

Project One: Random Ramsey Theory

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1 Goal and setup

The project studies what happens when every edge of K_n is independently colored red or blue with probability $1/2$. For $k \geq 3$, let

$$M_k = M_k(n)$$

denote the number of monochromatic copies of K_k in the coloring.

There are two different questions throughout the project:

- (1) What does a *random* coloring usually do?
- (2) What is guaranteed in *every* coloring?

These questions are related, but they are not the same. The experiments address the first question directly; Ramsey theory and extremal arguments address the second.

1.1 A general expected-value formula

A fixed copy of K_k has $\binom{k}{2}$ edges. There are

$$2^{\binom{k}{2}}$$

possible red–blue colorings of these edges, and exactly two are monochromatic: all red and all blue. Therefore

$$\mathbb{P}(K_k \text{ is monochromatic}) = \frac{2}{2^{\binom{k}{2}}} = 2^{1-\binom{k}{2}}.$$

Since K_n contains $\binom{n}{k}$ copies of K_k , linearity of expectation gives

$$\boxed{\mathbb{E}[M_k] = \binom{n}{k} 2^{1-\binom{k}{2}}.} \tag{1}$$

No independence between different K_k 's is needed for this expectation calculation.

For $k = 3$,

$$\mathbb{E}[M_3] = \frac{1}{4} \binom{n}{3},$$

and for $k = 4$,

$$\mathbb{E}[M_4] = \frac{1}{32} \binom{n}{4}.$$

1.2 General Pitch 1: random K_5 and monochromatic triangles

Since $R(3) = 6$, K_5 can be colored with no monochromatic triangle. However, such a coloring is rare if the edges are colored uniformly at random.

In fact, a red–blue coloring of K_5 has no monochromatic triangle exactly when one color class is a 5-cycle (and the other color class is its complementary 5-cycle). There are

$$\frac{(5-1)!}{2} = 12$$

labeled 5-cycles. Since K_5 has 10 edges and therefore $2^{10} = 1024$ two-colorings,

$$\mathbb{P}(K_5 \text{ has no monochromatic } K_3) = \frac{12}{1024} = \frac{3}{256} \approx 0.01172.$$

Thus a random coloring of K_5 has at least one monochromatic triangle with probability about 98.83%, even though Ramsey’s theorem does not force one until K_6 .

2 Step 3: program

I used one program for the experiments. It randomly colors every edge of K_n red or blue, checks all k -vertex subsets, and counts how many induce a monochromatic K_k . The experiment function repeats this 100 times unless a different number of trials is requested.

```
import random
import itertools
from collections import Counter

random.seed(42)

def random_coloring(n):
    color = {}
    for i in range(n):
        for j in range(i + 1, n):
            color[(i, j)] = random.randint(0, 1)
    return color

def get_color(color, i, j):
    if i > j:
        i, j = j, i
    return color[(i, j)]

def is_monochromatic(vertices, color):
    edge_colors = []
    for i, j in itertools.combinations(vertices, 2):
        edge_colors.append(get_color(color, i, j))
    return all(c == edge_colors[0] for c in edge_colors)

def count_mono(n, k):
    color = random_coloring(n)
    count = 0
    for vertices in itertools.combinations(range(n), k):
        if is_monochromatic(vertices, color):
            count += 1
    return count

def run_experiment(n, k, trials=100):
```

```

results = []
for _ in range(trials):
    results.append(count_mono(n, k))
return results

def summarize(results):
    freq = Counter(results)
    max_frequency = max(freq.values())
    modes = sorted(x for x in freq if freq[x] == max_frequency)
    average = sum(results) / len(results)
    return average, modes, max_frequency

```

3 Step 4: monochromatic triangles

For Step 4 I used $k = 3$ and ran 100 random colorings for each $n = 5, 6, \dots, 10$.

```

for n in range(5, 11):
    results = run_experiment(n, 3, 100)
    average, modes, frequency = summarize(results)
    print("n =", n)
    print("average =", average)
    print("most common =", modes)
    print("times seen =", frequency)
    print()

```

3.1 Experimental results

Graph	Experimental average	Exact $\mathbb{E}[M_3]$	Most common count	Times seen
K_5	2.72	2.50	3	36/100
K_6	5.01	5.00	4	30/100
K_7	8.59	8.75	9	18/100
K_8	13.68	14.00	9, 13, 15	12/100 each
K_9	21.29	21.00	23	12/100
K_{10}	30.07	30.00	30	12/100

