The Muffin Problem

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How it Began

A Recreational Math Conference
(Gathering for Gardner)
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I found a pamphlet:

The Julia Robinson Mathematics Festival:
A Sample of Mathematical Puzzles
Compiled by Nancy Blachman

which had this problem, proposed by Alan Frank:

How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ where nobody gets a tiny sliver?
## 5 Muffins, 3 Students, Proc by Picture

<table>
<thead>
<tr>
<th>Person</th>
<th>Color</th>
<th>What they Get</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td>RED</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Bob</td>
<td>BLUE</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Carol</td>
<td>GREEN</td>
<td>$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$</td>
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</tbody>
</table>

**Smallest Piece:** $\frac{1}{3}$
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

Work on it with your neighbor.
YES WE CAN!

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Smallest Piece: $\frac{5}{12}$
Can We Do Better?

The smallest piece in the above solution is \( \frac{5}{12} \). 

*Is there a procedure with a larger smallest piece?*

*Work on it with your neighbor*
NO WE CAN’T!
There is a procedure for 5 muffins, 3 students where each student gets \(\frac{5}{3}\) muffins, smallest piece \(N\). We want \(N \leq \frac{5}{12}\).

**Case 0:** Some muffin is uncut. Cut it \((\frac{1}{2}, \frac{1}{2})\) and give both \(\frac{1}{2}\)-sized pieces to whoever got the uncut muffin. (Note \(\frac{1}{2} > \frac{5}{12}\).) Reduces to other cases. (**Henceforth:** All muffins cut into \(\geq 2\) pieces.)
NO WE CAN’T!
There is a procedure for 5 muffins, 3 students where each student gets \( \frac{5}{3} \) muffins, smallest piece \( N \). We want \( N \leq \frac{5}{12} \).

Case 0: Some muffin is uncut. Cut it \( \left( \frac{1}{2}, \frac{1}{2} \right) \) and give both \( \frac{1}{2} \)-sized pieces to whoever got the uncut muffin. (Note \( \frac{1}{2} > \frac{5}{12} \).) Reduces to other cases. (Henceforth: All muffins cut into \( \geq 2 \) pieces.)

Case 1: Some muffin is cut into \( \geq 3 \) pieces. Then \( N \leq \frac{1}{3} < \frac{5}{12} \). (Henceforth: All muffins cut into 2 pieces.)
5 Muffins, 3 People–Can’t Do Better Than $\frac{5}{12}$

NO WE CAN’T!
There is a procedure for 5 muffins, 3 students where each student gets $\frac{5}{3}$ muffins, smallest piece $N$. We want $N \leq \frac{5}{12}$.

**Case 0:** Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both $\frac{1}{2}$-sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases. *(Henceforth: All muffins cut into $\geq 2$ pieces.)*

**Case 1:** Some muffin is cut into $\geq 3$ pieces. Then $N \leq \frac{1}{3} < \frac{5}{12}$. *(Henceforth: All muffins cut into 2 pieces.)*

**Case 2:** All muffins are cut into 2 pieces. 10 pieces, 3 students: *Someone* gets $\geq 4$ pieces. He has some piece

$$\leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12}$$

Great to see $\frac{5}{12}$
What Else Was in the Pamphlet?

The pamphlet also had asked about

1. 4 muffins, 7 students.
2. 12 muffins, 11 students.
3. a few others
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This seemed like a nice exercise and it was.
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1. 4 muffins, 7 students.
2. 12 muffins, 11 students.
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This seemed like a nice exercise and it was.

There can’t be much more to this.
If there is not much more to this then how come

https://www.amazon.com/
Mathematical-Muffin-Morsels-Problem-Mathematics/dp/9811215170
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The following happened:
  ▶ Find a technique that solves many problems (e.g.,
    Floor-Ceiling).
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The following happened:

- Find a technique that solves many problems (e.g., Floor-Ceiling).
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- Find a new technique which was interesting.
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The following happened:

▶ Find a technique that solves many problems (e.g., Floor-Ceiling).
▶ Come across a problem where the techniques do not work.
▶ Find a new technique **which was interesting**.
▶ Lather, Rinse, Repeat.
$f(m, s)$ be the smallest piece in the best procedure (best in that the smallest piece is maximized) to divide $m$ muffins among $s$ students so that everyone gets $\frac{m}{s}$.

We have shown $f(5, 3) = \frac{5}{12}$ here.

We have shown $f(m, s)$ exists, is rational, and is computable using a Mixed Int Program (in paper).
Amazing Results! / Amazing Theorems!

1. $f(43, 33) = \frac{91}{264}$.
2. $f(52, 11) = \frac{83}{176}$.
3. $f(35, 13) = \frac{64}{143}$.

All done by hand, no use of a computer by Co-author Erik Metz is a muffin savant!

Have General Theorems from which upper bounds follow. Have General Procedures from which lower bounds follow.
7 Muffins, 3 Students

Work on $f(7, 3)$ in groups in breakout rooms.
7 Muffins, 3 Students.
Get upper and lower bounds that match!
We first look at LIMITS on what we can expect.
7 Muffins, 3 Students: How to think about it

We first look at LIMITS on what we can expect.

1. If a muffin is uncut, can cut it in two.

2. If a muffin is cut in $\geq 3$ pieces then some piece $\leq \frac{1}{3}$. Unlikely.

3. 7 muffins, each one cut in two 2 pieces, so 14 pieces total.

4. 3 students, so some student gets $\geq \lceil \frac{14}{3} \rceil = 5$ pieces.

5. That student must get a piece $\leq \frac{7}{3} \times \frac{1}{5} = \frac{7}{15}$.

6. Great! We know $f(7, 3) \leq \frac{7}{15}$.

7. Can we show a protocol that gives $f(7, 3) \geq \frac{7}{15}$.

8. We tried. We failed. Darn :-(

Now what?
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6. That piece came from a muffin. Other piece is $\leq 1 - \frac{7}{12} = \frac{5}{12}$.
7. Great! We know $f(7,3) \leq \frac{5}{12}$.
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Want \( f(7, 3) \geq \frac{5}{12} \).
7 Muffins, 3 Students: How to think about protocol

Want \( f(7, 3) \geq \frac{5}{12} \).
Will be cutting some muffins \( \left( \frac{5}{12}, \frac{7}{12} \right) \).
Want $f(7, 3) \geq \frac{5}{12}$.
Will be cutting some muffins ($\frac{5}{12}, \frac{7}{12}$).
Can also cut some muffins ($\frac{6}{12}, \frac{6}{12}$).
7 Muffins, 3 Students: How to think about protocol

Want $f(7, 3) \geq \frac{5}{12}$.
Will be cutting some muffins $(\frac{5}{12}, \frac{7}{12})$.
Can also cut some muffins $(\frac{6}{12}, \frac{6}{12})$.
Need to know what combos of $\frac{5}{12}, \frac{6}{12}, \frac{7}{12}$ add to $\frac{7}{3} = \frac{28}{12}$.

