Chapter 1

$m \geq s$ then $f(m,s) \geq 1/3$

In this chapter we show that if $m \geq s$, then $f(m,s) \geq \frac{1}{3}$.

1.1 Example: $f(19,17) \geq \frac{1}{3}$

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We express $\frac{19}{17}$ as $\frac{57}{51}$ since other fractions will have a denominator of 51.

We initially divide all 19 muffins $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. There are now 57 pieces $\frac{1}{3}$-pieces. Since

$$\frac{1}{3} \times 3 < \frac{19}{17} < \frac{1}{3} \times 4$$

- The max number of pieces someone can get and have $< \frac{19}{17}$ is 3.
- The min number of pieces someone can get and have $> \frac{19}{17}$ is 4.

Hence we will give everyone either 3 or 4 $\frac{1}{3}$-pieces (which we will denote by $W = 3$ in the general technique). The only way to distribute 57 pieces so that everyone gets 3 or 4 pieces is to give 11 students 3 pieces and 6 students 4 pieces ($s_W = s_3 = 11$ and $s_{W+1} = s_4 = 6$ in the general technique). As usual a student who gets 3 (4) shares is called a *3-student (4-student)*.

We describe a process whereby students give pieces of muffins, called gifts, to other students so that, in the end, all students have $\frac{57}{51}$. Each gift leads to a change in how the muffins are cut in the first place; however, there will never be a muffin of size $< \frac{1}{3}$.

Each 4-student has $\frac{4}{3} = \frac{68}{51}$ and hence has to give (perhaps in several increments) $\frac{68}{51} - \frac{57}{51} = \frac{11}{51}$ to get *down to* $\frac{57}{51}$. Realize that if a 4-student gives $\frac{11}{51}$ to a 3-student, then the 3-student now has $\frac{51}{51} + \frac{11}{51} = \frac{62}{51} > \frac{57}{51}$.

Each 3-student has $\frac{51}{51}$ and hence has to receive $\frac{57}{51} - \frac{51}{51} = \frac{6}{51}$ to get *up to* $\frac{57}{51}$.

Call the 11 3-students $g_1, \ldots, g_{11}$.

Call the 6 4-students $f_1, \ldots, f_6$.

Notation 1.1. $x(f_1 \rightarrow g_1)$ means the following: f_1 gives x to g_1 by taking two $\frac{1}{3}$-pieces, combining them, cutting off a piece of size x, giving it to g_1 while keeping the rest. g_1 takes the piece given to him and combines it with a $\frac{1}{3}$ piece. Notice that in terms of pieces we are taking three pieces of size $\frac{1}{3}$ (2 from f_1 and 1 from g_1) and turning them into 1 piece of size $\frac{2}{3} - x$ and one of size $\frac{1}{3} + x$. Hence we can easily rearrange how the muffins are cut.

We need to make sure this procedure never results in a piece that is $< \frac{1}{3}$. In the above example (1) f_1 now has a piece of size $\frac{2}{3} - x$, hence we need $x \leq \frac{1}{3}$, (2) g_1 now has a piece of size $\frac{1}{3} + x$, which is clearly $\geq \frac{1}{3}$. Hence the only restriction is $x \leq \frac{1}{3}$.

(1) $\frac{11}{51}(f_1 \rightarrow g_1)$. Now f_1 has $\frac{57}{51}$. YEAH. However, g_1 has $\frac{62}{51}$.
(2) $\frac{5}{51}(g_1 \rightarrow g_2)$. Now g_1 has $\frac{62}{51} - \frac{5}{51} = \frac{57}{51}$. YEAH. However, g_2 has $\frac{51}{51} + \frac{5}{51} = \frac{56}{51}$.
(3) $\frac{1}{51}(f_2 \rightarrow g_2)$. Now g_2 has $\frac{57}{51}$. YEAH. However, f_2 has $\frac{67}{51}$.
(4) $\frac{10}{51}(f_2 \rightarrow g_3)$. Now f_2 has $\frac{57}{51}$. YEAH. However, g_3 has $\frac{61}{51}$.
(5) $\frac{4}{51}(g_3 \rightarrow g_4)$. Now g_3 has $\frac{57}{51}$. YEAH. However, g_4 has $\frac{55}{51}$.
(6) $\frac{2}{51}(f_3 \rightarrow g_4)$. Now g_4 has $\frac{57}{51}$. YEAH. However, f_3 has $\frac{66}{51}$.

