# The Muffin Problem: Complexity Issues

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(4 HS, 3 ugrad, 2 prof, 9 total)

# How it Began

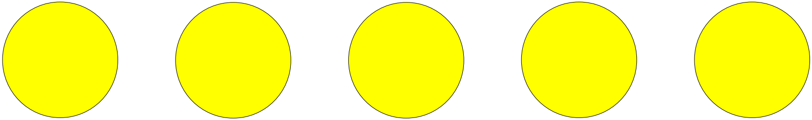
**A Recreational Math Conference**
**(Gathering for Gardner)**
**May 2016**

I found a pamphlet:

**The Julia Robinson Mathematics Festival:**
**A Sample of Mathematical Puzzles**
**Compiled by Nancy Blachman**

which had this problem, proposed by Alan Frank:

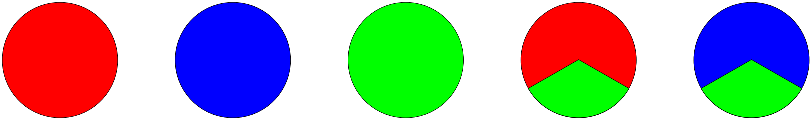
*How can you divide and distribute 5 muffins to 3 students so that every student gets* $\frac{5}{3}$ *where nobody gets a tiny sliver?*



# Five Muffins, Three Students, Proc by Picture

| Person | Color | What they Get |
|---|---|---|
| Alice | **RED** | $1 + \frac{2}{3} = \frac{5}{3}$ |
| Bob | **BLUE** | $1 + \frac{2}{3} = \frac{5}{3}$ |
| Carol | GREEN | $1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$ |

**Smallest Piece:** $\frac{1}{3}$



# Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

**Is there a procedure with a larger smallest piece?**

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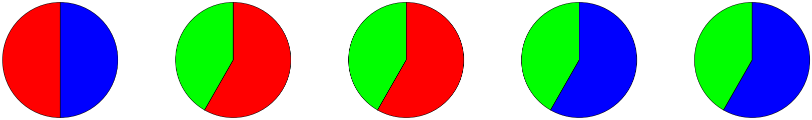
**Is there a procedure with a larger smallest piece?**

**YES WE CAN!**

# Five Muffins, Three People–Proc by Picture

| Person | Color | What they Get |
|---|---|---|
| Alice | **RED** | $\frac{6}{12}+\frac{7}{12}+\frac{7}{12}$ |
| Bob | **BLUE** | $\frac{6}{12}+\frac{7}{12}+\frac{7}{12}$ |
| Carol | GREEN | $\frac{5}{12}+\frac{5}{12}+\frac{5}{12}+\frac{5}{12}$ |

**Smallest Piece:** $\frac{5}{12}$



# Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

**Is there a procedure with a larger smallest piece?**

**NO WE CAN'T!**

# Five Muffins, Three People–Can't Do Better Than $\frac{5}{12}$

There is a procedure for 5 muffins,3 students where each student gets $\frac{5}{3}$ muffins, smallest piece $N$. We want $N \leq \frac{5}{12}$.

**Case 0:** Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both $\frac{1}{2}$-sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases.
(**Henceforth:** All muffins are cut into $\geq 2$ pieces.)

**Case 1:** Some muffin is cut into $\geq 3$ pieces. Then $N \leq \frac{1}{3} < \frac{5}{12}$.
(**Henceforth:** All muffins are cut into 2 pieces.)

**Case 2:** All muffins are cut into 2 pieces. 10 pieces, 3 students: **Someone** gets $\geq 4$ pieces. He has some piece

$$\leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12} \qquad \text{Great to see } \frac{5}{12}$$

# General Problem

*How can you divide and distribute $m$ muffins to $s$ students so that each students gets $\frac{m}{s}$ AND the MIN piece is MAXIMIZED?*

An *$(m, s)$-procedure* is a way to divide and distribute $m$ muffins to $s$ students so that each student gets $\frac{m}{s}$ muffins.

An $(m, s)$-procedure is *optimal* if it has the largest smallest piece of any procedure.

