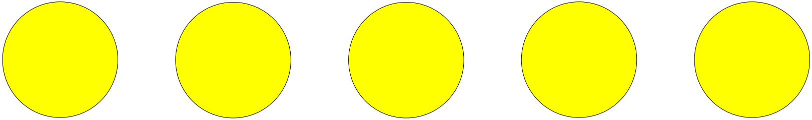
Muffin Problems

November 20, 2020

Five Muffins, Three Students

How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ where nobody gets a tiny sliver?



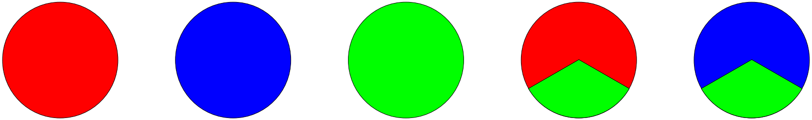
Credit where Credit is Due

Alan Frank came up with the problem and it was circulating in some math newsgroups around 2010, though I did not know this.

Five Muffins, Three Students, Proc by Picture

Person	Color	What they Get
Alice	**RED**	$1 + \frac{2}{3} = \frac{5}{3}$
Bob	**BLUE**	$1 + \frac{2}{3} = \frac{5}{3}$
Carol	GREEN	$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$

Smallest Piece: $\frac{1}{3}$



Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

Work on it with your Breakout Rooms Group

5 Muffins, 3 People–Proc by Picture

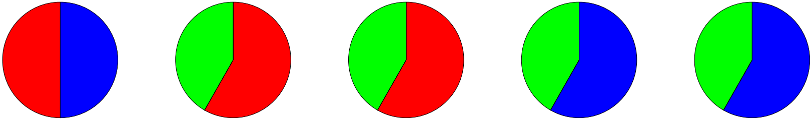
YES WE CAN!

5 Muffins, 3 People–Proc by Picture

YES WE CAN!

Person	Color	What they Get
Alice	**RED**	$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$
Bob	**BLUE**	$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$
Carol	GREEN	$\frac{5}{12} + \frac{5}{12} + \frac{5}{12} + \frac{5}{12}$

Smallest Piece: $\frac{5}{12}$



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5 Muffins, 3 People–Can't Do Better Than $\frac{5}{12}$

NO WE CAN'T!

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There is a procedure for 5 muffins,3 students where each student gets $\frac{5}{3}$ muffins, smallest piece N. We want $N \leq \frac{5}{12}$.

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There is a procedure for 5 muffins,3 students where each student gets $\frac{5}{3}$ muffins, smallest piece N. We want $N \le \frac{5}{12}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both $\frac{1}{2}$-sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases.

(**Henceforth:** All muffins are cut into ≥ 2 pieces.)

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(**Henceforth:** All muffins are cut into ≥ 2 pieces.)

Case 1: Some muffin is cut into ≥ 3 pieces. Then $N \le \frac{1}{3} < \frac{5}{12}$.
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Case 1: Some muffin is cut into ≥ 3 pieces. Then $N \leq \frac{1}{3} < \frac{5}{12}$.
(**Henceforth:** All muffins are cut into 2 pieces.)

Case 2: All muffins are cut into 2 pieces. 10 pieces, 3 students:
Someone gets ≥ 4 pieces. He has some piece

$$\leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12} \qquad \text{Great to see } \frac{5}{12}$$

General Problem

$\boldsymbol{f(m,s)}$ be the smallest piece in the best procedure (best in that the smallest piece is maximized) to divide m muffins among s students so that everyone gets $\frac{m}{s}$.

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$f(m,s)$ be the smallest piece in the best procedure (best in that the smallest piece is maximized) to divide m muffins among s students so that everyone gets $\frac{m}{s}$.

We have shown $f(5,3)=\frac{5}{12}$.

