Comments on
Big Ramsey Degrees of Countable Ordinals
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1 Comment on My Comments

1. Reading it on paper and marking it up and then typing this document really did find MANY MORE issues than my prior times when I edited directly.

2. Nathan will take these comments and, for each one either (a) do it, (b) do it in a different way that he does not need to ask me about, or (c) ask me a question about it.

   Nathn: send me an edited version of this document, where the comments that had (a) or (b) just delete, and the ones that are (c) state your question or thoughts.

3. To quote a part of our paper, or my suggestion of how to improve it, I often put the text in italics, even if the original was not.

4. While there are things that could be said better, and I say how to say them better, I don't see that there are many grammar or other such problems. Perhaps the report meant to say we could say things better.

2 Abstract

The very first sentence is incorrect.

Ramsey's Theorem states that for all finite colorings of an infinite set, there exists an infinite homogenous set.

Better

Recall that Ramsey's Theorem for countable sets states that, for all countable sets A, for all $n \in \mathbb{N}$, for all finite colorings of n-element subsets of A, there exists an infinite homogenous.

Make this correction or some variant of it and also correct the rest of the abstract.

3 Introduction

1. Definition 1.1. Here is a rewrite that I think is better for several reasons

   We use $\mathbb{N}$ to denote the set of natural numbers including 0. Let $n, c \in \mathbb{N}$ and S be a set. We use $[c]$ to denote $\{1, \ldots, c\}$. Note that if $c = 0$ then $[c] = \emptyset$. We use $\binom{S}{n}$ to denote the set of n-element subsets of S.

   The rest should be Definition 1.2.

2. Theorem 1.3 needs a reference and needs to be correct. The following from my open problems column with Natasha works:

Mašulović and Šobot [1] proved the following:

Theorem 3.1 *SAME AS WE HAVE EXCEPT THAT AT THE END SHOULD BE* $|\,\mathrm{COL}(\binom{H}{n})| \le 2^n$.

4 Definitions

1. The first and second sentence have some grammar issues AND should be a Definition.

   Here is a rewrite.

   Def 4.1 Let $\mathcal{A} = (A, \preceq_A)$ and $\mathcal{B} = (B, \preceq_B)$ be ordered sets.

   (a) $\mathcal{A}, \mathcal{B}$ are *order-equivalent*, denoted $\mathcal{A} \approx \mathcal{B}$, if there exists an order-preserving bijection $f\colon A \to B$ such that, for all $a_1, a_2 \in A$:

   $$a_1 \preceq_A a_2 \iff f(a_1) \preceq_B f(a_2).$$

   (b) The ordered set $\mathcal{A}+\mathcal{B}$ is defined to be $(A \sqcup B, \preceq)$, where $A \sqcup B$ is formally defined to be $(\{0\} \times A) \cup (\{1\} \times B)$, and where $\preceq$ is the dictionary ordering on $A \sqcup B$. Note that $+$ agrees with the definition of ordinal addition, and is not commutative in general.

2. The definition of $-\mathcal{A}$ is fine but needs to be in a definition environment.

3. The sentence beginning *Throughout* is to long and a bit odd to say we COULD define A.

   Possible rewrite:

   Throughout this paper, we conflate the notation for an ordered set and its underlying set. For example if A is an ordered set we can still use the notation $\binom{A}{2}$.

4. Definition 2.1. Make the first sentence into two sentences so you should use

   Def 4.2 Let S be an ordered set, $S' \subseteq S$, and $n, c, t \in \mathbb{N}$. Let $\mathrm{COL}\colon \binom{S}{n} \to [c]$ be a coloring. Then S' is *homogeneous* if $|\,\mathrm{COL}\left(\binom{S'}{n}\right)| = 1$. S' is *t-homogeneous* if $|\,\mathrm{COL}\left(\binom{S'}{n}\right)| \le t$. Similarly, S' is *S-t-homogeneous* if S' is t-homogeneous and $S' \approx S$.

5. Definition 2.3. We don't use the arrow notation so we should say that before Its okay to bring up notation that we are not going to use, but we need to tell the reader we are not going to use it earlier than we do.

   Better:

   Other sources use the Ramsey arrow notation modified for ordered sets. In that notation $T(n, S) = n$ is short-hand for $S \to (S)^n_{r,n}$ and $S \not\to (S)^n_{T(n,S),n-1}$ for all $r \in \mathbb{N}$.

6. Lemma 2.4 is long and badly written. $T(n, 0)$ is defined properly in the prior definition and we don't need to unravel all of that. So just have:

   The following is obvious. We include it for completeness

   Lemma 4.3 *For all ordered sets S, $T(0, S) = 1$.*

7. LATER- we will take the paragraph *Additionally* ... and expand it to a table of all that is known about various structures.

5 Summary of Results

1. Theorem 3.1. Here is a suggestion. Omit part 1 of the theorem and eliminate the enum-env so its just

   For all $n \in \mathbb{N}$, $T(n, \omega) = 1$.

2. The summary of results, item 2.

   Better to have

   Mašulović & Šobot [1] previously showed the following:

   (a) For all ordinals $\alpha < \omega^\omega$, $T(n, \alpha) < \infty$. The did not obtain exact values for $T(n, \alpha)$.
   (b) For all ordinals $\alpha \geq \omega^\omega$, $T(n, \alpha) = \infty$.

