i = \frac{(k + 1)(k + 2)}{2}$$
Inductive step

• Given that $P(k)$ is true, we will prove that $P(k + 1)$ is true.

\[
\sum_{i=1}^{k} i = \frac{k(k + 1)}{2} \Rightarrow \sum_{i=1}^{k+1} i = \frac{(k + 1)(k + 2)}{2}
\]
Inductive step

- Given that $P(k)$ is true, we will prove that $P(k + 1)$ is true.

\[
\sum_{i=1}^{k} i = \frac{k(k + 1)}{2} \Rightarrow \sum_{i=1}^{k+1} i = \frac{(k + 1)(k + 2)}{2}
\]

This is our goal!
Inductive step, contd.

• Starting from the LHS of the relation to prove, we have:

\[ \sum_{i=1}^{k+1} i = 1 + 2 + \cdots + k + (k + 1) \]

\[ \sum_{i=1}^{k+1} i = \frac{(k + 1)(k + 2)}{2} \]
Inductive step, contd.

• Starting from the LHS of the relation to prove, we have:

\[
\sum_{i=1}^{k+1} i = 1 + 2 + \cdots + k + (k + 1) = \sum_{i=1}^{k} i + (k + 1) \quad (1)
\]
Inductive step, contd.

• Starting from the LHS of the relation to prove, we have:

\[ \sum_{i=1}^{k+1} i = 1 + 2 + \cdots + k + (k + 1) = \sum_{i=1}^{k} i + (k + 1) \quad (1) \]

• From the Inductive Hypothesis, we have that

\[ \sum_{i=1}^{k} i = \frac{k(k + 1)}{2} \quad (2) \]
Inductive step, contd.

- Starting from the LHS of the relation to prove, we have:

\[
\sum_{i=1}^{k+1} i = 1 + 2 + \cdots + k + (k + 1) = \sum_{i=1}^{k} i + (k + 1) \quad (1)
\]

- From the Inductive Hypothesis, we have that

\[
\sum_{i=1}^{k} i = \frac{k(k + 1)}{2} \quad (2)
\]
Inductive step, contd.

• Starting from the LHS of the relation to prove, we have:

\[ \sum_{i=1}^{k+1} i = 1 + 2 + \cdots + k + (k + 1) = \sum_{i=1}^{k} i + (k + 1) \quad (1) \]

• From the Inductive Hypothesis, we have that

\[ \sum_{i=1}^{k} i = \frac{k(k + 1)}{2} \quad (2) \]

• Substituting (2) into (1) yields (next slide):

\[ \sum_{i=1}^{k+1} i = \frac{(k + 1)(k + 2)}{2} \]
Inductive step, contd.

\[
\sum_{i=1}^{k+1} i = \frac{k(k + 1)}{2} + (k + 1) = \frac{k(k + 1)}{2} + \frac{2(k + 1)}{2} = \frac{(k + 2)(k + 1)}{2} = RHS
\]
Inductive step, contd.

\[
\sum_{i=1}^{k+1} i = \frac{k(k + 1)}{2} + (k + 1) = \frac{k(k + 1)}{2} + \frac{2(k + 1)}{2} = \frac{(k + 2)(k + 1)}{2} = RHS
\]

• So, when \( P(k) \) is true, \( P(k + 1) \) was also proven true.
• We conclude that \( P(n) \) is true \( \forall n \geq 0. \) \( \square \)
And one for you!

\[ \sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6} \]
Inductive Base

• For $n = 0$, $LHS = \sum_{i=1}^{0} i^2 = 0$

• $RHS=\frac{0(0+1)(2*0+1)}{2} = 0$

• Since $LHS = RHS$, $P(0)$ holds and we are done.
Inductive Base

• For $n = 0$, $LHS = \sum_{i=1}^{0} i^2 = 0$
• $RHS = \frac{0(0+1)(2\cdot0+1)}{2} = 0$
• Since LHS = RHS, $P(0)$ holds and we are done.

