1 Quantum Games

We discuss two games such that if the players play the game with quantum resources they can provably do better than if they play the game with classical resources. For further discussion of these games, and other games with this property, see the survey of Brunner et al. [3].

1.1 The CHSH Game

Clauser et al. [5] invented the CHSH GAME as a realizable experiment that can differentiate quantum from classical computing.

Problem 1.1. *The* CHSH GAME

INSTANCE: Alice gets bit x, Bob gets bit y. Before they get their bits they can discuss strategy.
QUESTION: Alice outputs bit a, Bob outputs bit b. Alice and Bob win iff $x \wedge y = a \oplus b$.

Clauser et al. [5] proved the following (see also Aaronson [1, Chapter 13] for an exposition).

Theorem 1.

1. *If Alice and Bob play the* CHSH GAME *with classical resources (a) there is a deterministic strategy where they win with probability* 0.75 *(both always output 0), (b) there is no strategy, deterministic or randomized, that does better.*

2. *If Alice and Bob play the the* CHSH GAME *with quantum resources (they prepare entangled qubits before the game begins) then (a) there is a strategy where they win with probability* 0.85 *(this is complicated), (b) there is no strategy that does better.*

This game is of interest to us since it is a case where the quantum world is provably different from the classical world. Note that the gap between the classical and quantum is $0.85 - .0.75 = 0.1$.

1.2 Magic Square Game

Cabello [4] defined the Magic Square Game, though he did not call it that. For more information on it also see the survey on Quantum pseudo-telepathy [2, Section 5] by Brassard et al., or the Wikipedia entry on Quantum pseudo-telepathy [6].

Problem 1.2. *The* MAGIC SQUARE GAME *(*MS GAME*)*

INSTANCE: Alice gets $i \in \{1, 2, 3\}$, Bob gets $j \in \{1, 2, 3\}$. They interpret i as the row of a 3×3 matrix and j as the column of a 3×3 matrix. Alice and Bob get to discuss strategy ahead of time.

QUESTION: Alice and Bob both output a three-bit sequence. Alice's sequence is used as the ith row of a matrix. Bob's sequence is used as the jth column of a matrix. If the following three conditions hold then Alice and Bob win, else they lose. (a) The values in row i add to an even number, (b) The values in column j add to an odd number. (c) Alice and Bob's values are consistent (they agree at (i, j)).

Theorem 2.

1. *If Alice and Bob play the the* MS GAME *with classical resources (a) there is a deterministic strategy where they win with probability $\frac{8}{9} = 0.88\ldots$, (b) there is no strategy, deterministic or randomized, that does better.*

2. *If Alice and Bob play the the* CHSH GAME *with quantum resources (they prepare entangled qubits before the game begins) then there is a strategy that wins with probably 1 (so always wins).*

This game is of interest to us since it is a case where the quantum world is provably different from the classical world. Note that the gap between the classical and quantum is $1.0 - 0.88\ldots = 0.11\ldots$.

Is there an interesting version of the MS GAME on $k \times k$ matrices for $k \geq 4$? The following exercise shows that the answer is no.

Exercise 1.

1. *Give a* 4×4 *matrix* M *of 0's and 1's such that every row sums to an even number and every column sums to an odd number.*

2. *Show that there is a classical strategy for the* 4×4 MS GAME *that wins with probability 1. (Hint: Use Part 1.)*

3. *Show that, for all* $k \geq 4$*, there is a classical strategy for the* $k \times k$ MS GAME *that wins with probability 1.*

1.3 Comparing The CHSH Game with The MS Game

We give two reasons why the MS GAME game is better for distinguishing classical and quantum computation, and one reason why the CHSH GAME game is better.

Two reasons why the MS Game is better

1. The gap between classic players and quantum players is bigger for the MS GAME. The gap is $0.11\ldots - 0.10 = 0.011\ldots$.

2. For the MS GAME the quantum players *always* win. This is better for repeated experiments. Assume the game is played n times.

   (a) For the MS GAME:
   If Alice and Bob are classical then the expected number of wins is $8n/9$.
   If Alice and Bob are quantum then the expected number of wins is n.
   So if Alice and Bob lose just once, then they must be classical.

   (b) For the CHSH GAME
   If Alice and Bob are classical then the expected number of wins is $0.75n$.
   If Alice and Bob are classical then the expected number of wins is $0.85n$.
   These two cases are harder to distinguish since a lose by Alice and Bob could happen in either case.

One reason why the CHSH Game is better. Quantum computers (in 2023) are noisy. The computations are not that reliable, though there is a lot of work on quantum error correction to try to alleviate this. The CHSH GAME game is simpler and uses fewer operation, hence less noise. The calculations done above for the MS GAME were assuming an error-free quantum computer which is not a reality yet.

We have no strong opinion on which one to use in 2023; however, we suspect that as time goes on quantum computers will be less noisy so the MS GAME will be better.

References

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