The observed averages agree closely with the exact expectation

$$\mathbb{E}[M_3] = \frac{1}{4} \binom{n}{3}.$$

For large n ,

$$\mathbb{E}[M_3] \sim \frac{n^3}{24}.$$

3.2 Step 4(i): expected number of monochromatic triangles

A particular triangle has three edges. Of its $2^3 = 8$ possible edge colorings, exactly two are monochromatic. Thus the probability that a fixed triangle is monochromatic is $1/4$, and therefore

$$\mathbb{E}[M_3] = \frac{1}{4} \binom{n}{3} \sim \frac{n^3}{24}.$$

3.3 Step 4(ii): how many triangles are guaranteed in every coloring?

This can be determined asymptotically by a short counting argument. For a vertex v , let r_v be its red degree and b_v its blue degree, so

$$r_v + b_v = n - 1.$$

A non-monochromatic triangle is counted exactly twice in

$$\sum_v r_v b_v,$$

because exactly two of its vertices are incident to one red and one blue edge inside the triangle. Therefore the number N of non-monochromatic triangles is

$$N = \frac{1}{2} \sum_v r_v b_v.$$

Since $r_v b_v \leq (n-1)^2/4$,

$$N \leq \frac{n(n-1)^2}{8}.$$

Hence every coloring has at least

$$\begin{aligned} M_3 &= \binom{n}{3} - N \\ &\geq \binom{n}{3} - \frac{n(n-1)^2}{8} \\ &= \frac{n(n-1)(n-5)}{24}. \end{aligned}$$

Thus

$$M_3 \geq \left(\frac{1}{24} - o(1) \right) n^3. \quad (2)$$

This leading constant is attainable. Split the vertices into two almost equal groups, color all edges inside the two groups red, and all edges between the groups blue. The blue graph is bipartite, so it has no blue triangle; the only monochromatic triangles are the red triangles inside the two parts. Their number is

$$\binom{\lfloor n/2 \rfloor}{3} + \binom{\lceil n/2 \rceil}{3} \sim \frac{n^3}{24}.$$

Therefore

$$\boxed{M_{3,\min}(n) \sim \frac{n^3}{24} = \frac{1}{4} \binom{n}{3}.} \quad (3)$$

So the conjecture suggested by the original data is correct at the leading-order level. In fact, Goodman determined the exact finite- n minimum in 1959.

4 A general guaranteed bound from Ramsey numbers

There is a simple double-counting observation that applies to any known Ramsey number.

Proposition 1. *Suppose $R(k) = r$. Then every red-blue coloring of K_n , with $n \geq r$, contains at least*

$$\boxed{\frac{\binom{n}{k}}{\binom{r}{k}}}$$

monochromatic copies of K_k .

Proof. Every set of r vertices contains a monochromatic K_k , since $R(k) = r$. Count pairs (S, H) where S is an r -vertex subset and H is a monochromatic K_k contained in S .

There are at least $\binom{n}{r}$ such pairs, because every S contains at least one H . On the other hand, each fixed monochromatic K_k is contained in exactly $\binom{n-k}{r-k}$ choices of S . If there are M_k monochromatic copies of K_k , then

$$M_k \binom{n-k}{r-k} \geq \binom{n}{r}.$$

Using

$$\frac{\binom{n}{r}}{\binom{n-k}{r-k}} = \frac{\binom{n}{k}}{\binom{r}{k}},$$

we obtain the result. □

For $R(3) = 6$ this gives the weaker, but very general, bound

$$M_3 \geq \frac{\binom{n}{3}}{\binom{6}{3}} = \frac{1}{20} \binom{n}{3}.$$

The special triangle argument above improves the asymptotic constant from $1/20$ to the optimal $1/4$.

For $R(4) = 18$, the same argument gives the useful rigorous bound

$$\boxed{M_4 \geq \frac{\binom{n}{4}}{\binom{18}{4}} = \frac{1}{3060} \binom{n}{4}}. \tag{4}$$

This already proves, using only $R(4) = 18$, that the number of monochromatic K_4 's forced in every coloring is $\Omega(n^4)$.

5 Step 5: monochromatic copies of K_4

For Step 5 I used $k = 4$ and ran 100 random colorings for each $n = 10, 11, \dots, 25$.

```
for n in range(10, 26):
    results = run_experiment(n, 4, 100)
    average, modes, frequency = summarize(results)
    print("n =", n)
    print("average =", average)
    print("most common =", modes)
    print("times seen =", frequency)
    print()
```

5.1 Experimental results

Graph	Experimental average	Exact $\mathbb{E}[M_4]$	Most common count(s)	Times seen
K_{10}	7.21	6.56	4	14/100
K_{11}	10.78	10.31	6	12/100
K_{12}	17.55	15.47	13	10/100
K_{13}	23.20	22.34	19	8/100
K_{14}	31.76	31.28	30	7/100
K_{15}	43.19	42.66	31	7/100
K_{16}	56.54	56.88	46	6/100
K_{17}	77.77	74.38	70	6/100
K_{18}	96.19	95.63	87	5/100
K_{19}	119.06	121.13	106, 125	4/100 each
K_{20}	146.59	151.41	137	5/100
K_{21}	184.98	187.03	174	5/100
K_{22}	229.94	228.59	203	4/100
K_{23}	276.81	276.72	268, 307, 321	3/100 each
K_{24}	328.71	332.06	288, 320, 388	3/100 each
K_{25}	409.05	395.31	346, 358, 418, 430	3/100 each

Again the random averages are close to the exact expected values from (1).