• You could also start from $n = 1$! LHS = RHS in both cases
  • $n = 0$ sometimes makes the math easier (RHS in this case)
Inductive Hypothesis

• Suppose that \( n = k \geq 0. \) (Or 1 in the alternative scenario)
• We will then assume \( P(k) \), i.e:

\[
\sum_{i=1}^{k} i^2 = \frac{k(k + 1)(2k + 1)}{6}
\]
Inductive Step

• We will now attempt to prove $P(k + 1)$, i.e.

$$\sum_{i=1}^{k+1} i^2 = \frac{(k + 1)(k + 2)(2k + 3)}{6}$$

Careful with factoring please!!!
Inductive Step

• We will now attempt to prove $P(k + 1)$, i.e.

\[
\sum_{i=1}^{k+1} i^2 = \frac{(k + 1)(k + 2)(2k + 3)}{6}
\]

• By leveraging associativity of sum, the LHS can be written as follows:

\[
\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k + 1)^2
\]

Careful with factoring please!!!
Inductive Step

• We will now attempt to prove $P(k + 1)$, i.e.

$$\sum_{i=1}^{k+1} i^2 = \frac{(k + 1)(k + 2)(2k + 3)}{6}$$

• By leveraging associativity of sum, the LHS can be written as follows:

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k + 1)^2$$

Careful with factoring please!!!

We can apply the I.H here!
Inductive Step

• By I.H, we can now write:

\[
\sum_{i=1}^{k+1} i^2 = \frac{k(k + 1)(2k + 1)}{6} + (k + 1)^2
\]

• Remember: we want this to be equal to

\[
\frac{(k+1)(k+2)(2k+3)}{6}
\]

• We will fearlessly manipulate the algebra until it does!
Inductive Step - Algebra

\[
\frac{k(k + 1)(2k + 1)}{6} + (k + 1)^2 = \frac{k(k + 1)(2k + 1)}{6} + \frac{6(k + 1)^2}{6} = \\
\frac{(k + 1)[k(2k + 1) + 6(k + 1)]}{6} = \frac{6}{6}[2k^2 + 7k + 6]
\]

*If only we could prove that \(2k^2 + 7k + 6 = (k + 2)(2k + 3)\), we’d be done!*

*But…. \((k + 2)(2k + 3) = 2k^2 + 3k + 4k + 6 = 2k^2 + 7k + 6!\)

*So we’re done.*
And one with more than 1 variable!

- Prove that the sum of the first \( n \) terms of a geometric sequence with \( m \in (\mathbb{R} - \{1\}) \) and \( a_0 = 1 \) is equal to \( \frac{m^n - 1}{m - 1} \).

- Symbolically:

\[
\sum_{i=0}^{n-1} m^i = \frac{m^n - 1}{m - 1}
\]
And one with more than 1 variable!

• Prove that the sum of the first \( n \) terms of a geometric sequence with \( m \in (\mathbb{R} - \{1\}) \) and \( a_0 = 1 \) is equal to \( \frac{m^n - 1}{m - 1} \).

• Symbolically:

\[
\sum_{i=0}^{n-1} m^i = \frac{m^n - 1}{m - 1}
\]

• In this instance, we have two variables, \( m \) and \( n \), and it’s spectacularly easy to confuse ourselves about which variable we will be focusing on.

  • So, we will explicitly say, at the beginning of our proof, that we will be performing a proof by induction on \( n \).
Proof

• Proof: We attempt to prove $P(n)$, $\forall n \in \mathbb{N}$. We proceed via induction on $n$.
• Inductive base: We attempt to prove $P(0)$.

$$P(0): \sum_{i=0}^{0-1} m^i = \frac{m^0 - 1}{m-1} \iff \sum_{i=0}^{-1} m^i = \frac{m^0 - 1}{m-1} \iff 0 = 0$$

So $P(0)$ is true.

• Inductive hypothesis: Suppose $n = k \geq 0$. We assume $P(k)$, i.e

$$\sum_{i=0}^{k-1} m^i = \frac{m^k - 1}{m - 1}$$
**Proof (contd.)**

- **Inductive step:** We will attempt to prove $P(k + 1)$, i.e.

  $$
  \sum_{i=0}^{k} m^i = \frac{m^{k+1} - 1}{m - 1}
  $$

From the LHS to the RHS:

- $LHS = \sum_{i=0}^{k} m^i = \sum_{i=0}^{k-1} m^i + m^k = \frac{m^k - 1}{m - 1} + m^k = \frac{m - 1 + m^k (m - 1)}{m - 1} = \frac{m^{k+1} - 1}{m - 1} = RHS$
Feel free to either take notes from the second video, discuss what you learned, or just skip to the third video altogether.
THIRD VIDEO: COIN PROBLEMS!
A coin problem

• We will prove that every dollar amount $\geq 4$ cents can be exclusively paid for by 2 and/or 5 cent coins.
Theorem expressed in quantifiers

• All quantifiers implicitly assumed over \( \mathbb{N} \).

\[(\forall n \geq 4)(\exists n_1, n_2)[n = 2n_1 + 5n_2]\]
**Inductive base**

• The least amount of money we are required to prove the statement for is 4¢, so we will attempt to prove $P(4)$.

