The Referees Report and My Responses to it

Below is the referees report, which I transcribed into LaTeX, AND my comments on every correction that was suggested. My comments are in all caps so as to make them clearly different from the referees comments.

Most of my comments are DONE meaning I made the correction. Others explain what I did. For some of the comments I didn't quite know what the referee wanted, and I said so.

1 General Commentary

The version R1 of your paper *The Complexity of Grid Coloring* is a significant improvement of the associated basic version. Most of your ideas are now understandable expressed. However, the version R1 does not satisfies the requirements for a paper to publish within a journal due to remaining unclear statements and obviously wrong details. I hope that my hints help you to prepare a final version of this paper that is useful for the readers.

1. One key issue, you should think about, is related to your statement on Page 4:

This shows that GCE reduces to SAT but not that SAT reduces to GCE.

You state for your very partial coloring using a 3-SAT formula that ϕ a bijection to the grid coloring problem:

$$\phi \in \textit{3SAT iff } (N, M, c, \chi) \in \textit{GCE}.$$

Why this bijection is not valid for the SAT formula that expresses all requirements and all restrictions of the grid-coloring problem?

RESPONSE: I FIRST RESTATE HERE WHAT IS IN THE PAPER:

BEGIN QUOTING PAPER

We make one observation about GCE and SAT before our proof. It is an easy exercise to express the question $(N, M, c, \chi) \in$ GCE as a SAT formula. (This was the starting point for the work of Steinback and Posthoff with χ being the empty function.) This shows that GCE reduces to SAT but not that SAT reduces to GCE. Hence this reduction does not help us obtain a lower bound on the complexity of GCE.

END QUOTING PAPER

(a) THE REDUCTION GCE $\leq$ SAT WORKS FOR ANY (N, M, c, χ), NOT JUST THOSE THAT COME OUT OF THE SAT $\leq$ GCE REDUCTION.

(b) I DO NOT HAVE A BIJECTION. THE EASY REDUCTION GCE $\leq$ SAT

(c) I WONDER IF I MISUNDERSTAND YOUR CONCERN.

2. Based on the possible check of one complete coloring of the grid in a time $O(N^2M^2)$ (or your provided faster solution $O(c(NM)^{3/2})$ and the mentioned bijection between a SAT formula and the grid coloring problem it follows that the gird coloring problem is NP-complete. In this case your construction of a 3-SAT formula for a partial grid coloring problem is not needed.

   RESPONSE: I THINK YOU ARE REFERRING TO THIS:

   It is an easy exercise to express the question $(N, M, c, \chi) \in$ GCE as a SAT formula.

   BY THIS I MEAN THAT THERE IS AN EASY CONSTRUCTION THAT MAPS (N, M, c, ϕ) TO ϕ SUCH THAT

$$(N, M, c, \phi) \in \text{GCE IFF } \phi \in \text{SAT}.$$

   THIS IS NOT A BIJECTION.

   THIS DOES NOT SHOW THAT GCE IS HARD.

   IF GCE WAS IN P, THIS CONSTRUCTION WOULD NOT GET YOU THAT SAT IS IN P.

   TO SHOW A PROBLEM A IS NP-COMPLETE YOU NEED TO SHOW THAT SAT $\leq$ A (OR FOR SOME SOME NPC PROBLEM B, $B \leq A$.

2 Major Comment

1. Page 2: The search for OBS_4 has been completed with the found 4-coloring of $G_{12,21}$; hence, you should move the sentence: *These results completed the search for OBS_4.* to the end of item 5 of the enumeration.

   DONE

2. Page 3: You provide a very short specification of *GCE* in Definition 5 and declare that GCE stands for *Grid Coloring Extension*. Due to Definition 4 you restrict the meaning of the term *extendable* to the existence of χ to a total c-coloring of $G_{N,M}$. That means, "GCE" is a "Grid Coloring Extension" of a grid $G_{N,M}$ in which all cells are colored using one of the c colors and containing no monochromatic rectangle. Your Definition 5 can also be understood that GCE is the set of all rectangle-free c-colored grids $G_{N,M}$ and all selectable partial coloring χ.

