Big Ramsey Degrees of Countable Ordinals

Joanna Boyland*
Department of Mathematics
Harvard University
Cambridge, MA, 02138
jboyland@college.harvard.edu

William Gasarch*
Department of Computer Science
University of Maryland at College Park
College Park, MD, 20742
gasarch@umd.edu

Nathan Hurtig*
Department of Computer Science
Rose-Hulman Institute of Technology
Terre Haute, IN, 47803
hurtigna@rose-hulman.edu

Robert Rust*
Department of Mathematics
College of New Jersey
Ewing Township, NY, 08618
rustr2@tcnj.edu

Submitted: TODO; Accepted: TBD; Published: TBD

Abstract

Ramsey's theorem states that for all finite colorings of an infinite set, there exists an infinite homogeneous subset. What if we seek a homogeneous subset that is also order-equivalent to the original set? For some nonnegative integer a and an ordered set S, consider an arbitrary finite coloring of the a-subsets of S. The big Ramsey degree of a in S is the least integer t wuch that some subset S' order-equivalent to S where the coloring restricted to S' only uses t colors is guaranteed. Mašulović and Šobot (2019) showed that all countable ordinals less than ω^ω have finite big Ramsey degrees. We find exact big Ramsey degrees for all ordinals less than ω^ω. We also give an alternative proof for the big Ramsey degrees of the integers.

Mathematics Subject Classifications: 05D10, 03E10

1 Introduction

Definition 1. Let $(A, \preceq_A)$ and $(B, \preceq_B)$ be ordered sets. Then A, B are *order-equivalent*, denoted $A \approx B$, if there exists an order-preserving bijection $f\colon A \to B$; that is, for all $a_1, a_2 \in A$:

*Supported by NSF grant 2150382.

$$a_1 \preceq_A a_2 \iff f(a_1) \preceq_B f(a_2).$$

We might also say that A has order type B.

Notation 2. Let a, b be nonnegative integers and S be a set.

1. $[b]$ is $\{1, \ldots, b\}$. If $b = 0$ then $[b] = \emptyset$.
2. $\binom{S}{a}$ is the set of all a-element subsets of S. Often, we index the subset by the order of S.
3. Let $\mathrm{COL}\colon S \to [b]$ and $S' \subseteq S$. Then $\mathrm{COL}(S')$ is the codomain of COL restricted to S'. Hence $|\,\mathrm{COL}(S')|$ is the number of unique elements in the codomain of COL restricted to S'.

Definition 3. Let S be an ordered set with order type α, $S' \subseteq S$, a, b, t be nonnegative integers, and $\mathrm{COL}\colon \binom{S}{a} \to [b]$ be a coloring.

1. S' is *homogeneous* if $|\,\mathrm{COL}\left(\binom{S'}{a}\right)| = 1$. S' is *t-homogeneous* if $|\,\mathrm{COL}\left(\binom{S'}{a}\right)| \leqslant t$.
2. S' is *α-t-homogeneous* if $|\,\mathrm{COL}\left(\binom{S'}{a}\right)| \leqslant t$ and $S' \approx S$ (hence S' has order type α).

Notation.

1. ζ is the order type of the integers, ω is the order type of the naturals, and η is the order type of the rationals under their natural orderings.
2. Polynomials in ω are also order types: for example, $\omega^2 + \omega \cdot 3$ is ω^2 followed by 3 copies of ω.
3. For all of the order types above we use the notation given for both the order type and for the underlying set. For example, we write things like

$$\mathrm{COL}\colon \binom{\omega + 2}{a} \to [b].$$

When we use an order type α as a set, it's the set of all ordinals $\{\rho\colon \rho < \alpha\}$. For example, $\omega + 2 = \{0, 1, \ldots, \omega, \omega + 1\}$.

Definition 4. Let S be an ordered set and α be its order type. For nonnegative integers a, $T(a, S)$ is the least nonnegative integer t such that, for all nonnegative integers b, for all colorings $\mathrm{COL}\colon \binom{S}{a} \to [b]$, there exists some $S' \subseteq S$ such that S' is α-t-homogeneous. Note that t is independent of b. $T(a, S)$ is called the *big Ramsey degree* of $\binom{S}{a}$. In other literature, for example, Zucker [**?**], this is sometimes written as $S \to (S)^a_{r,T(a,S)}$ and $S \not\to (S)^a_{T(a,S),T(a,S)-1}$ for all nonnegative integers r.

This paper focuses on $T(a, \zeta)$ and $T(a, \alpha)$ where α is an ordinal that is less than ω^ω. We do not consider $T(a, \eta)$, however, the interested reader should know the following:

Theorem 5.

1. $T(2, \eta) = 2$. *This was first proven by Galvin, unpublished.*

2. *For all nonnegative integers a, $T(a, \eta)$ exists. This was first proven by Laver [**?**].*

3. *$T(a, \eta)$ is the coefficient of x^{2a+1} in the Taylor series for the tangent function, hence*

$$T(a, \eta) = \frac{B_{2a+1}(-1)^{a+1}(1 - 4^{a+1})}{(2(a+1))!}$$

*where B_{2a+1} is the $(2a+1)$th Bernoulli number. This was proven by Devlin [**?**]. See also Vuksanovic [**?**] and Halpern & Lauchli [**?**].*

Note 6. The notion of $T(a, S)$ has been defined on structures other than orderings. We give an example. Let $R = (\mathbb{N}, E)$ be the Rado graph. $T(a, R)$ is the least number t such that, for all b, for all colorings $\mathrm{COL} : \binom{\mathbb{N}}{a} \to [b]$, there exists $H \subseteq \mathbb{N}$ where both $|\,\mathrm{COL}\left(\binom{H}{a}\right)| \leqslant t$ and the graph induced by H is isomorphic to R. The numbers $T(a, R)$ are known but complicated; however, $T(2, R) = 2$. See Dobrinen [**?**] for references and other examples.

2 Summary of Results

Ramsey's Theorem on $\mathbb{N}$ gives an infinite 1-homogeneous subset of $\mathbb{N}$, so applying it to ω gives a ω-1-homogeneous set. Hence Theorem **??** follows.

Theorem 7. $T(a, \omega) = 1$ *for all nonnegative integers a.*

What happens for other ordered sets? In this paper we do the following.

1. In Section **??** we show that $T(a, \zeta) = 2^a$. This can be obtained by the result due to Mašulović and Šobot [**?**] that $T(a, \omega + \omega) = 2^a$. We give a simpler and more direct proof.