1. Cut 4 muffins $(\frac{5}{12}, \frac{7}{12})$.
2. Cut 3 muffins $(\frac{6}{12}, \frac{6}{12})$.
3. Give 1 student 4 pieces of size $\frac{7}{12}$.
4. Give 2 students 2 pieces of size $\frac{5}{12}$ and 3 pieces of size $\frac{6}{12}$. 

Want $f(7, 3) \geq \frac{5}{12}$.

Will be cutting some muffins ($\frac{5}{12}, \frac{7}{12}$).

Can also cut some muffins ($\frac{6}{12}, \frac{6}{12}$).

Need to know what combos of $\frac{5}{12}, \frac{6}{12}, \frac{7}{12}$ add to $\frac{7}{3} = \frac{28}{12}$.

Need to know what combos of 5, 6, 7 add to 28.

$7 + 7 + 7 + 7 = 28$

$5 + 5 + 6 + 6 + 6 = 28$
Want \( f(7, 3) \geq \frac{5}{12} \).

Will be cutting some muffins \( \left( \frac{5}{12}, \frac{7}{12} \right) \).

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7 Muffins, 3 Students: How to think about protocol

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Will be cutting some muffins $(\frac{5}{12}, \frac{7}{12})$.

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1. Cut 4 muffins $(\frac{5}{12}, \frac{7}{12})$.
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8 Muffins, 3 Students: How to think about it

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1. If a muffin is uncut, can cut it in two.
2. If a muffin is cut in $\geq 3$ pieces then some piece $\leq \frac{1}{3}$. Unlikely that thats a good idea.
3. 8 muffins, each one cut in two 2 pieces, so 16 pieces total.
8 Muffins, 3 Students: How to think about it

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2. If a muffin is cut in $\geq 3$ pieces then some piece $\leq \frac{1}{3}$. Unlikely that that’s a good idea.
3. 8 muffins, each one cut in two 2 pieces, so 16 pieces total.
4. 3 students, so some student gets $\geq \lceil \frac{16}{3} \rceil = 6$ pieces. That student must get a piece $\leq \frac{8}{3} \times \frac{1}{6} = \frac{4}{9}$.
5. 3 students, so some student gets $\leq \lfloor \frac{16}{3} \rfloor = 5$ pieces. That student must get a piece $\geq \frac{8}{3} \times \frac{1}{5} = \frac{4}{9}$. So there is some piece of size $\leq 1 - \frac{4}{9} = \frac{7}{9}$.
6. Great! We know $f(8, 3) \leq \min \{ \frac{4}{9}, \frac{7}{15} \} = \frac{4}{9}$.
7. Can we show a protocol that gives $f(8, 3) \geq \frac{4}{9}$?
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5. 3 students, so some student gets $\leq \lfloor \frac{16}{3} \rfloor = 5$ pieces. That student must get a piece $\geq \frac{8}{3} \times \frac{1}{5} = \frac{8}{15}$. So there is some piece of size $\leq 1 - \frac{8}{15} = \frac{7}{15}$.

6. Great! We know $f(8, 3) \leq \min\{\frac{4}{9}, \frac{7}{15}\} = \frac{4}{9}$.
We first look at LIMITS on what we can expect.

1. If a muffin is uncut, can cut it in two.

2. If a muffin is cut in $\geq 3$ pieces then some piece $\leq \frac{1}{3}$. Unlikely that thats a good idea.

3. 8 muffins, each one cut in two 2 pieces, so 16 pieces total.

4. 3 students, so some student gets $\geq \left\lceil \frac{16}{3} \right\rceil = 6$ pieces. That student must get a piece $\leq \frac{8}{3} \times \frac{1}{6} = \frac{4}{9}$.

5. 3 students, so some student gets $\leq \left\lfloor \frac{16}{3} \right\rfloor = 5$ pieces. That student must get a piece $\geq \frac{8}{3} \times \frac{1}{5} = \frac{8}{15}$. So there is some piece of size $\leq 1 - \frac{8}{15} = \frac{7}{15}$.

6. Great! We know $f(8, 3) \leq \min\{\frac{4}{9}, \frac{7}{15}\} = \frac{4}{9}$.

7. Can we show a protocol that gives $f(8, 3) \geq \frac{4}{9}$?
8 Muffins, 3 Students: How to think about protocol

Want \( f(8, 3) \geq \frac{4}{9} \).
8 Muffins, 3 Students: How to think about protocol

Want $f(8, 3) \geq \frac{4}{9}$.
Will be cutting some muffins $(\frac{4}{9}, \frac{5}{9})$. 
Want \( f(8, 3) \geq \frac{4}{9} \).
Will be cutting some muffins \((\frac{4}{9}, \frac{5}{9})\).
\(\frac{1}{2}\) was helpful last time so let's also include \(\frac{4.5}{9}\).
Need to know what combos of \(\frac{4}{9}, \frac{4.5}{9}, \frac{5}{9}\) add to \(\frac{8}{3} = \frac{24}{9}\).
Want \( f(8, 3) \geq \frac{4}{9} \).
Will be cutting some muffins \((\frac{4}{9}, \frac{5}{9})\).
\( \frac{1}{2} \) was helpful last time so lets also include \( \frac{4.5}{9} \).
Need to know what combos of \( \frac{4}{9}, \frac{4.5}{9}, \frac{5}{9} \) add to \( \frac{8}{3} = \frac{24}{9} \).
Need to know what combos of 4, 4.5, 5 add to 24
\[
4 + 4 + 4 + 4 + 4 + 4 = 24
\]
\[
4.5 + 4.5 + 5 + 5 + 5 = 24
\]
8 Muffins, 3 Students: How to think about protocol