(7) $\frac{9}{51}(f_3 \to g_5)$. Now f_3 has $\frac{57}{51}$. YEAH. However, g_5 has $\frac{60}{51}$.
(8) $\frac{3}{51}(g_5 \to g_6)$. Now g_5 has $\frac{57}{51}$. YEAH. However, g_6 has $\frac{54}{51}$.
(9) $\frac{3}{51}(f_4 \to g_6)$. Now g_6 has $\frac{57}{51}$. YEAH. However, f_4 has $\frac{65}{51}$.
(10) $\frac{8}{51}(f_4 \to g_7)$. Now f_4 has $\frac{57}{51}$. YEAH. However, g_7 has $\frac{59}{51}$.
(11) $\frac{2}{51}(g_7 \to g_8)$. Now g_7 has $\frac{57}{51}$. YEAH. However, g_8 has $\frac{53}{51}$.
(12) $\frac{4}{51}(f_5 \to g_8)$. Now g_8 has $\frac{57}{51}$. YEAH. However, f_5 has $\frac{64}{51}$.
(13) $\frac{7}{51}(f_5 \to g_9)$. Now f_5 has $\frac{57}{51}$. YEAH. However, g_9 has $\frac{58}{51}$.
(14) $\frac{1}{51}(g_9 \to g_{10})$. Now g_9 has $\frac{58}{51}$. YEAH. However, g_{10} has $\frac{52}{51}$.
(15) $\frac{5}{51}(f_6 \to g_{10})$. Now g_{10} has $\frac{57}{51}$. YEAH. However, f_6 has $\frac{63}{51}$.
(16) $\frac{6}{51}(f_6 \to g_{11})$. Now f_6 has $\frac{57}{51}$. YEAH. However, g_{11} has $\frac{57}{51}$. OH. thats a good thing!

YEAH- we are done.

Note that the first x was $\frac{11}{51} \leq \frac{1}{3}$ and the remaining x were all $\leq \frac{11}{51} \leq \frac{1}{3}$. Hence all pieces in the final procedure are $\geq \frac{1}{3}$.
End of Example

Theorem 1.2. *For all* $m \geq s$, $f(m, s) \geq \frac{1}{3}$.

Proof. Divide all the muffins into $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Let W be such that

$$\frac{1}{3} \times W \leq \frac{m}{s} \leq \frac{1}{3}(W + 1).$$

Give some students W $\frac{1}{3}$-pieces and some $(W + 1)$ $\frac{1}{3}$-pieces. How many students? Let s_W (s_{W+1}) be the number of students who get W $(W + 1)$ $\frac{1}{3}$-pieces. Then:

$$W s_W + (W + 1)s_{W+1} = 3m$$
$$s_W + s_{W+1} = s$$

These equations have a unique solution and unique value of W if s does not divide $3m$. If s does divide $3m$ there will be more than one possible value of W; however, we can pick one arbitrarily. So we give s_W students W $\frac{1}{3}$-pieces and s_{W+1} students $W + 1$ $\frac{1}{3}$-pieces.

By the definition of W:

$$0 \leq \frac{m}{s} - \frac{W}{3} \leq \frac{1}{3} \tag{1.1}$$

$$0 \leq \frac{W+1}{3} - \frac{m}{s} \leq \frac{1}{3} \tag{1.2}$$

Now we will need to smooth out the distribution so that everyone receives $\frac{m}{s}$. We will do this by a sequence of moves of the form $x(f_i \to g_j)$ or $x(g_i \to g_j)$, as defined in the example.