2. In Sections **??**, **??**, **??**, **??**, and **??** we determine $T(a, \alpha)$ for all ordinals $\alpha < \omega^\omega$. Mašulović and Šobot [**?**] previously showed for all $\alpha \geqslant \omega^\omega$ that $T(a, \alpha)$ is not finite. They also showed for $\alpha < \omega^\omega$ that $T(a, \alpha)$ is finite; however, they did not obtain the exact values of $T(a, \alpha)$.

3 $T(a, \zeta) = 2^a$

For a warmup we first prove $T(1, \zeta) = 2$ and $T(2, \zeta) = 4$.

Example 8. $T(1, \zeta) = 2$.

Proof. Let b be an arbitrary nonnegative integer.

We first prove $T(1, \zeta) \leqslant 2$. Let $\mathrm{COL}\colon \zeta \to [b]$. Let $\mathrm{COL}'\colon \omega \to [b]^2$ be defined by

$$\mathrm{COL}'(x) = (\mathrm{COL}(-x), \mathrm{COL}(x)).$$

By Theorem **??** there exists a ω-1-homogeneous set H'. Let the color of the homogeneous set be (c_1, c_2). Then the set $H = -H' + H'$ is ζ-2-homogeneous, made up of the colors c_1 and c_2. Because COL was arbitrary, $T(1, \zeta) \leqslant 2$.

We now prove $T(1, \zeta) \geqslant 2$. Let $\mathrm{COL}\colon \zeta \to [2]$ be the coloring that colors all nonnegative integers RED and all negative integers BLUE. There is no ζ-1-homogeneous set, as neither color extends indefinitely in both directions. Therefore $T(1, \zeta) \geqslant 2$, and with the previous result, $T(1, \zeta) = 2$. □

Example 9. $T(2, \zeta) = 4$.

Proof. Let b be an arbitrary nonnegative integer.

We first prove $T(2, \zeta) \leqslant 4$. Let $\mathrm{COL}\colon \binom{\zeta}{2} \to [b]$. Let $\mathrm{COL}'\colon \binom{\omega}{2} \to [b]^4$ be defined by

$$\mathrm{COL}'(x, y) = (\mathrm{COL}(-x, -y), \mathrm{COL}(-x, y), \mathrm{COL}(x, -y), \mathrm{COL}(x, y)).$$

By **??** there exists a homogeneous set H'. Let the color of the homogeneous set be (c_1, c_2, c_3, c_4). Index H' as $\{h_0 < h_1 < \cdots\}$. Then the set

$$H = \{-h_i \colon i \text{ is even}\} \cup \{h_i \colon i \text{ is odd}\}$$

is ζ-4-homogeneous, outputting colors c_1, c_2, c_3, and c_4. There is nothing special about partitioning H' into odd and even indices; any partition of H' into any two disjoint infinite sets suffices. We do this to filter out edges of the form $(-x, x)$, which were not considered by COL' and therefore might be any color. Therefore $T(2, \zeta) \leqslant 4$.

We now prove $T(2, \zeta) \geqslant 4$. Let $\mathrm{COL}\colon \binom{\zeta}{2} \to [4]$ be the coloring

$$\mathrm{COL}(x, y) = \begin{cases} 1 & \text{if } x, y \geqslant 0 \\ 2 & \text{if } x \geqslant 0,\ y < 0, \text{ and } |x| \leqslant |y| \\ 3 & \text{if } x \geqslant 0,\ y < 0, \text{ and } |x| > |y| \\ 4 & \text{if } x < 0,\ y < 0 \end{cases}$$

We leave it to the reader to show there is no ζ-3-homogeneous set. The key idea of the proof is that if we suppose some set doesn't express a color under COL, then it cannot be order-equivalent to ζ. Therefore $T(2, \zeta) \geqslant 4$, and with the previous result, $T(2, \zeta) = 4$. □

Theorem 10. *For all nonnegative integers a, $T(a,\zeta) = 2^a$.*

Proof. Let b be an arbitrary nonnegative integer.

We first prove $T(a,\zeta) \leqslant 2^a$. Let $\mathrm{COL}\colon \binom{\zeta}{a} \to [b]$ be an arbitrary coloring. Let $\mathrm{COL}'\colon \binom{\omega}{a} \to [b]^{(2^a)}$ be defined by

$$\mathrm{COL}'(x_1,\ldots,x_a) = (\mathrm{COL}(x_1,\ldots,x_a), \mathrm{COL}(-x_1,x_2,\ldots,x_a),\ldots,\mathrm{COL}(-x_1,\ldots,-x_a))$$

The output of COL' goes through all 2^a ways to negate a subset of the numbers. Note that COL' considers the color of every a-element subset where the absolute values of the elements are all different.

By Theorem **??** there exists some homogeneous set H'. Index H' as $\{h_0 < h_1 < \cdots\}$. Then the set

$$H = \{-h_i \colon i \text{ is even}\} + \{h_i \colon i \text{ is odd}\}$$

is ζ-2^a-homogeneous, as the absolute values of every element in H are all different. Therefore $T(a,\zeta) \leqslant 2^a$.

We now prove $T(a,\zeta) \geqslant 2^a$. We describe a coloring $\mathrm{COL}\colon \binom{\zeta}{a} \to [2^a]$. The codomain is not be the set $[2^a]$ exactly; however it is a set of that size.

1. Let $\{x_1,\ldots,x_a\} \in \binom{\zeta}{a}$.

2. Let $(i_1,\ldots,i_a)$ such that

$$|x_{i_1}| \leqslant |x_{i_2}| \leqslant \cdots \leqslant |x_{i_a}|.$$

   We give an example that also shows how to deal with ambiguity. Suppose we are given $\{-7,4,0,7\}$. We order the absolute values as $|0| \leqslant |4| \leqslant |-7| \leqslant |7|$ so we have indices $(3,2,1,4)$. Note that we ordered $|-7| \leqslant |7|$. This is our convention.

3. Let s_{i_j} be "+" if $x_{i_j} \geqslant 0$ and "−" if $x_{i_j} < 0$. In our example, $(x_{i_1},x_{i_2},x_{i_3},x_{i_4}) = (0,4,-7,7)$ so $(s_{i_1},s_{i_2},s_{i_3},s_{i_4}) = (+,+,-,+)$.

4. The color is $(s_{i_1},\ldots,s_{i_a})$.

We use 2^a colors, as there are a elements each either "+" or "−". We leave it to the reader to show that there is no ζ-(2^a-1)-homogeneous set.

□

4 $T(a,\omega\cdot k)$

As noted in Theorem **??**, $T(a,\omega) = 1$. In this and later sections we look at ordinals larger than ω. For simplicity in stating results, we save the big Ramsey degrees of successor ordinals such as $\omega+1, \omega+2, \ldots$ for Section **??**.