3. QUESTION: DID [1] or some other source have ANY upper bounds? If so we should include them. This might be a question for Natasha.

6 Big Ramsey degree of ζ

1. The title of the section should have *degree* as *Degree*. This is a global problem- check this for all section headings.

2. Example 4.1 and Example 4.2 should be Theorem 4.1 and Theorem 4.2

3. Example 4.1. You use colors 1 and 2. We should use RED and BLUE. We should generally use RED and BLUE unless we need to use numbers for some good reason like it makes a notation easier.

4. Example 4.2.

   (a) *Label H as $\{h_0 < h_1 < \cdots\}$*
       better
       Let $H = \{h_0 < h_1 < \cdots\}$.
       You later use *Index* intead of *Label*. Look for all uses of the word *Index* or *index* and make use *Let* instead.

(b) When we say

Let $h_1, h_2 \in H$

this is confusing since H has things like $-h_2$ in it.

Better to say

Let $s_i h_i, s_j h_j \in H$ *where* $s_1, s_2 \in \{-1, 1\}$.

Then fix the rest of it as well..

(c) The sentence

We could have partitioned H' *on something than than ...*

serves no purpose. Just remove it.

(d) The proof that $T(2, \zeta) \geq 4$ seems like a lot of words for not a lot of content. KEEP the stuff up to and including the sentence

Let $H \subseteq \zeta$ *with* $H \sim \zeta$.

THEN do the following.

We show that $|\,\mathrm{COL}(\binom{H}{2})| = 4$.

i. Let $x, y \in H$ *such that* $x, y \geq 0$. *Then* $\mathrm{COL}(x, y) = 1$.

ii. Let $x, y \in H$ *such that* $x \geq 0$, $y < 0$, *and* $|x| \leq |y|$. *Then* $\mathrm{COL}(x, y) = 2$.

iii. Let $x, y \in H$ *such that* $x \geq 0$, $y < 0$, *and* $|x| > |y|$. *Then* $\mathrm{COL}(x, y) = 3$.

iv. Let $x, y \in H$ *such that* $x < 0$, $y < 0$. *Then* $\mathrm{COL}(x, y) = 4$.

Hence $T(2, \zeta) \geq 4$. *Combine* $T(2, \zeta) \leq 4$ *and we obtain* $T(2, \zeta) = 4$.

5. Theorem 4.3.

(a) There are a few places where we have to comment on absolute values of elements being the same. I think it would make the proof easier if we did the following.

i. When proving $T(n, \zeta) \leq 2^n$ the FIRST thing we do is kick out the odd positives and the even negatives, so that COL is defined on sets of numbers that never have, say, $\{-10, -7, 1, 3, 10\}$.

ii. When proving $T(n, \zeta) \geq 2^n$ only define COL on sets where the absolute values are different, and for all of the rest just color it some already used color. This will alleviate the need for $<^*$.

6. You have

$$\mathrm{COL}(x_1, \ldots, x_n) = (\mathrm{COL}(x_1, \ldots, x_n), \mathrm{COL}(-x_1, x_2, \ldots, x_n), \ldots, \mathrm{COL}(-x_1, \ldots, -x_n)).$$

Are we assuming $x_1 < x_2 < \cdots < x_n$. I think we should so that this is well defined. This is also an issue for Theorem 4.2 so fix it there also.

7 Big Ramsey Degrees of finite multiples of ω

1. The paragraph before Theorem 5.1.

   Our first result shows $T(n, \omega \cdot k) = k^n$ for most n, k

   Better:

   Our first result shows that $T(n, \omega \cdot k) = k^n$ for all n, k where at least one is nonzero.

2. In the proof of Theorem 5.1.

   (a) When you define COL$'$, are we assuming $x_1 < \cdots < x_n$? This is a global issue so check it everywhere.

   (b) Possible typo: when you define H you have g_i on the first two lines but h_i on the last line.

   (c) The sentence

   Note that the use of modulus was only ...

   I find more confusing than clarifying. Get rid of it.

   (d) GREAT to have an example with $k = 3$. You should have it right after the general case, which will be easier since the last item tells you to get rid of the sentence between them. I leave it to you to figure out where to but the $H \sim \omega \cdot k$.

   (e) *The coloring is inspired by Theorem 5.1, although we use it to prove a different bound.*

   This confused me since I thought you meant the bound would not be k^n.

   Better:

   The coloring is inspired by the proof of Theorem 5.1.

   (f) The definition of the ordering is badly written.

   Better:

   We define the color of the elements

   $$e = \{(3, 12), (50, 2), (110, 12), (110, 7777), (117, 3)\}$$

   as follows:

   - *Order the ordered pairs by the second coordinate.*
   - *If two elements have the same second coordinate then order by the first coordinate.*
     So at this point in our example we have

     $$((50, 2), (117, 3)(3, 12), (110, 12), (110, 7777).$$

   - *Remove the second coordinates to get*

     $$(50, 117, 3, 110, 110).$$

     That sequence is the color.

8 A big Ramsey degree of ω^2

1. The title should be *The Big* instead of *A big*

BILL SEND THE ABOVE TO NATHAN ON FRIDAY SEPT 15, 2023, at 12:20PM.

References

[1] D. Mašulović and B. Šobot. Countable ordinals and big Ramsey degrees, 2019. `https://arxiv.org/pdf/1904.03637.pdf`.