We will assume s_{W+1} and s_W are relatively prime (this only comes up in Claim 3 below). This is fine because if they have a common factor d, we can just use the procedure for the $\frac{s_{W+1}}{d}$, $\frac{s_W}{d}$ case repeated d times.

Call the s_W W-students $g_1, \ldots, g_{s_W}$.

Call the s_{W+1} $(W+1)$-students $f_1, \ldots, f_{s_{W+1}}$.

Claim 1:

(1) If $s_{W+1} < s_W$ then $\frac{W+1}{3} - \frac{m}{s} > \frac{m}{s} - \frac{W}{3}$.
(2) If $s_W < s_{W+1}$ then $\frac{W+1}{3} - \frac{m}{s} > \frac{m}{s} - \frac{W}{3}$.

Proof of Claim 1:

$$s_{W+1} \times \frac{W+1}{3} + s_W \times \frac{W}{3} = m$$

$$s_{W+1} \times \left(\frac{m}{s} + \frac{W+1}{3} - \frac{m}{s}\right) + s_W\left(\frac{m}{s} + \frac{W}{3} - \frac{m}{s}\right) = m$$

$$\left(s_{W+1} + s_W\right)\frac{m}{s} + s_{W+1}\left(\frac{W+1}{3} - \frac{m}{s}\right) + s_W\left(\frac{W}{3} - \frac{m}{s}\right) = m$$

$$s \times \frac{m}{s} + s_{W+1}\left(\frac{W+1}{3} - \frac{m}{s}\right) + s_W\left(\frac{W}{3} - \frac{m}{s}\right) = m$$

$$\frac{W+1}{3} - \frac{m}{s} = \frac{s_W}{s_{W+1}}\left(\frac{m}{s} - \frac{W}{3}\right)$$

Both parts follow.

End of Proof of Claim 1

We give the procedure to obtain $f(m,s) \leq \frac{1}{3}$. There are two cases.

Case 1: $s_{W+1} < s_W$. Hence by Claim 1 $\frac{W+1}{3} - \frac{m}{s} > \frac{m}{s} - \frac{W}{3}$.

(1) Let $x = \frac{W+1}{3} - \frac{m}{s}$. Note that $x \leq \frac{1}{3}$. Do $x(f_1 \rightarrow g_1)$. Now f_1 has $\frac{m}{s}$. YEAH. However, g_1 has $\frac{W}{3} + \frac{W+1}{3} - \frac{m}{s} > \frac{m}{s}$. (This is where we use $s_{W+1} < s_W$, or more accurately the consequence of that from Claim 1.)

(2) Let $x = \frac{2W+1}{3} - 2 \times \frac{m}{s}$. Do $x(g_1 \rightarrow g_2)$. Now g_1 has $\frac{m}{s}$. YEAH.

(3) If g_2 has $> \frac{m}{s}$ then g_2 gives enough to g_3 so that g_2 has $\frac{m}{s}$. Keep up this chain of $g_1, g_2, g_3, \ldots$ until there is a g_i such that g_i end up with $< \frac{m}{s}$ (though more than the $\frac{W}{3}$ that g_i had originally). This happens because g_{i-1} gives g_i what it can, so g_{i-1} ends with exactly $\frac{m}{s}$, but its just not enough for g_i to have $\frac{m}{s}$ as well :-(.

(4) Do $x(f_2 \rightarrow g_i)$ where x is such that g_i will now have $\frac{m}{s}$.

(5) Do $x(f_2 \rightarrow g_{i+1})$ where x is such that f_2 will now have $\frac{m}{s}$. Repeat the same chain of g_i's as in step 3.

(6) Repeat the above steps until you are done.

We need to show that (1) there is never a piece of size $< \frac{1}{3}$, and (2) the process ends with every student getting $\frac{m}{s}$.

Claim 2: The first gift is $\leq \frac{1}{3}$ and no gift is larger.