Theorem 11. *For nonnegative integers a, k, $T(a, \omega \cdot k) \leqslant k^a$.*

Proof. Let a, k be arbitrary nonnegative integers and

$$\mathrm{COL}\colon \binom{\omega \cdot k}{a} \to [b]$$

be an arbitrary coloring for some nonnegative integer b. Define $\mathrm{COL}'\colon \binom{\omega}{a} \to [b]^{(k^a)}$ with

$$\begin{aligned}\mathrm{COL}'(x_1, x_2, \ldots, x_a) = (&\mathrm{COL}(x_1, \ldots, x_a), \mathrm{COL}(\omega + x_1, x_2, \ldots, x_a), \ldots, \\ &\mathrm{COL}(\omega \cdot (k-1) + x_1, \ldots, \omega \cdot (k-1) + x_a))\end{aligned}$$

where COL' maps a elements of ω to the colorings of all k^a valid ways to have each element as a low-dimension coefficient within one of the k copies of ω. Apply Theorem **??** with COL' to find some $N \approx \omega$ where

$$\mathrm{COL}'\left(\binom{N}{a}\right)$$

only expresses one tuple Y containing $|Y| = k^a$ colors. Index N as $\{n_0 < n_1 < \cdots\}$ and let

$$H = \{n_i \colon i \equiv 0 \mod k\} + \cdots + \{\omega \cdot (k-1) + n_i \colon i \equiv k-1 \mod k\}.$$

Now $H \approx \omega \cdot k$. Then

$$\mathrm{COL}\left(\binom{H}{a}\right)$$

only expresses k^a colors: for any selection of a elements from H, its color was considered in COL' so it must be one of the k^a colors in Y. □

Theorem 12. *For nonnegative integers a, k, $T(a, \omega \cdot k) \geqslant k^a$.*

Proof. We give a k^a-coloring of $\binom{\omega \cdot k}{a}$ that has no $(k^a - 1)$-homogeneous $H \approx \omega \cdot k$. We represent $\omega \cdot k$ as

$$\omega \cdot k = X_1 + \cdots + X_k$$

where each $X_i \approx \omega$. We write each X_i to be a set of disjoint nonnegative integers. We represent an element of $\omega \cdot k$ by the ordered pair (i, x) where the element is in X_i and within X_i, it is the number x. In standard notation, we would write the element as $\omega \cdot i + x$.

Before giving the coloring we give an example with $a = 5$ and $k = 200$. To color the element

$$\{(3, 12), (50, 2), (110, 7), (110, 7777), (117, 3)\}$$

we do the following:

1. Order the ordered pairs by their second coordinates. So we have

$$((50, 2), (117, 3), (110, 7), (3, 12), (110, 7777)).$$

   Because all of the X_i's are disjoint the second coordinates are all different, so there is never a tie.

2. The color is the sequence of first coordinates. So the color is

$$(50, 117, 110, 3, 110).$$

Notice that the number of colors is the number of 5-tuple where each number is in $\{0, \ldots, 199\}$. Hence there are 200^5 colors.

In general, given $\{(i_1, x_1), \ldots, (i_a, x_a)\}$:

1. Order the ordered pairs by their second coordinates.

2. The element's color is the sequence of first coordinates.

Notice that the number of colors is the number of a-tuples where each number is in $\{1, \ldots, k\}$. Hence there are k^a colors. We leave it to the reader to show that there can be no $(\omega \cdot k)$-$(k^a - 1)$-homogeneous H. The key idea of the proof, much like the previous lower bounds in this paper, is supposing that one of the k^a colors is not expressed by some set, and using that to show that the set cannot be order-equivalent to $\omega \cdot k$. $\square$

Theorem 13. *For nonnegative integers a, k, $T(a, \omega \cdot k) = k^a$.*

Proof. By Theorem **??**, $T(a, \omega \cdot k) \leqslant k^a$. By Theorem **??**, $T(a, \omega \cdot k) \geqslant k^a$. The result follows. $\square$

5 $T(2, \omega^2)$

This section provides a concrete example involving high-dimension ordinals. We treat ω^2 as an infinite addition of copies of ω:

$$\begin{array}{rrrrl} 0, & 1, & 2, & 3, & \ldots \\ \omega + 0, & \omega + 1, & \omega + 2, & \omega + 3, & \ldots \\ \omega \cdot 2 + 0, & \omega \cdot 2 + 1, & \omega \cdot 2 + 2, & \omega \cdot 2 + 3, & \ldots \\ \omega \cdot 3 + 0, & \omega \cdot 3 + 1, & \omega \cdot 3 + 2, & \omega \cdot 3 + 3, & \ldots \\ \vdots & & & & \end{array}$$

Every element of ω^2 is a linear expression in ω with non-negative integer coefficients. For example, $\omega \cdot 17 + 8 \in \omega^2$.

Example 14. $T(2, \omega^2) = 4$.

Proof. We first prove $T\left(2, \omega^2\right) \leqslant 4$.

Let

$$\mathrm{COL}\colon \binom{\omega^2}{2} \to [b]$$

be an arbitrary coloring of for some nonnegative integer b. We define four functions f_1, f_2, f_3, f_4 from domain $\binom{\omega}{4}$ to codomain $\binom{\omega^2}{2}$ and then use them to define a coloring from $\binom{\omega}{4}$ to $[b]^4$. In what follows, we index variables as $x_1 < x_2 < x_3 < x_4$.

$f_1 \colon \binom{\omega}{4} \to \binom{\omega^2}{2}$ is defined by

$$f_1(x_1, x_2, x_3, x_4) = \{\omega \cdot x_1 + x_2, \omega \cdot x_3 + x_4\}.$$

$f_2 \colon \binom{\omega}{4} \to \binom{\omega^2}{2}$ is defined by

$$f_2(x_1, x_2, x_3, x_4) = \{\omega \cdot x_1 + x_3, \omega \cdot x_2 + x_4\}.$$

$f_3 \colon \binom{\omega}{4} \to \binom{\omega^2}{2}$ is defined by

$$f_3(x_1, x_2, x_3, x_4) = \{\omega \cdot x_1 + x_4, \omega \cdot x_2 + x_3\}.$$

$f_4 \colon \binom{\omega}{4} \to \binom{\omega^2}{2}$ is defined by

$$f_4(x_1, x_2, x_3, x_4) = \{\omega \cdot x_1 + x_2, \omega \cdot x_1 + x_3\}.$$

