We now summarize and put the finishing touches on the proof.

Our goal was to, given a 3CNF formula

$$\phi(x_1, \ldots, x_n) = C_1 \wedge \cdots \wedge C_m,$$

with n free variables and m clauses, output an instance of GCE such that

$$\phi \in \text{3SAT iff } (N, M, c, \chi) \in \text{GCE}.$$

We have described the partial coloring χ. For the sake of completeness we now specify N, M, c. We first find the dimensions of the core grid.

Let o_i be maximum of the number of occurrence of x_i and $\overline{x}_i$.

$m \le \sum_{i=1}^{n} o_i \le 3m$.

The minimum case is of all of the o_i's are 1. Hence the sum is $\ge m$.

Since o_i is the number of times one of $\{x_i, \overline{x}_i\}$ occurs the sum is bounded by the number of occurrence of variables. Since the formula is in 3CNF form, the number of occurrence of variables is $3m$.

Number of rows in the core grid The literal column will have o_i rows labeled x_i and o_i rows labeled $\overline{x}_i$. Hence the literal column will have $\sum_{i=1}^{n} 2o_i$ blank cells. Every clause C induces a row that is all D's except for two T's under the columns labeled C. Hence we get m additional rows. Therefore the number of rows in the core grid is

$$N' = m + \sum_{i=1}^{n} 2o_i = m + 2\sum_{i=1}^{n} o_i \le m + 3m = 3m$$

Number of columns in the core grid Each variable x_i induces a rectangle of height $2o_i$ and length $4o_i - 2$. See Figure **??** for an example with $o_1 = 1$, Figure **??** for an example with $o_1 = 2$, and Figure 1 for an example with $o_1 = 3$. Each clause adds 2 columns. Therefore the number of columns in the core grid is

$$M' = 2m + \sum_{i=1}^{n}(4o_i - 2) = 2m - 2n + 4\sum_{i=1}^{n} o_i \le 2m - 2n + 4\times 3m = 14m - 2n \le 14m.$$

The Number of Blank Cells, T-Cells, F-Cells, and Colors

The first column has $\sum_{i=1}^{n} 2o_i$ blank cells. Each column labeled with a clause has 1 blank cell. Hence the number of blank cells is

$\overline{x}_1$		D	D	D	D	D	D	T	F	T	F
x_1		D	D	D	D	D	D	T	F	D	D
$\overline{x}_1$		D	D	D	D	T	F	T	F	D	D
x_1		D	D	T	F	T	F	D	D	D	D
$\overline{x}_1$		T	F	T	F	D	D	D	D	D	D
x_1		T	F	D	D	D	D	D	D	D	D

Figure 1: Three Occurrence of x_1

$$B = m + \sum_{i=1}^{n} 2o_i = m + 2\sum_{i=1}^{n} o_i$$

Each column that is not labeled with a clause has one T and one F. Each column labeled with a clause has one T and one F. Hence the number of cells labeled with a T or an F is

$$2M' \leq 28m$$

Every cell that is neither blank, T, or F has a distinct color. Hence the number of new colors that are not T or F is

$$E = N'M' - B - 2M' \leq N'M' \leq 42m^2$$

and the total number of colors is

$$c = E + 2 \leq 42m^2 + 2.$$

The real values of N, M

We now deal with the non-core part of the grid. For every color that is not T or F we add one row and one column to the grid (see Part I of the construction). Hence

$$M = M' + E \leq 14m + 42m^2 = O(m^2)$$

$$N = N' + E \leq 3m + 42m^2 = O(m^2)$$