FC Thm: If $m > s$ and s does not divide m then

$$f(m,s) \le \mathrm{FC}(m,s) = \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1 - \frac{m}{s\lfloor 2m/s\rfloor}\right\}\right\}.$$

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Case 2: Every muffin is cut into 2 pieces, so $2m$ pieces.

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Alice gets $\ge \lceil \frac{2m}{s} \rceil$ pieces. $\exists$ piece $\le \frac{m}{s} \times \frac{1}{\lceil 2m/s\rceil} = \frac{m}{s\lceil 2m/s\rceil}$.

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The other piece from that muffin is of size $\le 1 - \frac{m}{s\lfloor 2m/s\rfloor}$.

THREE Students

We only look at when 3 does not divide the number of muffins.
Here is what the FC theorem tells us:

m	$f(m,3) \leq$
4	$1/3$
5	$5/12$
7	$5/12$
8	$4/9$
10	$4/9$

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Try to find the protocol for EIGHT muffins, THREE students, everyone gets $\frac{8}{3}$, and smallest piece is $\frac{4}{9}$.

Solve in Breakout Rooms Groups

$f(8,3) = \frac{4}{9}$

By FC $f(8,3) \leq \frac{4}{9}$.

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We show $f(8,3) \geq \frac{4}{9}$.

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By FC $f(8,3) \le \frac{4}{9}$.

We show $f(8,3) \ge \frac{4}{9}$.

1. Divide 8 muffins $\{\frac{4}{9}, \frac{5}{9}\}$.
2. Give 2 students $\{\frac{5}{9}, \frac{5}{9}, \frac{5}{9}, \frac{4}{9}\}$.
3. Give 1 student $\{\frac{4}{9}, \frac{4}{9}, \frac{4}{9}, \frac{4}{9}, \frac{4}{9}, \frac{4}{9}\}$.

FOUR Students

We only look at when 4 and m have no common factors.
Here is what the FC theorem tells us:

m	$f(m,4) \leq$
5	$3/8$
7	$5/12$
9	$7/16$
11	$9/20$

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m	$f(m,4) \leq$
5	$3/8$
7	$5/12$
9	$7/16$
11	$9/20$

Try to find the protocol for SEVEN muffins, FOUR students, everyone gets $\frac{7}{4}$, and smallest piece is $\frac{5}{12}$.
Solve in Breakout Rooms Groups

$f(7,4) = \frac{5}{12}$

By FC $f(7,4) \le \frac{5}{12}$.

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We show $f(7,4) \geq \frac{5}{12}$.

$f(7,4) = \frac{5}{12}$

By FC $f(7,4) \le \frac{5}{12}$.

We show $f(7,4) \ge \frac{5}{12}$.

1. Divide 6 muffins $\{\frac{5}{12}, \frac{7}{12}\}$.
2. Divide 1 muffins $\{\frac{6}{12}, \frac{6}{12}\}$.
3. Give 2 students $\{\frac{7}{12}, \frac{7}{12}, \frac{7}{12}\}$.
4. Give 2 student $\{\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{6}{12}\}$.

FC Conjecture

The following is true:

- If $m \geq 4$ and $m \equiv 1, 2 \pmod 3$ then $f(m, 3) = \mathrm{FC}(m, 3)$.
- If $m \geq 5$ and $m \equiv 1, 3 \pmod 4$ then $f(m, 4) = \mathrm{FC}(m, 4)$.

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If $m \geq s + 1$ and s, m rel prime then $f(m, s) = \mathrm{FC}(m, s)$.

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If $m \geq s + 1$ and s, m rel prime then $f(m, s) = \mathrm{FC}(m, s)$.

Vote YES, NO, Unknown to Science (UN).

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FC Conjecture
If $m \geq s + 1$ and s, m rel prime then $f(m, s) = \mathrm{FC}(m, s)$.