$\mathrm{COL}' \colon \binom{\omega}{4} \to [b]^4$ is defined by

$$\mathrm{COL}'(X) = (\mathrm{COL}(f_1(X)), \mathrm{COL}(f_2(X)), \mathrm{COL}(f_3(X)), \mathrm{COL}(f_4(X))).$$

Apply Theorem **??** on COL' to find some $N \approx \omega$ where $|\,\mathrm{COL}'\left(\binom{N}{4}\right)| = 1$. Enumerate N as $N = \{x_0, x_1, \ldots\}$ with $x_0 < x_1 < \cdots$. Let

$$\begin{aligned} H = \ & \omega \cdot x_1 + x_2, \omega \cdot x_1 + x_6, \omega \cdot x_1 + x_{10}, \ldots, \\ & \omega \cdot x_3 + x_4, \omega \cdot x_3 + x_{12}, \omega \cdot x_3 + x_{20}, \ldots, \\ & \omega \cdot x_5 + x_8, \omega \cdot x_5 + x_{24}, \omega \cdot x_5 + x_{40}, \ldots, \\ & \vdots \end{aligned}$$

Then $H \approx \omega^2$, as it's an ω-sized concatenation of sets order-equivalent to ω.

For any edge $\{\omega \cdot y_1 + y_2, \omega \cdot y_3 + y_4\} \in \binom{H}{2}$ with $\omega \cdot y_1 + y_2 < \omega \cdot y_3 + y_4$, either $y_1 \neq y_3$ or $y_1 = y_3$.

- If $y_1 \neq y_3$, then $y_1 < y_3$ by the ordering of the two elements and $y_2 \neq y_4$ by the construction of H. We also have $y_1 < y_2$, $y_1 < y_4$, and $y_3 < y_4$ by the construction of H. Then either $y_1 < y_2 < y_3 < y_4$, $y_1 < y_3 < y_2 < y_4$, or $y_1 < y_3 < y_4 < y_2$. In each of the three cases, $f_1(y_1, y_2, y_3, y_4) \in Y$, $f_2(y_1, y_3, y_2, y_4) \in Y$, and $f_3(y_1, y_3, y_4, y_2) \in Y$ respectively so $\mathrm{COL}(\{\omega \cdot y_1 + y_2, \omega \cdot y_3 + y_4\}) \in Y$.

- If $y_1 = y_3$, then $y_2 < y_4$ by the ordering of the elements and so $y_1 = y_3 < y_2 < y_4$ by the construction of H. Because $f_4(y_1, y_2, y_4, 1) \in Y$ (note that f_4 "wastes" its 4th argument), $\mathrm{COL}(\{\omega \cdot y_1 + y_2, \omega \cdot y_3 + y_4\}) \in Y$.

In all cases, $\mathrm{COL}(\{\omega \cdot y_1 + y_2, \omega \cdot y_3 + y_4\}) \in Y$ so

$$\mathrm{COL}\left(\binom{H}{2}\right) \subseteq Y$$

with $|Y| = 4$. Because $H \approx \omega^2$ and COL was arbitrary, $T(2, \omega^2) \leqslant 4$.

We now prove $T(2, \omega^2) \geqslant 4$. Let $\mathrm{COL}\colon \binom{\omega^2}{2} \to [4]$ with

$$\mathrm{COL}(\omega \cdot x_1 + x_2, \omega \cdot x_3 + x_4) = \begin{cases} 1 & x_1 < x_2 < x_3 < x_4 \\ 2 & x_1 < x_3 < x_2 < x_4 \\ 3 & x_1 < x_3 < x_4 < x_2 \\ 4 & x_1 = x_3 < x_2 < x_4 \text{ or otherwise} \end{cases}$$

where $\omega \cdot x_1 + x_2 < \omega \cdot x_3 + x_4$. Color 4 could be formatted as simply "otherwise", but the specific part that makes color 4 present in any order-equivalent subset is the $x_1 = x_3 < x_2 < x_4$, as the "otherwise" case can be filtered out without breaking order-equivalence. One such set order-equivalent to ω^2 without any "otherwise" edges would be H from the first part of this proof. We leave it to the reader to show that there exists no 3-homogeneous order-equivalent subset.

□

6 Strong Colorings

We use a concept called *strong colorings* to prove general results about $T(a, \omega^d)$ and beyond. The concept behind strong colorings is built on the ideas of Blass et al. [?]. We motivate the concept by looking at the proof of Example **??**.

The proof of Example **??** used four functions f_1, f_2, f_3, f_4. These functions were specifically chosen to cover H in a way where the color of every edge in H was in the output of some f_1, f_2, f_3, f_4. We note a function that *was not* used:

$f\colon \binom{\omega}{4} \to \binom{\omega^2}{2}$ defined by

$$f(x_1, x_2, x_3, x_4) = \mathrm{COL}(\{\omega \cdot x_1 + x_3, \omega \cdot x_2 + x_3\}).$$

We didn't use f in the lower bound proof because f didn't cover *any* edges in H: we constructed H in a way where distinct copies of ω had distinct finite coefficients. Since $x_1 \neq x_2$, the elements $\omega \cdot x_1 + x_3$ and $\omega \cdot x_2 + x_3$ couldn't both be from H no matter the values of x_1, x_2, and x_3.

We could have designed H differently to require more than 4 functions to cover it, but that would have weakened the upper bound result of Example **??**. We define a notion of colorings that f_1, f_2, f_3, f_4 qualify but f does not. We also show how to count these colorings, and how these colorings are linked to big Ramsey degrees.

Definition 15. For integers $a, d, k \geqslant 0$, we say there are a *elements* in each edge of $\binom{\omega^d \cdot k}{a}$. For $1 \leqslant q \leqslant a$, we denote each element as

$$\omega^d \cdot y_q + \omega^{d-1} \cdot x_{q,d-1} + \omega^{d-2} \cdot x_{q,d-2} + \cdots + \omega^1 \cdot x_{q,1} + x_{q,0}.$$

This leads to a y_q variables in an edge of $\binom{\omega^d \cdot k}{a}$, each an integer ranging from $0 \leqslant y_q < k$ indexed with $1 \leqslant q \leqslant a$. We also have $a \cdot d$ x_{qn} variables, indexed with $1 \leqslant q \leqslant a$ and $0 \leqslant n < d$. Note that each x_{qn} could be any nonnegative integer.

A *strong coloring* is first defined as an assignment of the a y_q variables with nonnegative integers $0 \leqslant y_q < k$ for $1 \leqslant q \leqslant a$. Then, the x_{qn} variables in $\binom{\omega^d}{a}$ with $<$ or $=$ signs between them are permuted in a way that satisfies the below criteria.