$f(m, s)$ be the smallest piece in an optimal $(m, s)$-procedure.

We have shown $f(5, 3) = \frac{5}{12}$.

**Note**: $f(m, s) \geq \frac{1}{s}$: divide each muffin into $s$ pieces of size $\frac{1}{s}$ and give each student $m$ of them.

# Floor-Ceiling Theorem (Generalize $f(5,3) \le \frac{5}{12}$)

$$f(m,s) \le \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1-\frac{m}{s\lfloor 2m/s\rfloor}\right\}\right\}.$$

**Case 0:** Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin, so reduces to other cases.

**Case 1:** Some muffin is cut into $\ge 3$ pieces. Some piece $\le \frac{1}{3}$.

**Case 2:** Every muffin is cut into 2 pieces, so $2m$ pieces.

**Someone** gets $\ge \left\lceil \frac{2m}{s} \right\rceil$ pieces. $\exists$ piece $\le \frac{m}{s} \times \frac{1}{\lceil 2m/s\rceil} = \frac{\mathbf{m}}{\mathbf{s}\lceil \mathbf{2m/s}\rceil}$.

**Someone** gets $\le \left\lfloor \frac{2m}{s} \right\rfloor$ pieces. $\exists$ piece $\ge \frac{m}{s}\frac{1}{\lfloor 2m/s\rfloor} = \frac{m}{s\lfloor 2m/s\rfloor}$.

The other piece from that muffin is of size $\le 1 - \frac{m}{s\lfloor 2m/s\rfloor}$.

**Notation:** $\mathrm{FC}(m,s)$ is the upper bound given by this Theorem.

# Duality

$f(5,3) \geq \frac{5}{12}$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

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$f(3,5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

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4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

**Theorem** $f(m,s) = \frac{m}{s} f(s,m)$. (Ind by Erich Friedman)

# Conventions

**We know and use the following:**

1. By duality can assume $m > s$
2. If $s$ divides $m$ then $f(m,s) = 1$ so assume $s$ does not divide $m$.
3. All muffins are cut in $\geq 2$ pcs. Replace uncut muff with 2 $\frac{1}{2}$'s
4. If assuming $f(m,s) > \alpha > \frac{1}{3}$, assume all muffin in $\leq 2$ pcs.
5. $f(m,s) > \alpha > \frac{1}{3}$, so exactly 2 pcs, is common case.

**We do not know this but still use:** $f(m,s)$ only depends on $\frac{m}{s}$.
All of our techniques that hold for $(m,s)$ hold for $(Am, As)$.
For particular numbers, we only look at $m, s$ rel prime.

$f(m,s)$:

Does it exist?
If so then is it Rational?
If yes and yes, is it computable?

# Exists? Rational? Computable?

**Exists:** Does $f(m,s)$ always exist? Plausible that

$$(\forall\epsilon > 0)[\exists \text{ Proc for (24,11) with all pieces } \geq \frac{19}{44} - \epsilon]$$

but

$$(\forall)\text{Proc for (24,11) some piece } \leq \frac{19}{44}$$

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**Do not worry- does not happen! Yeah!**

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**Do not worry- does not happen! Yeah!**

**Solvable:** Plausible that $f(m,s)$ is not computable.

**Do not worry- There is a procedure to compute $f(m,s)$**

# Exists, Solvable, Rational: LP does not quite work

We try to formula $f(m,s)$ as a Linear Program
For $1 \le i \le m$, $1 \le j \le s$, $x_{ij}$ is the fraction of $M_i$ that goes to $S_j$.

**Constraints:**
$M_i$ is size 1: $x_{i1} + \cdots + x_{im} = 1$.

$S_j$ gets $\frac{m}{s}$: $x_{1j} + \cdots + x_{sj} = \frac{m}{s}$.

Sanity: $0 \le x_{ij} \le 1$

New Var $z$ is min of the $x_{ij}$: $\quad (\forall i)(\forall j)[z \le x_{ij}]$

**Objective Function:** Maximize $z$

This does not work. **Discuss**

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This does not work. **Discuss**
Possible that $M_1$ gives NONE to $S_2$. So $x_{12} = 0$. What do do?