   I HAVE ELIMINATED THE DEF OF EXTENDIBLE (DEF 4 IN LAST VERSION) SINCE I ONLY USE IT ONCE. I HAVE ELABORATED THE DEF OF GCE (DEF 5) IN LAST VERSION. I HAVE ALSO ELABORATED SLIGHLY, EARLIER ON, THE NOTION OF A PROPER COLORING.

   THE DEFINITION OF GCE REALLY IS ASKING IF THERE IS A PROPER EXTENSION.

3. On the same page you write below of Definition 6:

   $$\textit{Clearly } GCE \in NTIME(O((NM)^4)).$$

   A grid with a certain property does not require any time; hence, obviously you implicitly associated a task with the three letters *GCE* that can be:

   - check whether a chosen extension of χ does not contain any monochromatic rectangle, or
   - check whether χ can be extended such that at least one extension of χ does not contain any monochromatic rectangle.

   Based on your statement:

   $$\textit{Clearly } GCE \in NTIME(O((NM)^4)).$$

it can be assumed that you mean the first task number one of the list given above. You see, a more precise Definition 5 is needed. Assuming the meaning of GCE corresponds to the first interpretation given above, your unexplained statement

Clearly $GCE \in NTIME(O((NM)^4))$

is a very weak upper bound of this task. To check all rectangles of the grid for equal colors all pairs of rows $\binom{N}{2}$ and all pairs of columns $\binom{M}{2}$ must be selected where each selected rectangle can be checked in constant time. This leads to

$$GCE \in NTIME(O(N^2M^2)) = NTIME(O((NM)^2)).$$

Is your exponent 4 only a typo or have you another interpretation of your very large upper bound?

THE 4 IS A TYPO. I HAVE CORRECTED IT AND PUT IN THE PROOF OF THAT

$$GCE \in NTIME(O(N^2M^2)) = NTIME(O((NM)^2)).$$

4. Page 4: Two parentheses are missing in Theorem 1:

   $GCE \in NTIME(O(cMN)^{3/2})$. SHOULD BE $GCE \in NTIME(O(c(MN)^{3/2}))$.

   DONE

5. Theorem 2 is not clear due to the unclear definition of GCE. Your construction of a rectangle-free grid based on an arbitrary 3-SAT formula is now well explained; but it shows only that a 3-SAT formula can be mapped to huge grid (large numbers N, M of rows and columns) containing a very small and strongly restricted rectangle-free 2-coloring and almost all cell are colored by a huge number of different colors. The relation of this construction to your aim to find a lower bound of the grid-coloring problem is not clear.

   RESPONSE: I NOW INCLUDE A DISCUSSION OF IF MY RESULTS REALLY DO RELATE TO THE COMPLEXITY OF FINDING GRID COLORINGS. IT IS SECTION 5.

   What the NP-Completeness Result Does and Does Not Tell Us

6. In the proof of Theorem 2 you state:

   The output will be (N, M, c, χ)

   but you do not provide the output values N, M, and c. Furthermore, you state:

   $$\phi \in 3SAT \text{ iff } (N, M, c, \chi) \in GCE.$$

   (that means a bijection), but there are extendable grids that do not belong to your construction. For instance, using your construction a 3-SAT formula of a single clause would be mapped to a grid of N=7 rows, M=9 columns, and c=41 colors; however, there are many smaller extendable grids which do satisfy your stated bijection.

   I AGREE THAT THE MAPPING IS NOT A BIJECTION. HOWEVER, THE STATEMENT:

   $$\phi \in 3SAT \text{ iff } (N, M, c, \chi) \in GCE.$$

   DOES NOT MEAN IT IS A BIJECTION, NOR IS IT INTENDED TO BE.

   I AM SAYING THAT THERE IS A POLY TIME ALGORITHM THAT WILL, GIVEN ϕ IN 3CNF FORM, OUTPUT (N, M, c, ϕ) SUCH THAT THE IFF STATEMENT HOLDS. THIS DOES NOT MEAN I HAVE A BIJECTION.