1. If $d \geqslant 1$, $x_{i0} < x_{j0}$ for all $i < j$ (the element indices are ordered by their lowest-dimension variable). If $d = 0$, $y_i < y_j$ for all $i < j$.

2. $y_i \neq y_j \rightarrow x_{in} \neq x_{jn}$ for all n (Elements that have a different y value have all different x values).

3. $x_{qa} < x_{qb}$ for all $a > b$ (the high-dimension variables of each element are strictly less than the low-dimension variables).

4. $x_{ia} = x_{jb} \rightarrow a = b$ (only variables with the same dimension can be equal).

5. $x_{in} \neq x_{jn} \rightarrow x_{i,n-1} \neq x_{j,n-1}$ for all $n > 0$ (elements that differ in a high-dimension variable differ in all lower-dimension variables).

An example of a strong coloring for the expression $\binom{\omega^2}{2}$ would be

$$y_1 = 0,\ y_2 = 0,\ x_{11} = x_{21} < x_{10} < x_{20}.$$

Note that because $k = 1$ in the example, each y_q can only be assigned to 0.

We say that two strong colorings are equivalent if and only if their y_q values are all the same and they are logically equivalent; that is, identical up to permutation of variables within equivalence classes.

Definition 16. The *size* of a strong coloring as how many equivalence classes its x variables form: for example, $x_{11} = x_{21} < x_{10} < x_{20}$ would have size $p = 3$ regardless of its y variables. Clearly a strong coloring's size p can be no larger than $a \cdot d$, how many x variables $\binom{\omega^d}{a}$ has.

Definition 17.

1. $P_p\left(a, \omega^d \cdot k\right)$ is the number of strong colorings with size p there are for $\binom{\omega^d \cdot k}{a}$.

2. $P\left(a, \omega^d \cdot k\right)$ is the number of strong colorings there are for $\binom{\omega^d \cdot k}{a}$ regardless of size. It can be calculated as
$$\sum_{p=0}^{a \cdot d} P_p(a, \omega^d \cdot k).$$

Definition 18. We say that an edge *satisfies* a strong coloring if its y_q variables match and the x_{qn} variables match the ordering of the strong coloring. Note that some edges might not satisfy any strong colorings.

7 $T(a, \omega^d) = P(a, \omega^d)$

This section is devoted to the case where $k = 1$. When $k = 1$, each y_q in a strong coloring is forced to be 0. Then all y_q values are the same, so criterion **??** of Definition **??** is always satisfied. In this section, our proofs focus only on the permutation of the x_{qn} variables and the remaining criteria. When values for y_q are not specified, in this section they are assumed to be all 0 by default.

This section's aim is to show equality between big Ramsey degrees and strong colorings. To motivate this, we start with a recurrence that counts strong colorings.

Lemma 19. *For integers $a, d \geqslant 0$,*

$$P_p\left(a, \omega^d\right) = \begin{cases} 0 & d = 0 \land a \geqslant 2 \\ 1 & a = 0 \land p = 0 \\ 0 & a = 0 \land p \geqslant 0 \\ 1 & d = 0 \land a = 1 \land p = 0 \\ 0 & d = 0 \land a = 1 \land p \geqslant 1 \\ 1 & d = 1 \land a \geqslant 1 \land a = p \\ 0 & d = 1 \land a \geqslant 1 \land a \neq p \\ \sum\limits_{j=1}^{a} \sum\limits_{i=0}^{p-1} \binom{p-1}{i} P_i\left(j, \omega^{d-1}\right) P_{p-1-i}\left(a-j, \omega^d\right) & d \geqslant 2 \land a \geqslant 1 \end{cases}$$

Proof. First suppose $a \geqslant 2$ and $d = 0$. Because $k = 1$, we need $y_1, y_2 = 0$. But by criterion **??** of Definition **??**, since $d = 0$ we need $y_1 < y_2$, so no strong colorings are possible regardless of size p. This aligns with the first case of the result.

Suppose $a = 0$: now there are no y_q variables, and since $a \cdot d = 0$, there are no x variables to permute. Therefore criteria are vacuously satisfied. There is only one strong coloring, and it has size $p = 0$. This aligns with the second and third cases of the result.

When both $d = 0$ and $a \leqslant 1$, criterion **??** of Definition **??** can be vacuously satisfied with all $y_q = 0$ with because there are only one or zero y_q variables. Again, because $a \cdot d = 0$, there are no x variables to permute so there is only one strong coloring with size $p = 0$, aligning with the fourth and fifth cases of the result.

Now suppose $a \geqslant 1$ and $d = 1$. To ensure criteria **??** of Definition **??**, each of the a $x_{q,0}$ variables can only form one strong coloring $x_{1,0} < x_{2,0} < \ldots < x_{a,0}$ with size a so $P_a\left(a, \omega^d\right) = 1$ and $P_p\left(a, \omega^d\right) = 0$ for $p \neq a$. This aligns with the sixth and seventh cases of the result.

Finally, consider $a \geqslant 1, d \geqslant 2$. We prove the final case of our result by showing the process for combining strong colorings described below creates all possible strong colorings of an expression.

For arbitrary integers $a \geqslant 1$, $d \geqslant 2$, and $p \geqslant 0$, let $1 \leqslant j \leqslant a$ and $0 \leqslant i \leqslant p-1$ be integers. As we proceed through the process, we work with an example of $a = 4$, $d = 5$, $p = 13$, $j = 2$, and $i = 5$.

We create
$$\binom{p-1}{i} P_i\left(j, \omega^{d-1}\right) P_{p-1-i}\left(a-j, \omega^d\right)$$
strong colorings of size p by combining $P_i\left(j, \omega^{d-1}\right)$ strong colorings with size i and $P_{p-1-i}\left(a-j, \omega^d\right)$ strong colorings with size $p-1-i$, creating $\binom{p-1}{i}$ strong colorings from each pair.