# An Example where LP Does Not Work

$f(5,3) \geq \frac{5}{12}$

1. Divide $M_1$ $(\frac{6}{12}, \frac{6}{12})$.
2. Divide $M_2$ $(\frac{5}{12}, \frac{7}{12})$.
3. Divide $M_3$ $(\frac{5}{12}, \frac{7}{12})$.
4. Divide $M_4$ $(\frac{5}{12}, \frac{7}{12})$.
5. Divide $M_5$ $(\frac{5}{12}, \frac{7}{12})$.
6. Give $S_1$ $[\frac{6}{12}, \frac{7}{12}, \frac{7}{12}]$ From $M_1, M_2, M_3$
7. Give $S_2$ $[\frac{6}{12}, \frac{7}{12}, \frac{7}{12}]$ From $M_1, M_2, M_3$
8. Give $S_3$ $[\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12}]$ From $M_1, M_2, M_3, M_4$

$M_4$ gives NO piece to from $S_1$, so $x_{41} = 0$. But we don't want to count that.

# Exists, Solvable, Rational: Many LP's do work

**Theorem:** $f(m, s)$ exists, is rational, and can be computed.
**Proof One:** Formulate as an LP.
**Issue:** Some $x_{ij}$'s are 0. Don't want to count them for max $z$.

$(\forall) A \subseteq \{x_{ij}\}$ set all vars in $A$ to 0. Forms $LP_A$. Solve to get $z_A$.
Max of the $z_A$ is $f(m, s)$.

1. Since $0 \le x_{ij} \le 1$, for every $A$, $z_A$ exists.
2. Since all the coefficient in $\mathbb{Q}$, for all $A$, $z_A \in \mathbb{Q}$.
3. Since LP is computable, for all $A$, $z_A$ can be computed.
4. Since every $z_A$ exists, is rational, and is computable, $f(m, s)$ exist, is rational, and is computable.

**Note:** Would NEVER use this algorithm!

# Formulate as an MIP (Ind Discovery: Veit Elser)

**Theorem:** $f(m,s)$ exists, is rational, and can be computed.
**Proof Two:** Formulate as an LP as on prior slide.
**Issue:** Some $x_{ij}$'s are 0. Don't want to count them for max $z$.
Introduce new 0-1 valued variables $y_{ij}$ and constraints:

$$(1) \quad x_{ij} + y_{ij} \ge \frac{1}{s} \qquad (2) \quad x_{ij} + y_{ij} \le 1$$

1) By Eq (1) $x_{ij} = 0 \implies y_{ij} = 1$. Eq (2) satisfied.
2) By Eq (2) $x_{ij} > 0 \implies x_{ij} \ge \frac{1}{s} \implies y_{ij} = 0$. Eq (1) satisfied.

**Diff Constraints on $z$:** $(\forall i)(\forall j)[z \le x_{ij} + y_{ij}]$.
$x_{ij} = 0 \implies y_{ij} = 1 \implies$ Constraint is $z \le 1$ easily satisfied
$x_{ij} > 0 \implies y_{ij} = 0 \implies$ Constraint is $z \le x_{ij}$ as it should be

**Objective Function**: maximize $z$.

# Upper Bound on Complexity of Muffin Problem

Can compute $f(m,s)$ with MIP on $O(ms)$ variables and coefficients in $\{-m,\ldots,m\}$.
So time $2^{O(ms)}$.

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# Upper Bound on Complexity of Muffin Problem

Can compute $f(m, s)$ with MIP on $O(ms)$ variables and coefficients in $\{-m, \ldots, m\}$.
So time $2^{O(ms)}$.

MIP is worse than it sounds: input is of length $\lg m + \lg s$.
We have coded it up and used an MIP package.
**Empirical Observations:**

1. If we provided a very good upper bound then MIP sometimes worked well.
2. MIP had a much harder time when $m$ is prime. Do not know if this means anything.