Want $f(8,3) \geq \frac{4}{9}$.
Will be cutting some muffins $(\frac{4}{9}, \frac{5}{9})$. 
$\frac{1}{2}$ was helpful last time so let's also include $\frac{4.5}{9}$.
Need to know what combos of $\frac{4}{9}, \frac{4.5}{9}, \frac{5}{9}$ add to $\frac{8}{3} = \frac{24}{9}$.
Need to know what combos of 4, 4.5, 5 add to 24
$4 + 4 + 4 + 4 + 4 + 4 = 24$
$4.5 + 4.5 + 5 + 5 + 5 = 24$

1. Cut 6 muffins $(\frac{4}{9}, \frac{5}{9})$.
2. Cut 2 muffins $(\frac{4.5}{9}, \frac{4.5}{9})$.
3. Give 1 student six $\frac{4}{9}$ pieces.
4. Give 2 students two $\frac{4.5}{9}$ pieces and four $\frac{5}{9}$ pieces.
$f(3, 5) \geq \frac{1}{5}$.

Can we get $f(3, 5) > \frac{1}{5}$?

Work on it with your neighbor.
\[ f(3, 5) \geq \frac{1}{4} \]

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
2. Divide 1 muffin \([\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]\)
3. Give 4 students \((\frac{5}{20}, \frac{7}{20})\)
4. Give 1 students \((\frac{6}{20}, \frac{6}{20})\)
$$f(3, 5) \geq \frac{1}{4}$$

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
2. Divide 1 muffin \([\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]\)
3. Give 4 students \((\frac{5}{20}, \frac{7}{20})\)
4. Give 1 students \((\frac{6}{20}, \frac{6}{20})\)

Can we do better?
Work on it with your neighbor
NO WE CAN’T!

There is a procedure for 3 muffins, 5 students where each student gets $\frac{3}{5}$ muffins, smallest piece $N$. We want $N \leq \frac{1}{4}$. 
3 Muffins, 5 People–Can’t Do Better Than $\frac{1}{4}$

NO WE CAN’T!
There is a procedure for 3 muffins, 5 students where each student gets $\frac{3}{5}$ muffins, smallest piece $N$. We want $N \leq \frac{1}{4}$.

**Case 0:** Alice gets 1 piece of size $\frac{3}{5}$. Look at the rest of that muffin which totals to $\frac{2}{5}$. (1) That piece is cut. Have piece $\leq \frac{2}{5} \times \frac{1}{2} = \frac{1}{5}$, OR (2) That piece uncut. So someone gets a $\frac{2}{5}$-piece. Must also get a $\frac{1}{5}$ piece.

(Henceforth: All people get $\geq 2$ pieces.)
NO WE CAN’T!
There is a procedure for 3 muffins, 5 students where each student gets $\frac{3}{5}$ muffins, smallest piece $N$. We want $N \leq \frac{1}{4}$.

**Case 0:** Alice gets 1 piece of size $\frac{3}{5}$. Look at the rest of that muffin which totals to $\frac{2}{5}$. (1) That piece is cut. Have piece $\leq \frac{2}{5} \times \frac{1}{2} = \frac{1}{5}$, OR (2) That piece uncut. So someone gets a $\frac{2}{5}$-piece. Must also get a $\frac{1}{5}$ piece.
(Henceforth: All people get $\geq 2$ pieces.)

**Case 1:** Alice gets $\geq 3$ pieces. Then $N \leq \frac{3}{5} \times \frac{1}{3} = \frac{1}{5}$.
(Henceforth: Everyone gets 2 pieces.)
NO WE CAN’T!
There is a procedure for 3 muffins, 5 students where each student gets \( \frac{3}{5} \) muffins, smallest piece \( N \). We want \( N \leq \frac{1}{4} \).

**Case 0:** Alice gets 1 piece of size \( \frac{3}{5} \). Look at the rest of that muffin which totals to \( \frac{2}{5} \). (1) That piece is cut. Have piece \( \leq \frac{2}{5} \times \frac{1}{2} = \frac{1}{5} \), OR (2) That piece uncut. So someone gets a \( \frac{2}{5} \)-piece. Must also get a \( \frac{1}{5} \) piece.

(Henceforth: All people get \( \geq 2 \) pieces.)

**Case 1:** Alice gets \( \geq 3 \) pieces. Then \( N \leq \frac{3}{5} \times \frac{1}{3} = \frac{1}{5} \).

(Henceforth: Everyone gets 2 pieces.)

**Case 2:** Everyone gets 2 pieces. 10 pieces, 3 muffins: Some muffin gets \( \geq 4 \) pieces. So some piece is \( \leq \frac{1}{4} \).
$f(3, 5)$ and $f(5, 3)$

$f(5, 3) \geq \frac{5}{12}$

1. Divide 4 muffins $\left[\frac{5}{12}, \frac{7}{12}\right]$
2. Divide 1 muffin $\left[\frac{6}{12}, \frac{6}{12}\right]$
3. Give 2 students $\left(\frac{6}{12}, \frac{7}{12}, \frac{7}{12}\right)$
4. Give 1 student $\left(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12}\right)$
$f(3, 5)$ and $f(5, 3)$

$f(5, 3) \geq \frac{5}{12}$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$

2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$

3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$

4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

$f(3, 5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$

Duality Theorem: $f(m, s) = m \times s f(s, m)$. 
$f(3, 5)$ and $f(5, 3)$

$f(5, 3) \geq \frac{5}{12}$

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
3. Give 2 students \((\frac{6}{12}, \frac{7}{12}, \frac{7}{12})\)
4. Give 1 students \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)

$f(3, 5) \geq \frac{1}{4}$

1. Divide 2 muffin \([\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]\)
2. Divide 1 muffin \([\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]\)
3. Give 4 students \((\frac{5}{20}, \frac{7}{20})\)
4. Give 1 students \((\frac{6}{20}, \frac{6}{20})\)
$f(3, 5)$ and $f(5, 3)$

$f(5, 3) \geq \frac{5}{12}$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 student $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

$f(3, 5) \geq \frac{1}{4}$

1. Divide 2 muffins $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 student $(\frac{6}{20}, \frac{6}{20})$

$f(3, 5)$ proc is $f(5, 3)$ proc but swap Divide/Give and mult by $3/5$. 