Let τ_1 represent one of the $P_i\left(j, \omega^{d-1}\right)$ strong colorings of $\binom{\omega^{d-1}}{j}$ with size i, and τ_2 represent one of the $P_{p-1-i}\left(a-j, \omega^d\right)$ strong colorings of $\binom{\omega^d}{a-j}$ with size $p-1-i$. In our example, let
$$\begin{aligned}
&\tau_1\colon x_{1,3} = x_{2,3} < x_{1,2} = x_{2,2} < x_{1,1} = x_{2,1} < x_{1,0} < x_{2,0} \\
&\tau_2\colon x_{1,4} = x_{2,4} < x_{1,3} = x_{2,3} < x_{1,2} = x_{2,2} < x_{2,1} < x_{1,1} < x_{1,0} < x_{2,0}.
\end{aligned}$$

Then we can combine each τ_1 and τ_2 to form $\binom{p-1}{i}$ unique new strong colorings of size p: Reindex each variable $x_{q,n}$ of τ_2 to $x_{q+j,n}$, and permute the equivalence classes of the strong colorings together, preserving each strong coloring's original ordering of its own equivalence classes: there are $\binom{p-1}{i}$ ways to do this. In our example, after reindexing τ_2 we have
$$\begin{aligned}
&\tau_1\colon x_{1,3} = x_{2,3} < x_{1,2} = x_{2,2} < x_{1,1} = x_{2,1} < x_{1,0} < x_{2,0} \\
&\tau_2\colon x_{3,4} = x_{4,4} < x_{3,3} = x_{4,3} < x_{3,2} = x_{4,2} < x_{4,1} < x_{3,1} < x_{3,0} < x_{4,0}
\end{aligned}$$
and one of the $\binom{12}{5}$ permutations is
$$\begin{aligned}
x_{3,4} = x_{4,4} < x_{1,3} = x_{2,3} < x_{1,2} = x_{2,2} < x_{3,3} = x_{4,3} < x_{3,2} = x_{4,2} \\
< x_{1,1} = x_{2,1} < x_{4,1} < x_{1,0} < x_{3,1} < x_{3,0} < x_{2,0} < x_{4,0}.
\end{aligned}$$
This new strong coloring likely breaks criterion **??** of Definition **??**; for each $1 \leqslant q \leqslant a$, reindex each $x_{q,n}$ according to where $x_{q,0}$ is in the ordering of all $x_{i,0}$. In our example, we have $x_{1,0} < x_{3,0} < x_{2,0} < x_{4,0}$; after swapping indices 2 and 3 to enforce criterion **??** we have
$$\begin{aligned}
x_{2,4} = x_{4,4} < x_{1,3} = x_{3,3} < x_{1,2} = x_{3,2} < x_{2,3} = x_{4,3} < x_{2,2} = x_{4,2} \\
< x_{1,1} = x_{3,1} < x_{4,1} < x_{1,0} < x_{2,1} < x_{2,0} < x_{3,0} < x_{4,0}.
\end{aligned}$$

There are now $d \cdot (a-j) + (d-1) \cdot j = a \cdot d - j$ variables in the strong coloring. There are j variables of the form $x_{q_i,d-1}$ for $1 \leqslant i \leqslant j$ that are not in the strong coloring yet; insert one equivalence class $x_{q_1,d-1} = x_{q_2,d-1} = \cdots = x_{q_j,d-1}$ at the front of the new strong coloring, bringing its size to p. We insert $x_{1,4} = x_{3,4}$ in our example to get
$$\begin{aligned}
x_{1,4} = x_{3,4} < x_{2,4} = x_{4,4} < x_{1,3} = x_{3,3} < x_{1,2} = x_{3,2} < x_{2,3} = x_{4,3} < x_{2,2} = x_{4,2} \\
< x_{1,1} = x_{3,1} < x_{4,1} < x_{1,0} < x_{2,1} < x_{2,0} < x_{3,0} < x_{4,0}.
\end{aligned}$$

Each strong coloring is unique by the τ_1 and τ_2 used to create it because the process is invertible: we can remove the leading equivalence class and separate the remaining variables into τ_1 and τ_2 by whether their indices were in the leading equivalence class and reindexing. Here, τ_1 corresponds to indices 1 and 3 (not including the leading equivalence class) and is **bolded**, and τ_2 corresponds to indices 2 and 4 and is underlined.

$$x_{1,4} = x_{3,4} < \underline{x_{2,4} = x_{4,4}} < \mathbf{x_{1,3} = x_{3,3}} < \mathbf{x_{1,2} = x_{3,2}} < \underline{x_{2,3} = x_{4,3}} < \underline{x_{2,2} = x_{4,2}}$$
$$< \mathbf{x_{1,1} = x_{3,1}} < \underline{x_{4,1}} < \mathbf{x_{1,0}} < \underline{x_{2,1}} < \underline{x_{2,0}} < \mathbf{x_{3,0}} < \underline{x_{4,0}}.$$

Each strong coloring created by this process has the properties described by Definition **??**: Because the high-dimension equivalence class $x_{q_1,d-1} = x_{q_2,d-1} = \cdots = x_{q_j,d-1}$ was added at the start of the strong coloring, the high-dimension coefficients of each term are smaller than the low-dimension coefficients. Criterion **??** is satisfied by reindexing the variables. The remaining criteria are satisfied because τ_1 and τ_2 satisfied them and their internal orders were preserved in permuting the equivalence classes. Therefore this process does not overcount strong colorings.

Every strong coloring of $\binom{\omega^d}{a}$ is counted by this process: each can be mapped to some τ_1 and τ_2 that create it by a similar argument to proving that the process creates unique strong colorings. Every strong coloring of $\binom{\omega^d}{a}$ must have a leading equivalence class of $x_{q_1,d-1} = x_{q_2,d-1} = \cdots = x_{q_j,d-1}$ to satisfy Definition **??** (the equivalence class might only contain one variable); taking only the variables $x_{q,n}$ with indices appearing in that equivalence class (but not those variables in the equivalence class itself) forms τ_1, a strong coloring for $\binom{\omega^{d-1}}{j}$. The variables with q indices not in the equivalence class form τ_2, a strong coloring for $\binom{\omega^d}{a-j}$. The original strong coloring of $\binom{\omega^d}{a}$ is counted by interleaving τ_1 with τ_2 and inserting the leading equivalence class of $x_{q_1,d-1} = x_{q_2,d-1} = \cdots = x_{q_j,d-1}$. Therefore the final case of the result holds. $\square$

7.1 $T(a, \omega^d) \leqslant P(a, \omega^d)$

We use the following lemma to show that strong colorings bound big Ramsey degrees above.

Lemma 20. *For integers $a, d \geqslant 0$ and $N \approx \omega$, there exists some $H \subseteq \omega^d$ with $H \approx \omega^d$ where for all $e \in \binom{H}{a}$, e satisfies a strong coloring of $\binom{\omega^d}{a}$ and each coefficient in e is contained in N.*

Proof. Because $N \approx \omega$, we can index it $x_0, x_1, x_2, \ldots$ with $x_0 < x_1 < x_2 < \cdots$. We proceed by induction on d. When $d = 0$, $\omega^0 \approx 1$ and so $H = \{x_0\}$ suffices. When $d = 1$, $\omega^1 \approx \omega$ so $H = N$ suffices.

