## 1. A Non-HS Idea for $W(x^2; c)$

Theorem **??** gave an enormous upper bound on $W(x^2; 4)$. The proof was found by a computer program; however, it is a HS proof and human-verifiable. Four colors seems to be at the limit of what computers can find. That is, we have been unable to use a program to find a human-verifiable proof for a bound on $W(x^2; 5)$.

There is another possible approach. Usually a HS proof gives better bounds than a proof that uses advanced mathematics. However, our HS proof for $W(x^2; 4)$ gives such a large bound that its possible the advanced proofs, if looked at more carefully, will yield better bounds on $W(x^2; 4)$. It's also possible they will yield reasonable bounds for $W(x^2; c)$ for small value of $c$ such as $c = 5$.

We summarize the literature on the following problem: find the smallest possible function $a(n)$ such that, for large $n$, any $X \subseteq \{1, \ldots, n\}$ of density $\Omega(a(n))$ (that is, $|X| \geq \Omega(\frac{a(n)}{n})$) has two numbers that are a square apart. It is easy to see that, for large $n$, $W(x^2; O(\frac{1}{a(n)})) \leq n$ (which can be used to get a bound no $W(x^2; c)$). The proofs are asymptotic and not HS; however, it is possible the can be modified to give actual upper bounds on $W(x^2; c)$.

| $a(n)$ | Reference |
|---|---|
| 1 | Furstenberg [7] |
| 1 | Lyall [18] (simpler proof but not HS) |
| $\frac{(\log\log n)^{2/3}}{(\log n)^{1/3}}$ | Sárközy [8] |
| $(\log\log n)^{-c}$ | Green [19] |
| $\frac{1}{(\log n)^{c \log\log\log n}}$ | Bloom & Maynard [20] |

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