   RESPONSE: I NOW INCLUDE A WORKED OUT CALCULATION OF WHAT N, M, c ARE. IT BEGINS ON PAGE 14 AT THE BOTOOM WITH THE HEADING:

   Recap and the Actual Values of N, M, c

7. Page 5:

   Fig. 2 Cell (2,4) is colored (2,4) and nothing else can be

   should be

   Fig. 2 Cell (2,4) of the shown grid is colored (2,4) and no other cell can be colored with this color to get a rectangle-free grid

   DONE- though not in the way you suggested.

8. Page 6: A vertical line is missing in Fig. 3 and (2.4) must be moved to the right of this line.

   DONE

9. Page 8: The set of seven cases in the proof of Claim 2 are correct. The cases 1 to 4 are well ordered using the binary code in the first column. Exchanging the cases 5 and 6 would establish this order for all seven cases. This exchange requires also the change of TTF, TFT into TFT, TTF in the first sentence of the proof of Claim 2.

   DONE

10. Page 10: Figure 9 is not correct; you must replace the two D's below of C_3 into T's.

    DONE

11. In Claim 3 you restrict

    Let (N, M, c, χ) be the result of the reduction described above.

    so that your assumption of this claim is only true for a subset of extendable grids.

    YOU ARE CORRECT. WHY IS THIS A PROBLEM?

12. Page 11: In Claim 4 you restrict

    Let (N, M, c, χ) be the result of the reduction described above.

    so that your assumption of this claim is only true for a subset of extendable grids.

    YOU ARE CORRECT. WHY IS THIS A PROBLEM?

13. Page 12: Figure 11 is not correct; you must replace the two D's in the row of x_2 below of C_1 and C_2 into T's (to conform to your introduced rules).

    COMMENT- THIS IS A CORRECT CORRECTIONS; HOWEVER, THE ENTIRE EXAMPLE IS WRONG AND I HAVE REDONE IT.

    I WAS USING

    $$(x_1 \vee x_2 \vee \overline{x}_3) \wedge (x_1 \vee x_2 \vee \overline{x}_4) \wedge (\overline{x}_2 \vee x_3 \vee x_4)$$

    and $x_1 = T$, $x_2 = F$, $x_3 = T$, $x_4 = F$.

    BUT THIS DOES NOT GIVE THE MONO RECTANGLE THAT I NEED TO SHOW THE CONSTRUCTIONS DOES NOT WORK. I HAVE REDONE IT WITH

$$(x_1 \vee x_2 \vee \overline{x}_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge (\overline{x}_2 \vee x_3 \vee x_4)$$

and $x_1 = T$, $x_2 = F$, $x_3 = T$, $x_4 = F$.

WHICH DOES SHOW WHAT I INTENDED.

14. Page 13: Figure 12 is not correct; you must replace:

    columns C_1 C_1 row x_3: D F by D D DONE

    columns C_1 C_1 row $\overline{x}_3$: D D by D F DONE

    columns C_2 C_2 row x_4: D F by D D DID NOT DO SINCE NOW WTH NEW FML ITS CORRECT.

    columns C_2 C_2 row $\overline{x}_4$: D D by D F DID NOT DO SINCE NOW WITH NEW FML ITS CORRECT.

    columns C_3 C_3 row $\overline{x}x_2$: D D by F D DONE

    columns C_3 C_3 row x_3: D D by BLANK BLANK. DONE

    columns C_3 C_3 row $\overline{x}_3$ BLANK BLANK by D D. DONE

    columns C_3 C_3 row x_4: D D by D F. DONE

    columns C_3 C_3 row $\overline{x}_4$: D F by D D DONE

15. One coordinate in the proof of Claim 5 is wrong; you must replace:

    Since $\chi'(1,1) = T$, $\chi'(1,2) = T$

    SHOULD BE

    Since $\chi'(1,1) = T$, $\chi'(2,1) = T$

    DONE

16. Page 14: The paragraph before Section 5 is not clear due to missing explanations of used terms.

    DONE

17. Definition 7 is in the same manner unclear as Definition 5.

    DONE

18. Page 16: $f(S,i)$ is defined as a Boolean function and i as an integer (color); hence, $f(S,i) = i$ is wrong. I assume you mean: $f(S,i) = YES$.