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NO

FIVE Students

We only look at when 4 and m have no common factors.
Here is what the FC theorem tells us:

m	$f(m,5) \leq$
6	$2/5$
7	$1/3$
8	$2/5$
9	$2/5$
11	$11/25$

FIVE Students

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7	$1/3$
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9	$2/5$
11	$11/25$

I am sure you all could do protocols for $f(6,5)$, $f(7,5)$, $f(8,5)$, $f(9,5)$ that match the bounds here.

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History I tried to find a protocol for $f(11,5) \ge \frac{11}{25}$. I could not. I found a protocol with $f(11,5) \ge \frac{13}{30}$.

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History I tried to find a protocol for $f(11,5) \ge \frac{11}{25}$. I could not. I found a protocol with $f(11,5) \ge \frac{13}{30}$.

I showed that **if there is a protocol it must BLAH**. I showed there was NO such protocol with BLAH! and showed $f(11,5) \le \frac{13}{30}$.

Terminology: Buddy

Assume that in some protocol every muffin is cut into two pieces.

Let x be a piece from muffin M.
The *other piece* from muffin M is the *buddy of* x.

Note that the buddy of x is of size

$$1 - x.$$

$f(11,5) = \frac{13}{30}$, Easy Case Based on Muffins

There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece N. We want $N \le \frac{13}{30}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin. Reduces to other cases.

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Case 1: Some muffin is cut into ≥ 3 pieces. $N \le \frac{1}{3} < \frac{13}{30}$.

(**Negation of Case 0 and Case 1:** All muffins cut into 2 pieces.)

$f(11,5) = \frac{13}{30}$, Easy Case Based on Students

Case 2: Some student gets ≥ 6 pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$

$f(11,5)=\frac{13}{30}$, Easy Case Based on Students

Case 2: Some student gets ≥ 6 pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$

Case 3: Some student gets ≤ 3 pieces.
One of the pieces is

$$\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.$$

Look at the muffin it came from to find a piece that is

$$\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.$$

(**Negation of Cases 2 and 3:** Every student gets 4 or 5 pieces.)

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

- s_4 is number of students who get 4 pieces
- s_5 is number of students who get 5 pieces

$$\begin{aligned} 4s_4 + 5s_5 &= 22 \\ s_4 + s_5 &= 5 \end{aligned}$$

$s_4 = 3$: There are 3 students who have 4 shares.
$s_5 = 2$: There are 2 students who have 5 shares.

We call a share that goes to a person who gets 4 shares a **4-share**.
We call a share that goes to a person who gets 5 shares a **5-share**.

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4.1: Some 4-share is $\le \frac{1}{2}$.
Alice gets w, x, y, z and $w \le \frac{1}{2}$.
Since $w + x + y + z = \frac{11}{5}$ and $w \le \frac{1}{2}$

$$x + y + z \ge \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$

Let x be the largest of x, y, z

$$x \ge \frac{17}{10} \times \frac{1}{3} = \frac{17}{30}$$

Look at **buddy** of x.

$$B(x) \le 1 - x = 1 - \frac{17}{30} = \frac{13}{30}$$

GREAT! This is where $\frac{13}{30}$ comes from!

$f(11,5)=\frac{13}{30}$, Fun Cases

Case 4.2: All 4-shares are $>\frac{1}{2}$. There are $4s_4=12$ 4-shares.
There are ≥ 12 pieces $>\frac{1}{2}$. Can't occur.
So we are done!

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5. I presented at **Gathering for Gardner** in 2017, **Fun with Algorithms** in 2018, and **American Math Society Meeting** in 2019, and student groups.

But wait! There is more! Next Slide!

Fame and Fortune!

Where can I read about all this great stuff!

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In my book:
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I gave a talk on this at the MIT combinatorics seminar and I knew Alan Frank was in Boston so we agreed to meet. He gave me 5 muffins, 4 cut $\frac{5}{12}$-$\frac{7}{12}$ and one cut $\frac{6}{12}$-$\frac{6}{12}$. I gave him a free signed copy of my book.