Duality Theorem: $f(m, s) = m \times \frac{s}{f(s, m)}$. 
$f(3, 5)$ and $f(5, 3)$

$f(5, 3) \geq \frac{5}{12}$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

$f(3, 5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

$f(3, 5)$ proc is $f(5, 3)$ proc but swap Divide/Give and mult by 3/5.

**Duality Theorem:** $f(m, s) = \frac{m}{s} f(s, m)$. 
Conventions

We know and use the following:

1. By duality can assume $m > s$
2. If $s$ divides $m$ then $f(m, s) = 1$ so assume $s$ does not divide $m$.
3. All muffins are cut in $\geq 2$ pcs. Replace uncut muff with $2 \frac{1}{2}$’s
4. If assuming $f(m, s) > \alpha > \frac{1}{3}$, assume all muffin in $\leq 2$ pcs.
5. $f(m, s) > \alpha > \frac{1}{3}$, so exactly 2 pcs, is common case.

We do not know this but still use: $f(m, s)$ only depends on $\frac{m}{s}$.
All of our techniques that hold for $(m, s)$ hold for $(Am, As)$.
For particular numbers, we only look at $m, s$ rel prime.
FC Thm Generalizes $f(5, 3) \leq \frac{5}{12}$

$$f(m, s) \leq \text{FC}(m, s) = \max\left\{ \frac{1}{3}, \min\left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lceil 2m/s \rceil} \right\} \right\}.$$  

**Case 0:** Some muffin is uncut. Cut it ($\frac{1}{2}, \frac{1}{2}$) and give both halves to whoever got the uncut muffin, so reduces to other cases.

**Case 1:** Some muffin is cut into $\geq 3$ pieces. Some piece $\leq \frac{1}{3}$.

**Case 2:** Every muffin is cut into 2 pieces, so $2m$ pieces.

Someone gets $\geq \lceil \frac{2m}{s} \rceil$ pieces. $\exists$ piece $\leq \frac{m}{s} \times \frac{1}{\lceil 2m/s \rceil} = \frac{m}{s \lceil 2m/s \rceil}$.

Someone gets $\leq \lfloor \frac{2m}{s} \rfloor$ pieces. $\exists$ piece $\geq \frac{m}{s} \lfloor \frac{1}{2m/s} \rfloor = \frac{m}{s \lfloor 2m/s \rfloor}$.

The other piece from that muffin is of size $\leq 1 - \frac{m}{s \lceil 2m/s \rceil}$.
THREE Students

CLEVERNESS, COMP PROGS for the procedure.

FC Theorem for optimality.

\[ f(1, 3) = \frac{1}{3} \]

\[ f(3k, 3) = 1. \]

\[ f(3k + 1, 3) = \frac{3k-1}{6k}, \quad k \geq 1. \]

\[ f(3k + 2, 3) = \frac{3k+2}{6k+6}. \]

Note: A Mod 3 Pattern.

Theorem: For all \( m \geq 3 \), \( f(m, 3) = \text{FC}(m, 3). \)
FOUR Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

\[ f(4k, 4) = 1 \text{ (easy)} \]

\[ f(1, 4) = \frac{1}{4} \text{ (easy)} \]

\[ f(4k + 1, 4) = \frac{4k-1}{8k}, \quad k \geq 1. \]

\[ f(4k + 2, 4) = \frac{1}{2}. \]

\[ f(4k + 3, 4) = \frac{4k+1}{8k+4}. \]

Note: A Mod 4 Pattern.

Theorem: For all \( m \geq 4 \), \( f(m, 4) = FC(m, 4) \).

FC-Conjecture: For all \( m, s \) with \( m \geq s \), \( f(m, s) = FC(m, s) \).
FIVE Students

**CLEVERNESS, COMP PROGS** for procedures.

**FC Theorem** for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k + 1, 5) = \frac{5k+1}{10k+5}$. $f(11, 5)$?

For $k \geq 2$, $f(5k + 2, 5) = \frac{5k-2}{10k}$. $f(7, 5) = FC(7, 5) = \frac{1}{3}$

For $k \geq 1$, $f(5k + 3, 5) = \frac{5k+3}{10k+10}$

For $k \geq 1$, $f(5k + 4, 5) = \frac{5k+1}{10k+5}$

**Note:** A Mod 5 Pattern.

**Theorem:** For all $m \geq 5$ except $m=11$, $f(m, 5) = FC(m, 5)$. 
What About FIVE students, ELEVEN muffins?

\[ f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{5 \lceil \frac{22}{5} \rceil}, 1 - \frac{11}{5 \lfloor \frac{22}{5} \rfloor} \right\} \right\} = \frac{11}{25}. \]
What About FIVE students, ELEVEN muffins?

\[ f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{5 \lceil 22/5 \rceil}, 1 - \frac{11}{5 \lfloor 22/5 \rfloor} \right\} \right\} = \frac{11}{25}. \]

We tried to find a protocol to divide 11 muffins for 5 people, each gets \( \frac{11}{5} \), and smallest piece is size \( \frac{11}{25} = 0.44 \).
What About FIVE students, ELEVEN muffins?

\[ f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{5 \lceil 22/5 \rceil}, 1 - \frac{11}{5 \lfloor 22/5 \rfloor} \right\} \right\} = \frac{11}{25}. \]

We tried to find a protocol to divide 11 muffins for 5 people, each gets \( \frac{11}{5} \), and smallest piece is size \( \frac{11}{25} = 0.44 \).

We found a protocol with smallest piece \( \frac{13}{30} = 0.4333 \ldots \):

1. Divide 1 muffin \( \left( \frac{15}{30}, \frac{15}{30} \right) \).
2. Divide 2 muffins \( \left( \frac{14}{30}, \frac{16}{30} \right) \).
3. Divide 8 muffins \( \left( \frac{13}{30}, \frac{17}{30} \right) \).
4. Give 2 students \( \left[ \frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{14}{30} \right] \).
5. Give 1 students \( \left[ \frac{16}{30}, \frac{16}{30}, \frac{17}{30}, \frac{17}{30} \right] \).
6. Give 2 students \( \left[ \frac{15}{30}, \frac{17}{30}, \frac{17}{30}, \frac{17}{30} \right] \).
So Now What?

We have:

\[
\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666 \ldots
\]
So Now What?