# Fast(?) Algorithm:

**Input:** $(m, s), \frac{a}{b}$

**Output:**

$(m, s)$-proc, smallest piece $\frac{a}{b}$

OR

Gee, Could not find such a proc

# MATRIX Technique: $f(5,3) \ge \frac{5}{12}$

Want proc for $f(5,3) \ge \frac{5}{12}$.

1) **Guess** that the only piece sizes are $\frac{5}{12}, \frac{6}{12}, \frac{7}{12}$

2) **Muffin**=pieces add to 1: $\{\frac{6}{12}, \frac{6}{12}\}$, $\{\frac{5}{12}, \frac{7}{12}\}$. Vectors
$\{\frac{6}{12}, \frac{6}{12}\}$ is $(0,2,0)$, $m_1$ muffins of this type.
$\{\frac{5}{12}, \frac{7}{12}\}$ is $(1,0,1)$, $m_2$ muffins of this type.

3) **Student**=pieces add to $\frac{5}{3}$
$\{\frac{6}{12}, \frac{7}{12}, \frac{7}{12}\}$ is $(0,1,2)$, $s_1$ students of this type.
$\{\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12}\}$ is $(4,0,0)$, $s_2$ students of this type.

4) **Set up equations:**
$m_1(0,2,0) + m_2(1,0,1) = s_1(0,1,2) + s_2(4,0,0)$
$m_1 + m_2 = 5$
$s_1 + s_2 = 3$

**Natural Number Solution**: $m_1 = 1$, $m_2 = 4$, $s_1 = 2$, $s_2 = 1$

# MATRIX Technique

Want proc for $f(m,s) \ge \frac{a}{b}$.

1) **Guess** that the only piece sizes are $\frac{a}{b}, \ldots, \frac{b-a}{b}$

2) **Muffin**=pieces add to 1: Vectors $\vec{v_i}$. $x$ types.
$m_i$ muffins of type $\vec{v_i}$

3) **Student**=pieces add to $\frac{m}{s}$: Vectors $\vec{u_j}$. $y$ types.
$s_j$ students of type $\vec{u_j}$

4) **Set up equations:**
$m_1\vec{v_1} + \cdots + m_x\vec{v_x} = s_1\vec{u_1} + \cdots + s_y\vec{u_y}$
$m_1 + \cdots + m_x = m$
$s_1 + \cdots + s_y = s$

5) **Look for Nat Numb sol.** If find can translate into procedure.

# MATRIX Technique: Clear Fracs version

Want proc for $f(m,s) \ge \frac{a}{b}$. Clear Fracs version.

1) **Guess** that the only piece sizes are $a, \ldots, b-a$

2) **Muffin**=pieces add to $b \times 1 = b$: Vectors $\vec{v_i}$. $x$ types.
$m_i$ muffins of type $\vec{v_i}$

3) **Student**=pieces add to $b \times \frac{m}{s} = \frac{bm}{s}$: Vectors $\vec{u_j}$. $y$ types.
$s_j$ students of type $\vec{u_j}$

4) **Set up equations:**
$m_1\vec{v_1} + \cdots + m_x\vec{v_x} = s_1\vec{u_1} + \cdots + s_y\vec{u_y}$
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# Analysis of MATRIX$(m, s, \frac{a}{b})$

1) By **Dynamic Programming** can find sums in $O((b-2a)\frac{bm}{s})$.
2) **Empirical:** $b - 2a = O(s^2)$. So $O(bms)$.
3) **Empirical:** number of sums is **small**.
4) **Empirical:** If answer is $\frac{a}{b}$ then pieces all have denom $b$.
5) **Open:** Is MATRIX$(m, s, \frac{a}{b})$ poly in $a, b, m, s$?

$$f(m, s):$$

# Upper Bounds

# Floor-Ceiling Bound

Recall:

$$\mathrm{FC}(m,s) = \left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1-\frac{m}{s\lfloor 2m/s\rfloor}\right\}\right\}.$$

For all $m, s$, $f(m,s) \le \mathrm{FC}(m,s)$.