• For $n = 4$, we have 4¢. Since $4¢ = 2 \times 2¢$, we are done (we have shown that the amount of 4¢ can be exclusively paid for by using only 2 and/or 5 cent coins)
Inductive hypothesis

• Let \( n = k \geq 4 \).

• Assume \( P(k) \iff (\exists k_1, k_2)[k = 2k_1 + 5k_2] \)
Inductive step

• We will prove that $P(k) \Rightarrow P(k + 1)$, i.e. that we can pay an amount of money equal to $k + 1$ cents using only 2¢ or 5¢ coins.

• In terms of algebra, what we want to prove is:

$$(\exists k_3, k_4 \in \mathbb{N}) \ [k + 1 = 2k_3 + 5k_4]$$
Inductive step

• We will prove that $P(k) \Rightarrow P(k + 1)$, i.e that we can pay an amount of money equal to $k + 1$ cents using only 2¢ or 5¢ coins.

• In terms of algebra, what we want to prove is:

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Different variables from I.H!
Inductive Step (contd.)

• From the Inductive Hypothesis (I.H), we have that for some specific positive integers $k_1$ and $k_2$:

$$k = 2k_1 + 5k_2$$
Inductive Step (contd.)

• From the Inductive Hypothesis (I.H), we have that for some specific positive integers $k_1$ and $k_2$:

\[ k = 2k_1 + 5k_2 \]

1. Case #1: $k_1 \geq 2$
• I have at least 2 2¢ coins, so I can take away 2 2¢ coins and add one 5¢ coin
Inductive Step (contd.)

• From the Inductive Hypothesis (I.H), we have that for some specific positive integers $k_1$ and $k_2$:

\[ k = 2k_1 + 5k_2 \]

1. **Case #1:** $k_1 \geq 2$

• I have at least 2 2¢ coins, so I can take away two 2¢ coins and add one 5 ¢ coin

• By adding 1 on both sides of the I.H we obtain:

\[ k + 1 = 2k_1 + 5k_2 + 1 = 2k_1 + 5k_2 + (5 - 2 \times 2) = (2k_1 - 4) + (5k_2 + 5) = 2(k_1-2) + 5(k_2 + 1) = 2k_3 + 5k_4 \]
• From the Inductive Hypothesis (I.H), we have that for some specific positive integers $k_1$ and $k_2$:

$$k = 2k_1 + 5k_2$$

1. **Case #1: $k_1 \geq 2$**

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$k_1 - 2 \geq 0$ because $k_1 \geq 2$

In $\mathbb{N}$ by closure
2. Case #2: \( k_2 \geq 1 \)

- I have at least one 5¢ coin so I can take away one 5¢ coin and add three 2¢ coins.
- By adding 1 on both sides of the I.H we obtain:

\[
k + 1 = 2k_1 + 5k_2 + 1 = 2k_1 + 5k_2 + (3 \times 2 - 5) = 2(k_1 + 3) + 5(k_2 - 1) = 2k_3 + 5k_4
\]
2. Case #2: $k_2 \geq 1$

- I have at least one 5¢ coin so I can take away one 5¢ coin and add three 2¢ coins
- By adding 1 on both sides of the I.H we obtain:

$$k + 1 = 2k_1 + 5k_2 + 1 = 2k_1 + 5k_2 + (3 \times 2 - 5) = 2(k_1 + 3) + 5(k_2 - 1) = 2k_3 + 5k_4$$

$(k_1 + 3) \in \mathbb{N}$ by closure

$k_2 - 1 \geq 0$ because $k_2 \geq 1$
3. **Case #3:** \((k_1 \leq 1) \land (k_2 = 0)\)

- This case means that we have either 0 or 2¢ at our disposal.
- But this is not possible, since we want to prove the theorem only for values \(\geq 4¢\)
- So we’re done. □
A note about the penny problem

• Note that we proved the theorem for \( n \geq 4 \)

• Generally speaking, we can use induction to prove statements
  \[ P(n) \forall n \geq n_0, \text{ where } n_0 \in \mathbb{N}. \]

• Most of the time \( n_0 \) will be small (0, 1, 2, ...)
A note about the penny problem

• Note that we proved the theorem for \( n \geq 4 \)

• Generally speaking, we can use induction to prove statements \( P(n) \ \forall n \geq n_0, \) where \( n_0 \in \mathbb{N} \).