We have:

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff}= 0.006666\ldots$$

Options:

1. $f(11, 5) = \frac{11}{25}$. Need to find procedure.
2. $f(11, 5) = \frac{13}{30}$. Need to find new technique for upper bounds.
3. $f(11, 5)$ in between. Need to find both.
4. $f(11, 5)$ unknown to science!

Vote
So Now What?

We have:

\[ \frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666 \ldots \]

Options:

1. \( f(11, 5) = \frac{11}{25} \). Need to find procedure.
2. \( f(11, 5) = \frac{13}{30} \). Need to find new technique for upper bounds.
3. \( f(11, 5) \) in between. Need to find both.
4. \( f(11, 5) \) unknown to science!

**Vote** WE SHOW: \( f(11, 5) = \frac{13}{30} \). **Exciting** new technique!
Terminology: Buddy

Assume that in some protocol every muffin is cut into two pieces.

Let $x$ be a piece from muffin $M$. The other piece from muffin $M$ is the buddy of $x$.

Note that the buddy of $x$ is of size

$$1 - x.$$
There is a procedure for 11 muffins, 5 students where each student gets \( \frac{11}{5} \) muffins, smallest piece \( N \). We want \( N \leq \frac{13}{30} \).

**Case 0:** Some muffin is uncut. Cut it \( (\frac{1}{2}, \frac{1}{2}) \) and give both halves to whoever got the uncut muffin. Reduces to other cases.
There is a procedure for 11 muffins, 5 students where each student gets \( \frac{11}{5} \) muffins, smallest piece \( N \). We want \( N \leq \frac{13}{30} \).

**Case 0:** Some muffin is uncut. Cut it \( (\frac{1}{2}, \frac{1}{2}) \) and give both halves to whoever got the uncut muffin. Reduces to other cases.

**Case 1:** Some muffin is cut into \( \geq 3 \) pieces. \( N \leq \frac{1}{3} < \frac{13}{30} \).

**Negation of Case 0 and Case 1:** All muffins cut into 2 pieces.
$f(11, 5) = \frac{13}{30}$, Easy Case Based on Students

**Case 2:** Some student gets $\geq 6$ pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$
$f(11, 5) = \frac{13}{30}$, Easy Case Based on Students

**Case 2:** Some student gets $\geq 6$ pieces.

\[ N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}. \]

**Case 3:** Some student gets $\leq 3$ pieces.

One of the pieces is

\[ \geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}. \]

Look at the muffin it came from to find a piece that is

\[ \leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}. \]

(Beatification of Cases 2 and 3: Every student gets 4 or 5 pieces.)
Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note \( \leq 11 \) pieces are \( > \frac{1}{2} \).

- \( s_4 \) is number of students who get 4 pieces
- \( s_5 \) is number of students who get 5 pieces

\[
4s_4 + 5s_5 = 22
\]
\[
s_4 + s_5 = 5
\]

\( s_4 = 3 \): There are 3 students who have 4 shares.
\( s_5 = 2 \): There are 2 students who have 5 shares.

We call a share that goes to a person who gets 4 shares a 4-share.
We call a share that goes to a person who gets 5 shares a 5-share.
Case 4.1: Some 4-share is \( \leq \frac{1}{2} \).
Alice gets \( w, x, y, z \) and \( w \leq \frac{1}{2} \).
Since \( w + x + y + z = \frac{11}{5} \) and \( w \leq \frac{1}{2} \)

\[
x + y + z \geq \frac{11}{5} - \frac{1}{2} = \frac{17}{10}
\]

Let \( x \) be the largest of \( x, y, z \)

\[
x \geq \frac{17}{10} \times \frac{1}{3} = \frac{17}{30}
\]

Look at **buddy** of \( x \).

\[
B(x) \leq 1 - x = 1 - \frac{17}{30} = \frac{13}{30}
\]

GREAT! This is where \( \frac{13}{30} \) comes from!
\[ f(11, 5) = \frac{13}{30}, \text{ Fun Cases} \]

**Case 4.2:** All 4-shares are \( > \frac{1}{2} \). There are \( 4s_4 = 12 \) 4-shares. There are \( \geq 12 \) pieces \( > \frac{1}{2} \). Can’t occur.
Proof that \( f(11, 5) \leq \frac{13}{30} \) was an example of the INT method. We give a more sophisticated example
Assume (24, 11)-procedure with smallest piece $> \frac{19}{44}$. Can assume all muffin cut in two and all student gets $\geq 2$ shares. We show that there is a piece $\leq \frac{19}{44}$. 

Case 1: A student gets $\geq 6$ shares. Some piece $\leq 24 \times 6 < \frac{19}{44}$.

Case 2: A student gets $\leq 3$ shares. Some piece $\geq 24 \times 3 = \frac{8}{11}$. Buddy of that piece $\leq 1 - \frac{8}{11} \leq \frac{3}{11} < \frac{19}{44}$.

Case 3: Every muffin is cut in 2 pieces and every student gets either 4 or 5 shares. Total number of shares is 48.
Assume $(24, 11)$-procedure with smallest piece $> \frac{19}{44}$. Can assume all muffin cut in two and all student gets $\geq 2$ shares. We show that there is a piece $\leq \frac{19}{44}$.

**Case 1:** A student gets $\geq 6$ shares. Some piece $\leq \frac{24}{11 \times 6} < \frac{19}{44}$. 
More Sophisticated INT: $f(24, 11) \leq \frac{19}{44}$

Assume $(24, 11)$-procedure with smallest piece $> \frac{19}{44}$. Can assume all muffin cut in two and all student gets $\geq 2$ shares. We show that there is a piece $\leq \frac{19}{44}$.

**Case 1:** A student gets $\geq 6$ shares. Some piece $\leq \frac{24}{11 \times 6} < \frac{19}{44}$.

**Case 2:** A student gets $\leq 3$ shares. Some piece $\geq \frac{24}{11 \times 3} = \frac{8}{11}$. Buddy of that piece $\leq 1 - \frac{8}{11} \leq \frac{3}{11} < \frac{19}{44}$. 
Assume \((24, 11)\)-procedure with smallest piece \(> \frac{19}{44}\).
Can assume all muffin cut in two and all student gets \(\geq 2\) shares.
We show that there is a piece \(\leq \frac{19}{44}\).