1. Can compute $FC(m,s)$ in $O(\log m)$. Note: do not need to know the answer ahead of time.
2. For all $m \ge 3$, $f(m,3) = \mathrm{FC}(m,3)$.
3. For all $m \ge 4$, $f(m,4) = \mathrm{FC}(m,4)$.
4. For $3 \le s \le 60$, $s < m \le 70$, $m, s$ rel prime:
   4.1 There are 1360 cases total.
   4.2 For 927 of the $(m,s)$, $f(m,s) = \mathrm{FC}(m,s)$. $\sim 68\%$
   4.3 The cases not covered use **interesting** new techniques!

# Terminology: Buddy

Assume that in some protocol every muffin is cut into two pieces.

Let $x$ be a piece from muffin $M$.
The *other piece* from muffin $M$ is the *buddy of* $x$.

Note that the buddy of $x$ is of size

$$1 - x.$$

# Example of INT Technique: $f(24, 11) \leq \frac{19}{44}$

Assume $(24, 11)$-procedure with smallest piece $> \frac{19}{44}$.
Can assume all muffin cut in two and all student gets $\geq 2$ shares.
We show that there is a piece $\leq \frac{19}{44}$.

**Case 1:** A student gets $\geq 6$ shares. Some piece $\leq \frac{24}{11\times 6} < \frac{19}{44}$.

**Case 2:** A student gets $\leq 3$ shares. Some piece $\geq \frac{24}{11\times 3} = \frac{8}{11}$.
Buddy of that piece $\leq 1 - \frac{8}{11} \leq \frac{3}{11} < \frac{19}{44}$.

**Case 3:** Every muffin is cut in 2 pieces and every student gets either 4 or 5 shares. Total number of shares is 48.

# How many students get 4? 5? Where are the Shares?

*4-students:* a student who gets 4 shares. $s_4$ is the number of them.
*5-students:* a student who gets 5 shares. $s_5$ is the number of them.

*4-share:* a share that a 4-student who gets.
*5-share:* a share that a 5-student who gets.

$$\begin{aligned} 4s_4 + 5s_5 &= 48 \\ s_4 + s_5 &= 11 \end{aligned}$$

$s_4 = 7$. Hence there are $4s_4 = 4 \times 7 = 28$ 4-shares.
$s_5 = 4$. Hence there are $5s_5 = 5 \times 4 = 20$ 5-shares.

# Case 3.1 and 3.2: Too Big or Too Small

**Case 3.1:** There is a share $\geq \frac{25}{44}$. Then its buddy is

$$\leq 1 - \frac{25}{44} = \frac{19}{44}$$

**Case 3.2:** There is a share $\leq \frac{19}{44}$. Duh.
Henceforth assume that all shares are in

$$\left(\frac{19}{44}, \frac{25}{44}\right)$$

## Case 3.3: Some 5-shares $\geq \frac{20}{44}$

*5-share:* a share that a 5-student who gets.
**Claim:** If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.
**Proof:** Assume that Alice 5 pieces $A, B, C, D, E$ and $E \geq \frac{20}{44}$.
Since $A+B+C+D+E = \frac{24}{11}$ and $E > \frac{20}{44}$

$$A+B+C+D \leq \frac{24}{11} - \frac{20}{44} = \frac{76}{44}$$

Assume $A$ is the smallest of $A, B, C, D$.

$$A \leq \frac{76}{44} \times \frac{1}{4} = \frac{19}{44}$$

Henceforth we assume all 5-shares are in

$$\left(\frac{19}{44}, \frac{20}{44}\right).$$

## Case 3.4: Some 4-shares $\le \frac{21}{44}$

*4-share:* a share that a 4-student who gets.