Proof of Claim 2: Let $C = \frac{W+1}{3} - \frac{m}{s}$ which is the size of the first gift. By equation (2) $C \leq \frac{1}{3}$.

Assume that all gifts so far have been $\leq C$. We analyze the three kinds of gifts and show that in all cases the gift is $\leq C$.

- $x(f_i \rightarrow g_j)$ where (1) initially f_i has $> \frac{m}{s}$, g_j has $< \frac{m}{s}$, and (2) after the gift f_i has $\frac{m}{s}$. When this occurs it is f_i's first or second gift giving. (This happens in steps 1 and 5 above, and later as well.) Before the gift f_i has at least $\frac{m}{s}$ but at most $\frac{W+1}{3}$, so this gift has size at most $\frac{W+1}{3} - \frac{m}{s} = C$.

- $x(g_i \rightarrow g_{i+1})$ where (1) initially g_i has $> \frac{m}{s}$, g_j has $< \frac{m}{s}$, and (2) after the gift g_i has $\frac{m}{s}$. When this occurs, g_i has received a gift once and this is g_i's first time giving. (This happens in steps 2 and in the chain referred to in step 5.) Since g_i just received a gift of size $\leq C$ she has $\leq \frac{W}{3} + C$. Hence the gift is $\leq \frac{W}{3} - \frac{m}{s} + C \leq C$.
- $x(f_i \rightarrow g_j)$ where (1) initially f_i has $> \frac{m}{s}$, g_j has $< \frac{m}{s}$, and (2) after the gift g_j has $\frac{m}{s}$. This will be f_i's first time giving. (This happens in step 4 above.) Before the gift f_i has at least $\frac{W}{3}$ but at most $\frac{m}{s}$, so this gift has size at most $\frac{m}{s} - \frac{W}{3} \leq C$ (by Claim 1).

Claim 3: If s_W and s_{W+1} are relatively prime then the process terminates with all students having $\frac{m}{s}$.

Proof of Claim 3:

In each step all of the f_i have at least $\frac{m}{s}$. In each step the number of students who have the correct amount of muffin goes up. One may be worried that at some point we will try to do step 4 (for example) of the procedure and there will be no g_i left who need more muffin. But this is not possible because until the process terminates the f's always have more muffins than they need, so there is always a g with less muffins than they need.

One may also be worried that eventually we will get all of the f's to have $\frac{m}{s}$, but the g's will not all have $\frac{m}{s}$. This is not possible either, because whenever we only make gifts from f to g, there is no g with more than $\frac{m}{s}$.

Finally, if s_W and s_{W+1} are not relatively prime, it is possible that the procedure will terminate early because in step 5 the size of the donation x is 0. If this occurred it would mean that there is some subset of F f's and G g's each of which has exactly $\frac{m}{s}$, and only made donations amongst themselves. But then $\frac{F}{G} = \frac{s_{W+1}}{s_W}$, a contradiction.

End of Proof of Claim 3

Case 2: $s_W < s_{W+1}$. This is similar to Case 1 except that instead of f_1 giving g_1 so that f_1 has $\frac{m}{s}$, f_1 gives to g_1 so that g_1 has $\frac{m}{s}$. Hence we have a chain of f_i's instead of a chain of g_i's. □

1.2 Conjectures About Extensions

We first restate the main theorem:

Theorem 1.3. *For all $m \geq s$, if $V \geq 3$ then $f(m,s) \geq \frac{1}{3}$.*

What if $V = 4$? $V = 5$?

Conjecture 1.4. *There exists a function $a(V)$ such that the following is true: For all $m \geq s$, if $V \geq V$ then $f(m,s) \geq a(V)$.*

What might $a(V)$ look like? We know that $a(3) = \frac{1}{3}$ and empirically it seems that $\lim_{V\to\infty} a(V) = \frac{1}{2}$. One candidate is

$$a(V) = \frac{V+1}{2V+6}$$