**Case 1:** A student gets \(\geq 6\) shares. Some piece \(\leq \frac{24}{11 \times 6} < \frac{19}{44}\).

**Case 2:** A student gets \(\leq 3\) shares. Some piece \(\geq \frac{24}{11 \times 3} = \frac{8}{11}\).
Buddy of that piece \(\leq 1 - \frac{8}{11} \leq \frac{3}{11} < \frac{19}{44}\).

**Case 3:** Every muffin is cut in 2 pieces and every student gets either 4 or 5 shares. Total number of shares is 48.
How many students get 4? 5? Where are Shares?

4-students: a student who gets 4 shares. $s_4$ is the number of them.
5-students: a student who gets 5 shares. $s_5$ is the number of them.

4-share: a share that a 4-student who gets.
5-share: a share that a 5-student who gets.

\[
4s_4 + 5s_5 = 48 \\
\]
\[
s_4 + s_5 = 11 \\
\]
\[
s_4 = 7. \text{ Hence there are } 4s_4 = 4 \times 7 = 28 \text{ 4-shares.} \\
s_5 = 4. \text{ Hence there are } 5s_5 = 5 \times 4 = 20 \text{ 5-shares.} \\
\]
Case 3.1 and 3.2: Too Big or Too Small

Case 3.1: There is a share \( \geq \frac{25}{44} \). Then its buddy is \( \leq \frac{1 - 25}{44} = \frac{19}{44} \).

Case 3.2: There is a share \( \leq \frac{19}{44} \). Duh.

Henceforth assume that all shares are in \( \left( \frac{19}{44}, \frac{25}{44} \right) \).
Case 3.1 and 3.2: Too Big or Too Small

**Case 3.1:** There is a share $\geq \frac{25}{44}$. Then its buddy is

$$\leq 1 - \frac{25}{44} = \frac{19}{44}$$
Case 3.1 and 3.2: Too Big or Too Small

**Case 3.1:** There is a share $\geq \frac{25}{44}$. Then its buddy is
\[
\leq 1 - \frac{25}{44} = \frac{19}{44}
\]

**Case 3.2:** There is a share $\leq \frac{19}{44}$. Duh.
Henceforth assume that all shares are in
\[
\left( \frac{19}{44}, \frac{25}{44} \right)
\]
Case 3.3: Some 5-shares ≥ $\frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume that Alice 5 pieces $A, B, C, D, E$ and $E \geq \frac{20}{44}$.

Since $A + B + C + D + E = \frac{24}{11}$ and $E \geq \frac{20}{44}$

\[
A + B + C + D \leq \frac{24}{11} - \frac{20}{44} = \frac{76}{44}
\]

Assume $A$ is the smallest of $A, B, C, D$.

\[
A \leq \frac{76}{44} \times \frac{1}{4} = \frac{19}{44}
\]

Henceforth we assume all 5-shares are in $\left( \frac{19}{44}, \frac{20}{44} \right)$. 

\[
\left( \frac{19}{44}, \frac{20}{44} \right) \left[ \begin{array}{c} 19 \\ \frac{25}{44} \end{array} \right]
\]
Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume that Alice 5 pieces $A, B, C, D, E$ and $E \geq \frac{20}{44}$.
Since $A + B + C + D + E = \frac{24}{11}$ and $E \geq \frac{20}{44}$

$$A + B + C + D \leq \frac{24}{11} - \frac{20}{44} = \frac{76}{44}$$

Assume $A$ is the smallest of $A, B, C, D$.

$$A \leq \frac{76}{44} \times \frac{1}{4} = \frac{19}{44}$$

Henceforth we assume all 5-shares are in $\left(\frac{19}{44}, \frac{20}{44}\right)$.

( ?? 5-shs ) $\left[\begin{array}{c}
\frac{19}{44} \\
\frac{20}{44} \\
\frac{25}{44}
\end{array}\right]$
Case 3.4: Some 4-shares $\leq \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\leq \frac{21}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume that Alice 4 pieces $A, B, C, D$ and $D \leq \frac{21}{44}$.

Since $A + B + C + D = \frac{24}{11}$ and $D \leq \frac{21}{44}$

$$A + B + C \geq \frac{24}{11} - \frac{21}{44} = \frac{75}{44}$$

Assume $A$ is the largest of $A, B, C$.

$$A \geq \frac{75}{44} \times \frac{1}{3} = \frac{25}{44}$$

The buddy of $A$ is of size

$$\leq 1 - \frac{25}{44} = \frac{19}{44}$$

Henceforth we assume all 4-shares are in

$$\left( \frac{21}{44}, \frac{25}{44} \right).$$
Case 3.5:  All Shares in Their Proper Intervals

**Case 3.5:** 4-shares in \((\frac{21}{44}, \frac{25}{44})\), 5-shares in \((\frac{19}{44}, \frac{20}{44})\).

\[
\begin{array}{cccc}
\text{19} & \text{?? 5-shs} & \text{0 shs} & \text{21} \\
\text{20} & \text{?? 4-shs} & \text{25} \\
\end{array}
\]
Case 3.5: All Shares in Their Proper Intervals

**Case 3.5:** 4-shares in \((\frac{21}{44}, \frac{25}{44})\), 5-shares in \((\frac{19}{44}, \frac{20}{44})\).

\[
\begin{array}{ccc}
( & ?? & 5\text{-shs}) & 0 \text{ shs} & ( & ?? & 4\text{-shs}) \\
\frac{19}{44} & & \frac{20}{44} & & \frac{21}{44} & & \frac{25}{44}
\end{array}
\]

**Recall:** there are \(4s_4 = 4 \times 7 = 28\) 4-shares.

**Recall:** there are \(5s_5 = 5 \times 4 = 20\) 5-shares.
Case 3.5: All Shares in Their Proper Intervals

Case 3.5: 4-shares in \((\frac{21}{44}, \frac{25}{44})\), 5-shares in \((\frac{19}{44}, \frac{20}{44})\).

\[
\begin{pmatrix}
?? & 5\text{-shs} \\
\frac{19}{44} & \frac{20}{44} & \frac{21}{44} & \frac{25}{44}
\end{pmatrix}
\begin{pmatrix}
0 \text{ shs} \\
\end{pmatrix}
\begin{pmatrix}
?? & 4\text{-shs} \\
\end{pmatrix}
\]

Recall: there are \(4s_4 = 4 \times 7 = 28\) 4-shares.