**Claim:** If some 4-shares is $\le \frac{21}{44}$ then some share $\le \frac{19}{44}$.

**Proof:** Assume that Alice 4 pieces $A, B, C, D$ and $D \le \frac{21}{44}$.

Since $A + B + C + D = \frac{24}{11}$ and $D \le \frac{21}{44}$

$$A + B + C \ge \frac{24}{11} - \frac{21}{44} = \frac{75}{44}$$

Assume $A$ is the largest of $A, B, C$.

$$A \ge \frac{75}{44} \times \frac{1}{3} = \frac{25}{44}$$

The buddy of $A$ is of size

$$\le 1 - \frac{25}{44} = \frac{19}{44}$$

Henceforth we assume all 4-shares are in

$$\left(\frac{21}{44}, \frac{25}{44}\right).$$

# Case 3.5: All Shares in Their Proper Intervals

**Case 3.5:** 4-shares in $(\frac{21}{44}, \frac{25}{44})$, 5-shares in $(\frac{19}{44}, \frac{20}{44})$.

$$\underset{\frac{19}{44}}{(} \quad ??\ \text{5-shs} \quad \underset{\frac{20}{44}}{)[} \quad 0\ \text{shs} \quad \underset{\frac{21}{44}}{](} \quad ??\ \text{4-shs} \quad \underset{\frac{25}{44}}{)}$$

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**Recall:** there are $4s_4 = 4 \times 7 = 28$ 4-shares.
**Recall:** there are $5s_5 = 5 \times 4 = 20$ 5-shares.

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**Recall:** there are $4s_4 = 4 \times 7 = 28$ 4-shares.
**Recall:** there are $5s_5 = 5 \times 4 = 20$ 5-shares.

$$\underset{\frac{19}{44}}{(} \quad 20\ \text{5-shs} \quad \underset{\frac{20}{44}}{)[} \quad 0\ \text{shs} \quad \underset{\frac{21}{44}}{](} \quad 28\ \text{4-shs} \quad \underset{\frac{25}{44}}{)}$$

# More Refined Picture of What is Going On

$$\underset{\frac{19}{44}}{(}\quad \text{20 5-shs} \quad \underset{\frac{20}{44}}{)[}\quad \text{0 shs} \quad \underset{\frac{21}{44}}{](}\quad \text{28 4-shs} \quad \underset{\frac{25}{44}}{)}$$

**Claim 1:** There are no shares $x \in [\frac{23}{44}, \frac{24}{44}]$.

If there was such a share then buddy is in $[\frac{20}{44}, \frac{21}{44}]$.

# More Refined Picture of What is Going On

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**Claim 1:** There are no shares $x \in [\frac{23}{44}, \frac{24}{44}]$.

If there was such a share then buddy is in $[\frac{20}{44}, \frac{21}{44}]$.
The following picture captures what we know so far.

$$\underset{\frac{19}{44}}{(}\quad \text{20 5-shs} \quad \underset{\frac{20}{44}}{)[}\quad 0 \quad \underset{\frac{21}{44}}{](}\quad \text{8 S4-shs} \quad \underset{\frac{23}{44}}{)[}\quad 0 \quad \underset{\frac{24}{44}}{](}\quad \text{20 L4-shs} \quad \underset{\frac{25}{44}}{)}$$

S4= Small 4-shares
L4= Large 4-shares. L4 shares, 5-share: **buddies**, so $|\text{L4}|=20$.

$$\underset{\frac{19}{44}}{(} \quad 20\ \text{5-shs} \quad \underset{\frac{20}{44}}{)[} \quad 0 \quad \underset{\frac{21}{44}}{](} \quad 8\ \text{S4-shs} \quad \underset{\frac{23}{44}}{)[} \quad 0 \quad \underset{\frac{24}{44}}{](} \quad 20\ \text{L4-shs} \quad \underset{\frac{25}{44}}{)}$$

**Claim 2:** Every 4-student has at least 3 L4 shares.

If a 4-student had $\le 2$ L4 shares then he has

$$< 2 \times \left(\frac{23}{44}\right) + 2 \times \left(\frac{25}{44}\right) = \frac{24}{11}.$$

$$\underset{\frac{19}{44}}{(}\quad 20\ \text{5-shs}\quad \underset{\frac{20}{44}}{)[}\quad 0\quad \underset{\frac{21}{44}}{](}\quad 8\ \text{S4-shs}\quad \underset{\frac{23}{44}}{)[}\quad 0\quad \underset{\frac{24}{44}}{](}\quad 20\ \text{L4-shs}\quad \underset{\frac{25}{44}}{)}$$

**Claim 2:** Every 4-student has at least 3 L4 shares.

If a 4-student had $\leq 2$ L4 shares then he has

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**Contradiction:** Each 4-student gets $\geq 3$ L4 shares. There are $s_4 = 7$ 4-students. Hence there are $\geq 21$ L4-shares. But there are only 20.

