Grid Colorings that Avoid Mono Squares

Exposition by William Gasarch

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Notation

If $n \in \mathbb{N}$ then

$$[n] = \{1, \ldots, n\}.$$

Mono L's and Mono Squares

Def Let $G \in \mathbb{N}$ and $c \in N$. Assume we have a c-coloring of $[G] \times [G]$

1. A **mono L** is 3 points

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that are all the same color ($d \geq 1$). (This should be called an *mono isosceles right triangle* but we just call it a *mono L*.)

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2. A **mono Square** is 4 points

$$(x, y), (x + d, y), (x, y + d), (x + d, y + d)$$

   that are all the same color ($d \geq 1$). This is a square.

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4. More Colors: *For all c there exists $G = G(c)$ such that for all c-colorings of $[G] \times [G]$ there exists a mono square.* Proof needs a larger c' for $GG(c')$.

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View a 2-coloring of $[H] \times [H]$ *as a* 2^9*-coloring of the tiles.*

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Work on this with your neighbor.

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We won't prove this but I am sure any of you could prove it given what we have done so far. Would be messy.

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Go to Whiteboard for rest of proof.

What Else is Known

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There are also multi-dim versions.