• Most of the time \( n_0 \) will be small \((0, 1, 2, \ldots)\)

• If \( P(n_0) \land (\forall k \geq n_0)[P(k) \Rightarrow P(k + 1)] \) is true, then the inductive principle holds and we have the desired statement

\[
(\forall n \geq n_0)[P(n)]
\]
Another!

- Prove that every dollar amount equal to at least 112 cents can be paid for exclusively by 5 and 6 cent coins.
Another!

• Prove that every dollar amount equal to at least 112 cents can be paid for exclusively by 5 and 6 cent coins.
• Let’s do this one together.
A coin problem for you!

Prove to me that every dollar amount \( \geq 20 \) cents can be exclusively paid for through combinations of 5-cent coins and 6-cent coins!
FOURTH VIDEO: TREATING INEQUALITIES
Here’s one with an inequality!

• Prove that for all integers $n$ at least 4, $2^n < n!$

1. **I.B:** We will prove $P(4) \iff 2^4 < 4!$ Done.

2. **I.H:** For $n = k \geq 4$, we assume $P(k)$, i.e $2^k < k!$

3. **I.S:** We will prove $P(k) \implies P(k + 1)$, i.e

$$(2^k < k!) \implies (2^{k+1} < (k + 1)!$$
Inductive Step...

• Prove that for all integers $n$ at least 4, $2^n < n!$

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2. **I.H:** For $n = k \geq 4$, we assume $P(k)$, i.e $2^k < k!$

3. **I.S:** We will prove $2^{k+1} < (k + 1)!$
Inductive Step...

• Prove that for all integers $n$ at least 4, $2^n < n!$

1. **I.B:** We will **prove** $P(4) \iff 2^4 < 4!$ Done.

2. **I.H:** For $n = k \geq 4$, we **assume** $P(k)$, i.e. $2^k < k!$

3. **I.S:** We will **prove** $2^{k+1} < (k + 1)!$
   • From algebra, we have that $2^{k+1} = 2^k \cdot 2 \quad (1)$
Inductive Step...

• Prove that for all integers $n$ at least 4, $2^n < n!$

1. **I.B:** We will **prove** $P(4) \iff 2^4 < 4!$ Done.

2. **I.H:** For $n = k \geq 4$, we **assume** $P(k)$, i.e $2^k < k!$

3. **I.S:** We will **prove** $2^{k+1} < (k + 1)!$
   • From algebra, we have that $2^{k+1} = 2^k \cdot 2$ \hspace{1cm} (1)
   • From the I.H, we have that $2^k < k! \iff 2^k \cdot 2^2 > 0$ $\iff 2^k \cdot 2 < k! \cdot 2$ \hspace{1cm} (2)
Inductive Step...

• Prove that for all integers $n$ at least 4, $2^n < n!$

1. I.B: We will prove $P(4) \iff 2^4 < 4!$ Done.

2. I.H: For $n = k \geq 4$, we assume $P(k)$, i.e $2^k < k!$

3. I.S: We will prove $2^{k+1} < (k + 1)!$
   • From algebra, we have that $2^{k+1} = 2^k \cdot 2$ (1)
   • From the I.H, we have that $2^k < k! \iff 2^k \cdot 2 < k! \cdot 2$ (2)
   • Since $k \geq 4$, we have that $2 < k + 1 \iff k! \cdot 2 < k! (k + 1)$ (3)
Inductive Step...

• Prove that for all integers $n$ at least 4, $2^n < n!$

1. **I.B:** We will prove $P(4) \iff 2^4 < 4!$ Done.

2. **I.H:** For $n = k \geq 4$, we assume $P(k)$, i.e $2^k < k!$

3. **I.S:** We will prove $2^{k+1} < (k + 1)!$
   - From algebra, we have that $2^{k+1} = 2^k \cdot 2$ \hspace{2cm} (1)
   - From the I.H, we have that $2^k < k! \iff 2^k \cdot 2 < k! \cdot 2$ \hspace{2cm} (2)
   - Since $k \geq 4$, we have that $2 < k + 1 \iff k! \cdot 2 < k! \cdot (k + 1)$ \hspace{2cm} (3)
   - $(2) \Rightarrow 2^k \cdot 2 < (k + 1)! \iff 2^{k+1} < (k + 1)!$
An inequality problem for you!

• Using mathematical induction, prove that, for all naturals \( n \geq 3 \),

\[ 2n + 1 < 2^n \]