# INT Technique

INT is generalization of $f(24,11) \leq \frac{19}{44}$ proof.

**Definition:** Let $\mathrm{INT}(m,s)$ be the bound obtained.

1. INT proofs can get more complicated than this one.
2. $\mathrm{INT}(m,s)$ can be computed in $O(\frac{2m \log m}{s})$. Note: do not need to know the answer ahead of time.
3. For $1 \leq s \leq 60$, $s < m \leq 70$, $m,s$ rel prime:
   3.1 There are 1360 cases total.
   3.2 For 927 of the $(m,s)$, $f(m,s) = \mathrm{FC}(m,s)$. $\sim 68\%$
   3.3 For 268 of the $(m,s)$, $f(m,s) = \mathrm{INT}(m,s)$. $\sim 20\%$
   3.4 The cases not covered use **interesting** new techniques!

# Example of GAPS Technique: $f(31,19) \le \frac{54}{133}$

We show $f(31,19) \le \frac{54}{133}$.
Assume $(31,19)$-procedure with smallest piece $> \frac{54}{133}$.
By INT-technique methods obtain:
$s_3 = 14$, $s_4 = 5$.

$$\underset{\frac{54}{133}}{(}\quad \text{20 4-shs} \quad \underset{\frac{55}{133}}{)[}\quad 0 \quad \underset{\frac{59}{133}}{](}\quad \text{S3 shs} \quad \underset{\frac{74}{133}}{)[}\quad 0 \quad \underset{\frac{78}{133}}{](}\quad \text{20 L3-shs} \quad \underset{\frac{79}{133}}{)}$$

We just look at the 3-shares:

$$\underset{\frac{59}{133}}{(}\quad \text{S3 shs} \quad \underset{\frac{74}{133}}{)[}\quad 0 \quad \underset{\frac{78}{133}}{](}\quad \text{20 L3-shs} \quad \underset{\frac{79}{133}}{)}$$

# GAPS Technique: $f(31,19) \le \frac{54}{133}$

$$\underset{\frac{59}{133}}{(} \quad \text{S3 shs} \quad \underset{\frac{74}{133}}{)[} \quad 0 \quad \underset{\frac{78}{133}}{](} \quad \text{20 L3-shs} \quad \underset{\frac{79}{133}}{)}$$

1. $J_1 = (\frac{59}{133}, \frac{66.5}{133})$
2. $J_2 = (\frac{66.5}{133}, \frac{74}{133})$ $(|J_1| = |J_2|)$
3. $J_3 = (\frac{78}{133}, \frac{79}{133})$ $(|J_3| = 20)$

**Note:** Split the shares of size 66.5 between $J_1$ and $J_2$.

**Notation:** An $e(1,1,3)$ students is a student who has a $J_1$-share, a $J_1$-share, and a $J_3$-share.

Generalize to $e(i,j,k)$ easily.

# GAPS Technique: $f(31,19) \le \frac{54}{133}$

1. $J_1 = (\frac{59}{133}, \frac{66.5}{133})$
2. $J_2 = (\frac{66.5}{133}, \frac{74}{133})$ $(|J_1| = |J_2|)$
3. $J_3 = (\frac{78}{133}, \frac{79}{133})$ $(|J_3| = 20)$

1) Only students allowed: $e(1,2,3)$, $e(1,3,3)$, $e(2,2,2)$, $e(2,2,3)$.
All others have either $< \frac{31}{19}$ or $> \frac{31}{19}$.

2) No shares in $[\frac{61}{133}, \frac{64}{133}]$. Look at $J_1$-shares:
An $e(1,2,3)$-student has $J_1$-share $> \frac{31}{19} - \frac{74}{133} - \frac{79}{133} = \frac{64}{133}$.
An $e(1,3,3)$-student has $J_1$-share $< \frac{31}{19} - 2 \times \frac{78}{133} = \frac{61}{133}$.