    DONE

19. Page 16.

    You state:

    The algorithm has 2^s iterations...

    but do not clearly specify the algorithm your mean. It can be assumed that you mean the algorithm provided within the paragraph above; hence, the small change into

    The just now specified algorithm has 2^s iterations ...

    (or similar) brings more clearness.

    DONE

20. Page 15 - 17: You use the approach of dynamic programming in the proof of Lemma 4 to get a time complexity of $O(cuNM3^u)$. In the case of an empty partial coloring this complexity will be $O(cN^2M^23^{NM})$ which is an improvement for $c > 3$ in comparison to the check of all c^{NM} possible colorings that take $O((NM)^{3/2}c^{NM})$. However, due to step 2 of your dynamic programming algorithm on page 7 you need a space of $c * 2^u$ which will be $c * 2NM)$ for an empty coloring χ. Of course, you must store only Boolean values of the function $f(S,i)$ in this table, but even if you store the eight bits for three cell within one byte, a memory of c TB is already needed for $u = 43$. For the computation of a 4-coloring of $G_{18,18}$ starting with an empty χ already more than 10^{85} TB are needed; hence, the required memory is the limiting factor of your dynamic programming algorithm. You should insert a hint to this limitation into your paper.

    RESPONSE: In the first paragraph of the Fixed Parameter Section I state that I will look at what happens in the case of the empty 17×17 grid and $c = 4$ (I changed from 18 to 17 since 17 was the orignal motivation) at the END of the section. At the end of the section I do the calcuations to show how bad the time and space are.

    RESPONSE: I also include in the theorems themselves the space bound.

21. Page 17: The inequality on page 17 is wrong. The variable c is missing on the right-hand side.

    DONE

22. You should decide by yourself whether the statement

    The following is folklore

    should remain in the final version of your paper. A hint to obvious lemmas 5 and 6 is enough. A more clear proof of Lemma 5 should be given. The used binomial coefficient is an integer; a set i, j cannot be an element of an integer! A short well understandable proof of lemma 4 can be: Since every column has at least $c + 1$ cells, at least one color appears twice in each column. There are $\binom{c+1}{2}$ columns of $c + 1$ rows with different pairs of cells to which this selected color can been assigned; hence, a rectangle-free grid $G_{N,M}$ with $c + 1 \leq N$ rows can have maximal $M = c * \binom{c+1}{2}$ columns.

    RESPONSE:

    - I HAVE REPLACED 'FOLKLORE' WITH 'EASY'- SEE FOR YOURSELF.
    - WHEN YOU SAY "lemma 4" I THINK YOU MEAN "lemma 5"
    - I DID NOT WRITE $\{i, j\} \in \binom{c+1}{2}$, I WROTE $\{i, j\} \in \binom{[c+1]}{2}$ WHICH IS CORRECT- RECALL FROM THE NOTATION ON PAGE 1 THAT (1) IF $x \in \mathbb{N}$ THEN $[x]$ IS THE SET $\{1, \ldots, x\}$ and (2) IF A IS A SET THEN $\binom{A}{k}$ IS THE SET OF ALL k-SIZED SUBSETS OF A.
    - WITH ALL OF THAT IN MIND, I THINK MY PROOF AND YOUR PROOF ARE THE SAME, SO I HAVE TAKEN MINE AND GOTTEN RID OF THE NOTATION AND JARGON.

23. Page 17:

    The proof of Theorem 3 uses the result of Lemma 4; hence, a limiting factor is the available space. This limitation should be inserted into the Theorem!

    I HAVE INSERTED INTO THE THEOREMS THEMSELVES THE SPACE BOUND.

24. Page 18: For more clearness you should insert the variable u on the left-hand side of the upper inequality.

    RESPONSE: I DO NOT KNOW WHAT THIS IS REFERRING TO, THOUGH IT MAY BE THAT I MESSED UP TRANSCRIBING THE REPORT AND THIS IS TAKEN CARE OF IN THE NEXT COMMENT.

25. Page 18 The variable u is missing in the left term of the inequality in Step 5. (b):

    $O(cNM3^u)$ should be $O(cuNM3^u)$

    RESPONSE: I THINK YOU MEAN Step 4. IF SO THEN DONE.

26. Page 18:

    The proof of Theorem 4 uses the result of Lemma 4; hence, a limiting factor is the available space. This limitation should be inserted into the Theorem!