Recall: there are \(5s_5 = 5 \times 4 = 20\) 5-shares.

\[
\begin{pmatrix}
20 & 5\text{-shs} \\
\frac{19}{44} & \frac{20}{44} & \frac{21}{44} & \frac{25}{44}
\end{pmatrix}
\begin{pmatrix}
0 \text{ shs} \\
\end{pmatrix}
\begin{pmatrix}
28 & 4\text{-shs} \\
\end{pmatrix}
\]
More Refined Picture of What is Going On

\[
\begin{bmatrix}
\frac{19}{44} & 0 & \frac{20}{44} & \frac{21}{44} & \frac{25}{44}
\end{bmatrix}
\]

**Claim 1:** There are no shares \( x \in \left[ \frac{23}{44}, \frac{24}{44} \right] \).

If there was such a share then buddy is in \( \left[ \frac{20}{44}, \frac{21}{44} \right] \).
Claim 1: There are no shares \( x \in \left[ \frac{23}{44}, \frac{24}{44} \right] \).

If there was such a share then buddy is in \( \left[ \frac{20}{44}, \frac{21}{44} \right] \).

The following picture captures what we know so far.

\[
\begin{pmatrix}
\begin{array}{c}
20 	ext{ 5-shs} \\
19 \\
\end{array}
\end{pmatrix}
\begin{pmatrix}
0 \\
20 \\
\end{pmatrix}
\begin{pmatrix}
8 	ext{ S4-shs} \\
23 \\
\end{pmatrix}
\begin{pmatrix}
0 \\
24 \\
\end{pmatrix}
\begin{pmatrix}
20 	ext{ L4-shs} \\
25 \\
\end{pmatrix}
\]

S4 = Small 4-shares
L4 = Large 4-shares. L4 shares, 5-share: buddies, so \( |L4| = 20 \).
Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had $\leq 2$ L4 shares then he has

$$< 2 \times \left(\frac{23}{44}\right) + 2 \times \left(\frac{25}{44}\right) = \frac{24}{11}.$$
Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had $\leq 2$ L4 shares then he has

$$< 2 \times \left(\frac{23}{44}\right) + 2 \times \left(\frac{25}{44}\right) = \frac{24}{11}.$$

Contradiction: Each 4-student gets $\geq 3$ L4 shares. There are $s_4 = 7$ 4-students. Hence there are $\geq 21$ L4-shares. But there are only 20.
INT Technique

INT is generalization of $f(24, 11) \leq \frac{19}{44}$ proof.

**Definition:** Let $\text{INT}(m, s)$ be the bound obtained.

1. INT proofs can get more complicated than this one.
2. $\text{INT}(m, s)$ can be computed in $O\left(\frac{2m\log m}{s}\right)$. Note: do not need to know the answer ahead of time.
3. For $1 \leq s \leq 60$, $s < m \leq 70$, $m, s$ rel prime:
   3.1 There are 1360 cases total.
   3.2 For 927 of the $(m, s)$, $f(m, s) = \text{FC}(m, s)$. $\sim 68\%$
   3.3 For 268 of the $(m, s)$, $f(m, s) = \text{INT}(m, s)$. $\sim 20\%$
   3.4 The cases not covered use **interesting** new techniques!
Example of GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

We show $f(31, 19) \leq \frac{54}{133}$. Assume $(31, 19)$-procedure with smallest piece $\geq \frac{54}{133}$. By INT-technique methods obtain:

$s_3 = 14$, $s_4 = 5$.

$$
\begin{pmatrix}
\frac{54}{133} & 20 \text{ 4-shs} & 0 & \frac{22}{133} & 22 \text{ S3 shs} & 0 & \frac{20}{133} & 20 \text{ L3-shs} & 0
\end{pmatrix}
$$

We just look at the 3-shares:

$$
\begin{pmatrix}
\frac{59}{133} & 22 \text{ S3 shs} & 0 & \frac{74}{133} & 20 \text{ L3-shs} & 0 & \frac{78}{133} & 79 & \frac{133}{133}
\end{pmatrix}
$$
GAPS Technique: \( f(31, 19) \leq \frac{54}{133} \)

\[
\begin{pmatrix}
\frac{59}{133} & 22 \text{ S3 shs} & 0 \\
\frac{74}{133} & \frac{78}{133} & \frac{79}{133} & 20 \text{ L3-shs}
\end{pmatrix}
\]

1. \( J_1 = \left( \frac{59}{133}, \frac{66.5}{133} \right) \)
2. \( J_2 = \left( \frac{66.5}{133}, \frac{74}{133} \right) \) (\( |J_1| = |J_2| \))
3. \( J_3 = \left( \frac{78}{133}, \frac{79}{133} \right) \) (\( |J_3| = 20 \))  

**Note:** Split the shares of size 66.5 between \( J_1 \) and \( J_2 \).

**Notation:** An \( e(1, 1, 3) \) students is a student who has a \( J_1 \)-share, a \( J_1 \)-share, and a \( J_3 \)-share.  
Generalize to \( e(i, j, k) \) easily.
GAPS Technique: \( f(31, 19) \leq \frac{54}{133} \)

1. \( J_1 = (\frac{59}{133}, \frac{66.5}{133}) \)
2. \( J_2 = (\frac{66.5}{133}, \frac{74}{133}) \) (\(|J_1| = |J_2|\))
3. \( J_3 = (\frac{78}{133}, \frac{79}{133}) \) (\(|J_3| = 20\))

1) Only students allowed: \( e(1, 2, 3), e(1, 3, 3), e(2, 2, 2), e(2, 2, 3) \). All others have either \( < \frac{31}{19} \) or \( > \frac{31}{19} \).

2) No shares in \([\frac{61}{133}, \frac{64}{133}]\). Look at \( J_1 \)-shares:
An \( e(1, 2, 3) \)-student has \( J_1 \)-share \( > \frac{31}{19} - \frac{74}{133} - \frac{79}{133} = \frac{64}{133} \).
An \( e(1, 3, 3) \)-student has \( J_1 \)-share \( < \frac{31}{19} - 2 \times \frac{78}{133} = \frac{61}{133} \).

