π Upper Bounding Sums

June 29, 2026

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Can we use this sum to get an upper bound on π?

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Proof by Picture:

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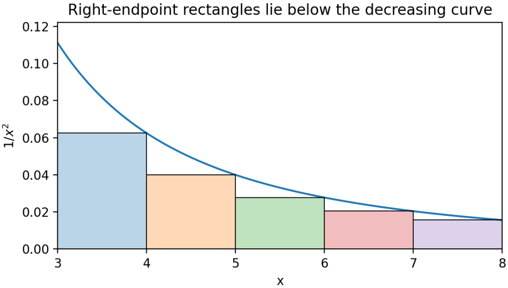
Proof by Picture:

Picture proof

For each integer $i \geq n+1$, draw the rectangle of width 1 over $[i-1,i]$ with height $1/i^2$. Because $y=1/x^2$ is decreasing, each rectangle lies entirely below the curve. Hence $\int_{i-1}^{i} 1/x^2 \, dx \geq 1/i^2$. Summing over $i=n+1,n+2,\ldots$ gives $\Sigma_{i=n+1}^{\infty} 1/i^2 \leq \int_{n}^{\infty} 1/x^2 \, dx$.



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$$\pi = \sqrt{6\sum_{i=1}^{\infty} \frac{1}{i^2}} = \sqrt{6\sum_{i=1}^{n} \frac{1}{i^2} + \sum_{i=n+1}^{\infty} \frac{1}{i^2}} \leq \sqrt{6\sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{n}}$$

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$$\sqrt{6\sum_{i=1}^{n}\frac{1}{i^2}} \leq \pi \leq \sqrt{6\sum_{i=1}^{n}\frac{1}{i^2}+\frac{1}{n}}$$

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Do the same for $\frac{333}{106}$ and $\frac{355}{113}$.

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Use this technique on other sums that involved π.

π Wrapping up the Project

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Discuss the inscribed-circumscribed method. Give data to show where the method shows $\frac{22}{7} \neq \pi$, $\frac{333}{106} \neq \pi$, $\frac{355}{113} \neq \pi$ and the value of n. Also note that it looks like for n-sided polygon you get $X(n)$ number of digits (I do not know the X but your data should show it.)

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Discuss other sum methods, or other methods you know. Same thing- Give data ETC.