5.2 Step 5(i): where does the probability of a monochromatic K_4 pass 0.9?

I used 10,000 random colorings for each n near the crossing point.

```
random.seed(42)

for n in range(5, 11):
    trials = 10000
    hits = 0
    for _ in range(trials):
        if count_mono(n, 4) > 0:
            hits += 1
    probability = hits / trials
    print(n, probability)
```

n	Estimated $\mathbb{P}(\text{at least one monochromatic } K_4)$
5	0.1369
6	0.3105
7	0.5583
8	0.7965
9	0.9419
10	0.9896

The first estimated probability above 0.9 occurs at $n = 9$, while the estimate at $n = 8$ is below 0.9. Therefore the experiment gives

$$m = 9.$$

This is an experimental determination based on 10,000 trials, rather than an exact enumeration of all colorings.

5.3 Step 5(ii): expected number of monochromatic K_4 's

A K_4 has six edges. Of its $2^6 = 64$ possible red–blue edge colorings, exactly two are monochromatic. Therefore

$$\mathbb{P}(K_4 \text{ is monochromatic}) = \frac{2}{64} = \frac{1}{32}.$$

Hence

$$\mathbb{E}[M_4] = \frac{1}{32} \binom{n}{4} \sim \frac{n^4}{768}. \quad (5)$$

5.4 Step 5(iii): how many K_4 's are guaranteed in every coloring?

The random-coloring experiment by itself cannot determine a universal minimum, because a deliberately constructed coloring can behave very differently from a typical random coloring.

Using only $R(4) = 18$, however, the double-counting proposition gives the rigorous statement

$$M_4 \geq \frac{1}{3060} \binom{n}{4} \sim \frac{n^4}{73440}.$$

Thus the answer is certainly of order n^4 .

The natural asymptotic problem is to define

$$c_4 = \lim_{n \rightarrow \infty} \frac{\min M_4(n)}{\binom{n}{4}}.$$

Unlike the triangle case, the exact value of c_4 is not known. Current research places it in the narrow range

$$0.0296 < c_4 < 0.030145. \quad (6)$$

The random-coloring density is

$$\frac{1}{32} = 0.03125,$$

which is larger than the best known upper bound in (6). Thus, unlike K_3 , a uniformly random coloring is not asymptotically optimal for minimizing monochromatic K_4 's.

Thus we can actually prove, rather than merely conjecture, that

$$M_{4,\min}(n) = \Theta(n^4).$$

What remains open is the exact leading constant. The experiments suggest a density near the random value $1/32$, while the known bounds show that the optimum is slightly smaller and lies very close to $0.03 \binom{n}{4}$.

6 Conclusions

The experiments and calculations give the following answers to Project One.

1. For a random red–blue coloring of K_n ,

$$\mathbb{E}[M_k] = \binom{n}{k} 2^{1-\binom{k}{2}}.$$

This formula works for every fixed k .

2. For triangles,

$$\mathbb{E}[M_3] = \frac{1}{4} \binom{n}{3} \sim \frac{n^3}{24},$$

and the minimum over *all* colorings has the same leading asymptotic behavior:

$$M_{3,\min}(n) \sim \frac{n^3}{24}.$$

Thus random colorings have the correct asymptotic monochromatic-triangle density.

3. For K_4 ,

$$\mathbb{E}[M_4] = \frac{1}{32} \binom{n}{4} \sim \frac{n^4}{768}.$$

The 10,000-trial experiment estimates that $n = 9$ is the first n for which the probability of at least one monochromatic K_4 is at least 0.9.

4. From $R(4) = 18$ alone,

$$M_4 \geq \frac{1}{3060} \binom{n}{4},$$

so every coloring has $\Omega(n^4)$ monochromatic K_4 's. The exact asymptotic leading constant is a much deeper problem and is not currently known.

5. More generally, if $R(k) = r$, then every coloring of K_n satisfies

$$M_k \geq \frac{\binom{n}{k}}{\binom{r}{k}}.$$

This provides one general way to turn a Ramsey theorem into a multiplicity statement.

References

- [1] A. W. Goodman, “On Sets of Acquaintances and Strangers at Any Party,” *American Mathematical Monthly* 66 (1959), 778–783.
- [2] O. Parczyk, S. Pokutta, C. Spiegel, and T. Szabó, “New Ramsey Multiplicity Bounds and Search Heuristics,” arXiv:2206.04036; later published in *Foundations of Computational Mathematics*.