3) No shares in \([\frac{69}{133}, \frac{72}{133}]\): \( x \in [\frac{69}{133}, \frac{72}{133}] \implies 1 - x \in [\frac{61}{133}, \frac{64}{133}] \).
GAPS Technique: \( f(31, 19) \leq \frac{54}{133} \)

1. \( J_1 = \left( \frac{59}{133}, \frac{61}{133} \right) \)
2. \( J_2 = \left( \frac{64}{133}, \frac{66.5}{133} \right) \)
3. \( J_3 = \left( \frac{66.5}{133}, \frac{69}{133} \right) \) (\( |J_2| = |J_3| \))
4. \( J_4 = \left( \frac{72}{133}, \frac{74}{133} \right) \) (\( |J_1| = |J_4| \))
5. \( J_5 = \left( \frac{78}{133}, \frac{79}{133} \right) \) (\( |J_5| = 20 \))

The following are the only students who are allowed.
\( e(1, 5, 5) \).
\( e(2, 4, 5) \).
\( e(3, 4, 5) \).
\( e(4, 4, 4) \).
GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

$e(1, 5, 5)$. Let the number of such students be $x$
$e(2, 4, 5)$. Let the number of such students be $y_1$
$e(3, 4, 5)$. Let the number of such students be $y_2$.
$e(4, 4, 4)$. Let the number of such students be $z$.

1) $|J_2| = |J_3|$, only students using $J_2$ are $e(2, 4, 5)$ – they use one share each, only students using $J_3$ are $e(3, 4, 5)$ – they use one share each. Hence $y_1 = y_2$. We call them both $y$.

2) Since $|J_1| = |J_4|$, $x = 2y + 3z$.

3) Since $s_3 = 14$, $x + 2y + z = 14$.

$(2y + 3z) + 2y + z = 14 \implies 4(y + z) = 14 \implies y + z = \frac{7}{2}$.
Contradiction.
**MATRIX Technique:  \( f(5, 3) \geq \frac{5}{12} \)**

Want proc for \( f(5, 3) \geq \frac{5}{12} \).

1) **Guess** that the only piece sizes are \( \frac{5}{12}, \frac{6}{12}, \frac{7}{12} \).

2) **Muffin** = pieces add to 1: \( \{ \frac{6}{12}, \frac{6}{12} \}, \{ \frac{5}{12}, \frac{7}{12} \} \). Vectors \( \{ \frac{6}{12}, \frac{6}{12} \} \) is \( (0, 2, 0) \), \( m_1 \) muffins of this type.
   \( \{ \frac{5}{12}, \frac{7}{12} \} \) is \( (1, 0, 1) \), \( m_2 \) muffins of this type.

3) **Student** = pieces add to \( \frac{5}{3} \)
   \( \{ \frac{6}{12}, \frac{7}{12}, \frac{7}{12} \} \) is \( (0, 1, 2) \), \( s_1 \) students of this type.
   \( \{ \frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12} \} \) is \( (4, 0, 0) \), \( s_2 \) students of this type.

4) **Set up equations:**
   \[ m_1(0, 2, 0) + m_2(1, 0, 1) = s_1(0, 1, 2) + s_2(4, 0, 0) \]
   \[ m_1 + m_2 = 5 \]
   \[ s_1 + s_2 = 3 \]

**Natural Number Solution:** \( m_1 = 1, \ m_2 = 4, \ s_1 = 2, \ s_2 = 1 \)
MATRX Technique

Want proc for $f(m, s) \geq \frac{a}{b}$.

1) **Guess** that the only piece sizes are $\frac{a}{b}, \ldots, \frac{b-a}{b}$

2) **Muffin** = pieces add to 1: Vectors $\vec{v}_i$. $\times$ types.
   $m_i$ muffins of type $\vec{v}_i$

3) **Student** = pieces add to $\frac{m}{s}$: Vectors $\vec{u}_j$. $\times$ types.
   $s_j$ students of type $\vec{u}_j$

4) **Set up equations**:
   
   \[ m_1 \vec{v}_1 + \cdots + m_x \vec{v}_x = s_1 \vec{u}_1 + \cdots + s_y \vec{u}_y \]
   
   \[ m_1 + \cdots + m_x = m \]
   
   \[ s_1 + \cdots + s_y = s \]

5) **Look for Nat Numb sol.** If find can translate into procedure.
Later Results by Other People

1. In Fall 2018 Scott Huddleston had code for an algorithm that, on input $m, s$, found $f(m, s)$ and the procedure REALLY FAST.

2. Jacob and Erik Understand WHAT his algorithm does and Jacob coded it up to make sure he understood it. Jacob’s code is also REALLY FAST.

3. Neither Scott, Bill, Jacob, or Erik had a proof that Scott’s algorithm was fast (poly in $m, s$).

4. Richard Chatwin independently came up with the same algorithm; however, he also has a proof that it works. Its on arXiv.

5. One corollary of the work: $f(m, s)$ only depends on $m/s$. 
I meet Alan Frank!

I emailed Alan Frank, the creator of the Muffin Problem and we planned to meet at the MIT combinatorics seminar where I was scheduled to give a talk.
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- He brought to the seminar 11 muffins:
  1 cut \( \left( \frac{15}{30}, \frac{15}{30} \right) \), 2 cut \( \left( \frac{14}{30}, \frac{16}{30} \right) \), 8 cut \( \left( \frac{13}{30}, \frac{17}{30} \right) \).
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  The five us of took pieces so we each got \( \frac{11}{5} \) muffins.
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► He does a Bike-For-Food Charity. I asked him if I should give $40.00 a year OR my Royalties. He chose the $40.00.
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- He does a Bike-For-Food Charity. I asked him if I should give $40.00$ a year OR my Royalties. He chose the $40.00.
  First Year Royalties: $41.00
I meet Alan Frank!

I emailed Alan Frank, the creator of the Muffin Problem and we planned to meet at the MIT combinatorics seminar where I was scheduled to give a talk.

▶ He was delighted that his innocent problem, that he viewed as recreational, has lead to so much math of interest.

▶ He brought to the seminar 11 muffins:
  1 cut \(\left(\frac{15}{30}, \frac{15}{30}\right)\), 2 cut \(\left(\frac{14}{30}, \frac{16}{30}\right)\), 8 cut \(\left(\frac{13}{30}, \frac{17}{30}\right)\).

The five of us took pieces so we each got \(\frac{11}{5}\) muffins.

▶ He does a Bike-For-Food Charity. I asked him if I should give $40.00 a year OR my Royalties. He chose the $40.00.
  First Year Royalties: $41.00
  Second Year Royalties: Don’t know yet but will be < $40.00.