1 Lewis- Sat paper

We have been dealing with connectives, $\wedge, \vee, \neg$. What if we used other connectives such as $\oplus$? Lewis [3] has shown a dichotomy theorem in this context.

Theorem 1. *Let C be a set of connectives. Let C-**SAT** be the problem of **SAT** where the formulas use those connectives.*

1. *If the function $x \wedge \neg y$ can be expressed with the connectives in C then C-**SAT** is **NP**-complete.*
2. *If the function $x \wedge \neg y$ cannot be expressed with the connectives in C then C-**SAT** is in **P**.*

Exercise 1. *Let C be the set of all binary connectives. For all $D \subseteq C$ determine if $x \wedge \neg y$ can be expressed with the connectives in D.*

2 Lewis-Quant

We will now look at sentences that have quantifiers.

Example 1. *Consider the sentence $\Phi = (\exists x)(\forall y)[E(x, y)]$. Is there a domain for the variables and a meaning of E such that this sentence is true? Yes. Let the domain be the vertices of a directed graph G such that there is a vertex x that has an edge to all of the other vertices including itself.*

Definition 1. *Let Φ be a sentence of the form*

$$(\mathsf{Q}_1 x_1)(\mathsf{Q}_2 x_2) \cdots (Q_p x_p)[\phi(R_1(x_1, \ldots, x_p), \ldots, R_q(x_1, \ldots, x_p)]$$

where each Q_i is a quantifier, ϕ is a Boolean formula, and the $R_1, \ldots, R_q$ have no meaning (for now). We have written the R's as if they have to have all of the variables as arguments to avoid messy notation; however, they may have fewer arguments.

1. *Φ is **satisfiable** if there is a domain D for the variables and an interpretation for the predicates $R_1, \ldots, R_q$ such that, using that domain and those predicates, the sentence is true.*
2. *If the quantifier prefix is a (possibly empty) string of $\exists$ followed by a (possibly empty) string of $\forall$ then ϕ is **of the form E^*A^***. Other ways of using E, E^*, A, A^* are easily defined.*
3. *A predicate that takes only one argument is called **unary**.*

Note that we do not allow the equal sign in our sentences. This matters a great deal, not just for complexity but even for decidability. We will give an example later.

Example 2.

1. *Let Φ be*

$$(\forall x)(\exists y)(\exists z)[E(x, y) \wedge \neg E(x, z)].$$

Φ says that every vertex has an edge to some vertex and a non-edge to some vertex. Φ is satisfiable by the directed 2-vertex graph where there is a directed edge from each vertex to the other. So each vertex has the other as a neighbor and itself for a non-neighbor.

2. *Let Φ be*
$$(\forall w)(\exists x)(\forall y)(\exists z)[R(w,x,y) \wedge \neg R(x,y,z)].$$
We leave it to the reader to determine if Φ is satisfiable.

Problem 2.1. *X-**SAT***

INSTANCE: A sentence Φ with quantifiers as in Definition 1. We soon restrict the form of the sentence (that will be the X).

QUESTION: Is Φ satisfiable?

NOTE:

1. *Monadic-**SAT**: The predicates are all unary.*
2. *Ackermann-**SAT**: Φ is of the form E^*AE^*.*
3. *Gödel-**SAT**: Φ is of the form E^*AAE^*.*
4. *Schönfinkel-Bernays-**SAT**: Φ is of the form E^*A^*.*

There are other variants depending on what else you allow in the language, such as the equals sign or functions. The study of which versions of X-**SAT** are decidable and for those that are, what is their complexity, has rich history (see Börger et al. [1]). Turing [6] showed that the case of X-**SAT** with no restrictions is undecidable. All of the restricted cases we consider were shown to be decidable a long time ago (a 1915 paper of Lowenheim [5]); however, complexity results began in the 1970's.

We present four complexity results due to Lewis [4] where the upper and lower bounds match. Perhaps surprisingly, the lower bounds do not need a hardness assumptions.

Theorem 2. *For the following problems there are matching upper and lower bounds, which we state:*

1. *Monadic-**SAT**:* $\mathrm{NTIME}(2^{\Theta(n/\log n)})$.
2. *Ackermann-**SAT**:* $\mathrm{DTIME}(2^{\Theta(n/\log n)})$.
3. *Gödel-**SAT**:* $\mathrm{NTIME}(2^{\Theta(n/\log n)})$.
4. *Schönfinkel-Bernays-**SAT**:* $\mathrm{NTIME}(2^{\Theta(n)})$.

We had mentioned earlier that the lack of an equal sign changes the complexity. Here is an example. Goldfarb [2] proved the following.

Theorem 3. *If the equal sign is allowed then Gödel-**SAT** is undecidable.*

References

[1] E. Börger, E. Grädel, and Y. Gurevich. *The Classical Decision Problem.* Perspectives in Mathematical Logic. Springer, 1997.

[2] W. D. Goldfarb. The unsolvability of the Gödel class with identity. *Journal of Symbolic Logic*, 49(4):1237–1252, 1984.
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[3] H. R. Lewis. Satisfiability problems for propositional calculi. *Math. Syst. Theory*, 13:45–53, 1979.
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[4] H. R. Lewis. Complexity results for classes of quantificational formulas. *Journal of Computing and System Sciences*, 21(3):317–353, 1980.
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[6] A. Turing. On computable numbers with an application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society*, 42:230–265, 1936. corrections ibid. 43 (1937) 544–546.