For $d \geqslant 2$, partition $N \setminus \{x_0\}$ into infinite sets order-equivalent to ω:

$$\begin{aligned} X_0 &= \{x_1, x_3, x_5, \ldots\} \\ X_1 &= \{x_2, x_{6,10}\,, \ldots\} \\ X_2 &= \{x_4, x_{12}, x_{20}, \ldots\} \\ X_3 &= \{x_8, x_{24}, x_{40}, \ldots\} \\ &\vdots \end{aligned}$$

Apply the inductive hypothesis on X_i for all $i \geqslant 1$, yielding $S_i \approx \omega^{d-1}$ for all $i \geqslant 1$. Then for all $i \geqslant 1$, for all $e \in \binom{S_i}{a}$, e satisfies a strong coloring of $\binom{\omega^{d-1}}{a}$ and each coefficient of e is contained in X_i. For all i, let $S_i = \{y_{i,0}, y_{i,1}, \ldots\}$. Then let

$$\begin{aligned} H = \ & \omega^{d-1}x_1 + y_{1,0}, \omega^{d-1}x_1 + y_{1,1}, \omega^{d-1}x_1 + y_{1,2}, \ldots, (\omega^{d-1} \text{ times}) \\ & \omega^{d-1}x_3 + y_{2,0}, \omega^{d-1}x_3 + y_{2,1}, \omega^{d-1}x_3 + y_{2,2}, \ldots, (\omega^{d-1} \text{ times}) \\ & \omega^{d-1}x_5 + y_{3,0}, \omega^{d-1}x_5 + y_{3,1}, \omega^{d-1}x_5 + y_{3,2}, \ldots, (\omega^{d-1} \text{ times}) \\ & \vdots \ (\omega \text{ times}) \end{aligned}$$

Then $H \approx \omega^d$. For any edge e in $\binom{S}{a}$, index its variables to satisfy criterion **??** of Definition **??** (this is possible because all low-dimension coefficients are distinct in H). Criterion **??** is satisfied inductively for variables with dimensions lower than $d-1$. Because $\min X_i = x_{2^i}$ for all i and $2i - 1 < 2^i$ for all integers $i \geqslant 1$, $x_{2i-1} < x$ for all $x \in X_i$ so criterion **??** is satisfied by e. Because X_0 is disjoint with all X_i with $i \geqslant 1$, criterion **??** is satisfied for variables with dimension $d-1$ and by induction, it's satisfied for lower dimensions. Because X_i is disjoint with X_j for all $i \neq j$, elements that differ in variables with dimension $d-1$ differ in all lower-dimension variables. The induction with the previous statement satisfies criterion **??**. Therefore e satisfies a strong coloring of $\binom{\omega^d}{a}$. The coefficients in e are contained in N by the construction of H from N. $\square$

Theorem 21. *For integers $a, d \geqslant 0$, $T\left(a, \omega^d\right) \leqslant P\left(a, \omega^d\right)$.*

Proof. Let $E = \binom{\omega^d}{a}$ and

$$\mathrm{COL}\colon E \to [b]$$

be an arbitrary coloring of E for some nonnegative integer b.

Enumerate the strong colorings of E from τ_1 to $\tau_{P(a,\omega^d)}$. The maximum size of any strong coloring of E is $a \cdot d$. For each τ_i, let

$$f_i\colon \binom{\omega}{a \cdot d} \to E$$

where if τ_i has size p, f_i maps X to the unique $e \in E$ where e satisfies τ_i and the p equivalence classes of e are made up of the p least elements of X. For example, one strong coloring of $\binom{\omega^2}{2}$ is

$$x_{11} = x_{21} < x_{10} < x_{20}.$$

The corresponding f_i would be $f_i\colon \binom{\omega}{4} \to \binom{\omega^2}{2}$ with

$$f_i(x_1, x_2, x_3, x_4) = \mathrm{COL}(\{\omega \cdot x_1 + x_2, \omega \cdot x_1 + x_3\})$$

where $x_1 < x_2 < x_3 < x_4$. Note that x_4 is "wasted" by f_i – this is because the example strong coloring has size 3, but the maximum sized strong coloring for $\binom{\omega^2}{2}$ has size 4.

Then, define $\mathrm{COL}'\colon \binom{\omega}{a \cdot d} \to [b]^{P(a,\omega^d)}$ with

$$\mathrm{COL}'(X) = (f_1(X), f_2(X), \ldots, f_{P(a,\omega^d)}(X))$$

and apply Theorem **??** to find some $N \approx \omega$ where

$$\mathrm{COL}'\left(\binom{N}{a \cdot d}\right)$$

expresses only one tuple Y containing $|Y| = P(a, \omega^d)$ colors.

Apply Lemma **??** to find some $H \approx \omega^d$ with the properties listed in Lemma **??**. Now we claim

$$\mathrm{COL}\left(\binom{H}{a}\right)$$

expresses at most $P(a, \omega^d)$ colors.

By Lemma **??**, each element $e \in \binom{H}{a}$ satisfies a strong coloring of E. Then for some arbitrary edge e, let e satisfy τ_i with size $p \leqslant a \cdot d$. Then take the p unique values in e, and if necessary, insert any new larger nonnegative integers from N to form a set of $a \cdot d$ values; denote this $X \in \binom{N}{a \cdot d}$. $\mathrm{COL}'(X) = Y$ so by the definition of COL', $\mathrm{COL}(e) \in Y$. Because $|Y| = P(a, \omega^d)$, $T(a, \omega^d) \leqslant P(a, \omega^d)$. □

7.2 $T(a, \omega^d) \geqslant P(a, \omega^d)$

Theorem 22. *For integers $a, d \geqslant 0$, $T\left(a, \omega^d\right) \geqslant P\left(a, \omega^d\right)$.*

Proof. If $P(a, \omega^d) = 0$, this is satisfied vacuously because $T(a, \omega^d) \geqslant 0$. Now suppose $P(a, \omega^d) \geqslant 1$. Let $E = \binom{\omega^d}{a}$. Note that all strong colorings of E are disjoint from each other. That is, for any edge $e \in E$, if e satisfies τ', then it does not satisfy any nonequivalent strong coloring of E. This is because if e were to satisfy two strong colorings τ_1 and τ_2, then τ_1 and τ_2 must share the same equivalence classes and order, so the strong colorings must be equivalent. Therefore, we can index them $\tau_1 \ldots \tau_{P(a,\omega^d)}$ and construct a coloring $\mathrm{COL}\colon E \to [P(a, \omega^d)]$ with

$$\mathrm{COL}(e) = \begin{cases} i & e \text{ satisfies } \tau_i \\ 1 & \text{otherwise} \end{cases}$$

Similar to Example ??, our coloring has two ways to output color 1, both through satisfaction of τ_1 and through the catch-all case. The part that forces color 1 to be present in all order-equivalent subsets is the satisfaction of τ_1.