3) No shares in $[\frac{69}{133}, \frac{72}{133}]$: $x \in [\frac{69}{133}, \frac{72}{133}] \implies 1 - x \in [\frac{61}{133}, \frac{64}{133}]$.

# GAPS Technique: $f(31, 19) \le \frac{54}{133}$

1. $J_1 = (\frac{59}{133}, \frac{61}{133})$
2. $J_2 = (\frac{64}{133}, \frac{66.5}{133})$
3. $J_3 = (\frac{66.5}{133}, \frac{69}{133})$ $(|J_2| = |J_3|)$
4. $J_4 = (\frac{72}{133}, \frac{74}{133})$ $(|J_1| = |J_4|)$
5. $J_5 = (\frac{78}{133}, \frac{79}{133})$ $(|J_5| = 20)$

The following are the only students who are allowed.
$e(1, 5, 5)$.
$e(2, 4, 5)$,
$e(3, 4, 5)$.
$e(4, 4, 4)$.

# GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

$e(1, 5, 5)$. Let the number of such students be $x$
$e(2, 4, 5)$. Let the number of such students be $y_1$
$e(3, 4, 5)$. Let the number of such students be $y_2$.
$e(4, 4, 4)$. Let the number of such students be $z$.
1) $|J_2| = |J_3|$,
only students using $J_2$ are $e(2, 4, 5)$ – they use one share each,
only students using $J_3$ are $e(3, 4, 5)$ – they use one share each.
Hence $y_1 = y_2$. We call them both $y$.

2) Since $|J_1| = |J_4|$, $x = 2y + 3z$.

3) Since $s_3 = 14$, $x + 2y + z = 14$.

$(2y + 3z) + 2y + z = 14 \implies 4(y + z) = 14 \implies y + z = \frac{7}{2}$.
Contradiction.

# GAPS Method

GAPS is generalization of $f(24,11) \le \frac{19}{44}$ proof.

1. GAPS proofs can get MUCH more complicated than this one.
2. GAPS needs to know the answer ahead of time. Can prob modify so that you do not.
3. GAPS fast in practice.
4. For $1 \le s \le 60$, $s < m \le 70$, $m, s$ rel prime:
   4.1 There are 1360 cases total.
   4.2 For 927 of the $(m,s)$, $f(m,s) = \mathrm{FC}(m,s)$. $\sim 68\%$
   4.3 For 268 of the $(m,s)$, $f(m,s) = \mathrm{INT}(m,s)$. $\sim 20\%$
   4.4 For 165 of the $(m,s)$, $f(m,s)$"$=$"$\mathrm{GAPS}(m,s)$. $\sim 12\%$
   4.5 FC, INT, GAPS took care of ALL cases.

$f(m,s)$"$=$"$\mathrm{GAPS}(m,s)$: GAPS only verifies upper bounds.
No such thing as $\mathrm{GAPS}(m,s)$, only $\mathrm{GAPS}(m,s,\alpha)$.

# Later Results by Other People

1. In Fall 2018 Scott Huddleston had code for an algorithm that, on input $m, s$, found $f(m, s)$ and the procedure REALLY FAST.
2. Jacob and Erik Understand WHAT his algorithm does and Jacob coded it up to make sure he understood it. Jacob's code is also REALLY FAST.
3. Neither Scott, Bill, Jacob, or Erik had a proof that Scott's algorithm was fast (poly in $m, s$).
4. Richard Chatwin independently came up with the same algorithm; however, he also has a proof that it works. Its on arixv.
5. One corollary of the work: $f(m, s)$ only depends on $m/s$.

# Accomplishment I Am Most Proud of

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Accomplishment I Am Most Proud of:

**Convinced**

- 4 High School students (Guang, Naveen, Naveen, Sunny)
- 3 college student (Erik, Jacob, Daniel)
- 1 professor (John D)

that the most important field of Mathematics is **Muffinry.**