    I HAVE INSERTED INTO THE THEOREMS THEMSELVES THE SPACE BOUND.

3 Minor Comments

1. Page 1: a full c-coloring?We show

   SHOULD BE a full c-coloring? We show

   SEEMS TO HAVE ALREADY BEEN FIXED.

2. Page 3: 2 Definition of The Grid Coloring Extension Problem

   SHOULD BE 2 Definition of the Grid Coloring Extension Problem

   DONE

3. Page 3 Then again—it may not. We discuss

   SHOULD BE Then again—it may not. We discuss

   SEEMS TO HAVE ALREADY BEEN FIXED.

4. Page 3 in Section 4

   SHOULD BE in Section 4.

   DONE

5. Page 4: It takes $O(MN)$ time to identify P

   SHOULD BE It takes $O(MN)$ time to identify P.

   DONE

6. Page 5 Steinback and Postoff

   SHOULD BE Steinbach and Posthoff

   DONE

7. Page 5 χ is a a partial c-coloring

   SHOULD BE χ is a partial c-coloring

   DONE

8. Page 5: uses the colors T, F

   SHOULD BE uses the colors T, F

   DONE

9. Page 5 The literls
   SHOULD BE The literals
   DONE

10. Page 5 The colors will be T, F
    SHOULD BE The colors will be T, F
    DONE

11. Page 6: Fig. 3 $(2, 4)$ and $(5, 3)$
    SHOULD BE Fig. 3 $(2, 4)$ and $(5, 3)$ within a sub-grid
    DONE

12. Page 6 Then we use the grid in Figure 3
    SHOULD BE Then we use the grid in Figure 3.
    DONE

13. Page 7: Fig. 4 Literal Gadget with three variables
    SHOULD BE Fig. 4 Literal gadget with four variables
    DONE

14. Page 7 Fig. 5 Clause Set Up
    SHOULD BE Fig. 5 Clause set up
    DONE

15. Page 7 for ease of use.We refer
    SHOULD BE for ease of use. We refer
    SEEMS TO HAVE ALREADY BEEN FIXED.

16. Page 8: Fig. 6 The Clause Gadget
    SHOULD BE Fig. 6 The clause gadget
    DONE

17. Page 8 Fig. 7 The Clause Gadget—easier to work with
    SHOULD BE Fig. 7 The clause gadget—easier to work with
    DONE

18. Page 11: Reformat the first line such that the text on this line satisfies the maximal width. (x_n)

    DONE

19. Page 11 $\overline{x_n}$

    SHOULD BE $\overline{x}_n$

    DONE

20. Page 11 for every clause$C$

    SHOULD BE for every clause C

    SEEMS TO HAVE ALREADY BEEN FIXED.

21. Page 11 Let$C$

    SHOULD BE Let C

    SEEMS TO HAVE ALREADY BEEN FIXED.

22. Page 11 the construction fails!We give

    SHOULD BE the construction fails! We give

    SEEMS TO HAVE ALREADY BEEN FIXED.

23. Page 13: the cell to the left has coordinate (2;1)

    SHOULD BE the cell to the right has coordinate (2;1)

    DONE

24. Page 13: Does Not Tell us

    SHOULD BE Does Not Tell Us

    DONE

25. Page 14: Can we do better? Yes.We will show

    SHOULD BE Can we do better? Yes. We will show

    SEEMS TO HAVE ALREADY BEEN FIXED.

26. Page 14 GCE is in time

    SHOULD BE GCE$_c$ is in time (two times)

    DONE

27. Page 15: coloringC_s
    SHOULD BE coloring C_s
    SEEMS TO HAVE ALREADY BEEN FIXED.
28. Page 15 We can determine if χ^*
    SHOULD BE We can determine whether χ^*
    DONE
29. Page 16: YES.We need
    SHOULD BE YES. We need
    SEEMS TO HAVE ALREADY BEEN FIXED.
30. Page 18: Step 3 in the proof of Theorem 4 is empty.
    SEEMS TO HAVE ALREADY BEEN FIXED.
31. Page 19: timesink
    SHOULD BE time-sink
    DONE