There is no ω^d-$(P(a,\omega^d)-1)$-homogeneous set: for all $H \approx \omega^d$ and for every strong coloring τ of E, there exists some $e \in \binom{H}{a}$ that satisfies τ.

For arbitrary $H \approx \omega^d$ and τ, we find z_{qn} where

$$\{\omega^{d-1} z_{1,d-1} + \cdots + \omega^1 z_{1,1} + z_{1,0}, \ldots, \omega^{d-1} z_{a,d-1} + \cdots + \omega^1 z_{a,1} + z_{a,0}\}$$

satisfies τ.

We do this by assigning values to each z_{qn} according to where the equivalence class that contains x_{qn} is found in τ, moving left to right in τ's permutation. By criterion ?? of Definition ??, each z_{qn} is assigned before $z_{q,n-1}$. As we do this, we ensure that if the leftmost unassigned value in τ is z_{qn}, then

$$\{\omega^{d-1} z_{q,d-1} + \cdots + \omega^{n+1} z_{q,n+1} + \omega^n c_n + \omega^{n-1} c_{n-1} + \cdots \\ + \omega^1 c_1 + c_0 \mid c_i \in \omega\} \cap H \approx \omega^{n+1}.$$

By criterion ?? of Definition ??, the leftmost variable in τ must be $x_{q,d-1}$. Before any values are assigned, it is clear that

$$\{\omega^{d-1} c_{d-1} + \cdots + \omega^1 c_1 + c_0 \mid c_i \in \omega\} = \omega^d$$

and because $H \subseteq \omega^d$, $\omega^d \cap H = H \approx \omega^d$.

By criterion ?? of Definition ??, all variables in an equivalence class must have the same dimension d. Let the leftmost equivalence class in τ be $x_{q_1,n} = x_{q_2,n} = \cdots = x_{q_m,n}$. By criterion ?? each $x_{q_i,\ell}$ for $1 \leqslant i \leqslant m$ and $\ell > n$ appeared to the left of this equivalence class and has already been assigned a value, and by criterion ?? the values for each dimension are equal: for all $\ell > n$ and $1 \leqslant i \leqslant m$, $z_{q_i,\ell} = z_{q_1,\ell}$.

By our previous steps,

$$\{\omega^{d-1} z_{q_1,d-1} + \cdots + \omega^{n+1} z_{j,q_1,n+1} + \omega^n c_n + \omega^{n-1} c_{n-1} + \cdots \\ + \omega^1 c_1 + c_0 \mid c_i \in \omega\} \cap H \approx \omega^{n+1}.$$

Then there exists some value z' where

$$\{\omega^{d-1} z_{q_1,d-1} + \cdots + \omega^{n+1} z_{q_1,n+1} + \omega^n z' + \omega^{n-1} c_{n-1} + \cdots \\ + \omega^1 c_1 + c_0 \mid c_i \in \omega\} \cap H \approx \omega^n$$

where z' is greater than all previously assigned (and therefore finite) z_{qn} values. Then for $1 \leqslant i \leqslant m$, assign $z_{q_i,n}$ to be z'.

We can repeat this process to find z_{qn} that satisfy every strong coloring of E for arbitrary $H \approx \omega^d$. Therefore for all $H \approx \omega^d$, $|\operatorname{COL}(\binom{S}{a})| \geqslant P(a,\omega^d)$ so $T(a,\omega^d) \geqslant P\left(a,\omega^d\right)$. $\square$

7.3 $T(a, \omega^d) = P(a, \omega^d)$

Theorem 23. *For all nonnegative integers a, d, $T(a, \omega^d) = P(a, \omega^d)$.*

Proof. By Theorem ??, $T(a, \omega^d) \leqslant P(a, \omega^d)$. By Theorem ??, $T(a, \omega^d) \geqslant P(a, \omega^d)$. The result follows. $\square$

8 $T(a, \omega^d \cdot k) = P(a, \omega^d \cdot k)$

We now use the theory we developed for the case $k = 1$ to prove results for arbitrary k. We first extend the recurrence from Lemma ??.

Lemma 24. *For integers $a, d, k \geqslant 0$,*

$$P_p\left(a, \omega^d \cdot k\right) = \begin{cases} 0 & d = 0 \wedge a > k \\ 1 & a = 0 \wedge p = 0 \\ 0 & a = 0 \wedge p \geqslant 0 \\ \binom{k}{a} & d = 0 \wedge 1 \leqslant a \leqslant k \wedge p = 0 \\ 0 & d = 0 \wedge 1 \leqslant a \leqslant k \wedge p \geqslant 1 \\ k^a & d = 1 \wedge a \geqslant 1 \wedge a = p \\ 0 & d = 1 \wedge a \geqslant 1 \wedge a \neq p \\ k \sum_{j=1}^{a} \sum_{i=0}^{p-1} \binom{p-1}{i} P_i\left(j, \omega^{d-1}\right) P_{p-1-i}\left(a - j, \omega^d k\right) & d \geqslant 2 \wedge a \geqslant 1 \end{cases}$$

Proof. First suppose $a > k$ and $d = 0$. We need $0 \leqslant y_q < k$ for all y_q, so at most k unique values of y_q variables are possible. But by criterion ?? of Definition ??, since $d = 0$ we need a unique values of y_q variables, so no strong colorings are possible regardless of size p. This aligns with the first case of the result.

Suppose $a = 0$: now there are no y_q variables, and since $a \cdot d = 0$, there are no x variables to permute so all criteria are vacuously satisfied. Because there are no y or x variables, there is only one strong coloring, and it has size $p = 0$. This aligns with the second and third cases of the result.

When both $d = 0$ and $a \leqslant k$, criterion ?? of Definition ?? can be satisfied with the assignments to the a y_q variables being any permutation of a unique values out of k possible integer values. This leads to $\binom{k}{a}$ feasible combinations. Again, because $a \cdot d = 0$, there are no x variables to permute so there are $\binom{k}{a}$ empty strong colorings with size $p = 0$, aligning with the fourth and fifth cases of the result.

Now suppose $a \geqslant 1$ and $d = 1$. To ensure criteria ?? of Definition ??, each of the a $x_{q,0}$ variables can only form one permutation $x_{1,0} < x_{2,0} < \ldots < x_{a,0}$ with size a. Because all x values are distinct and $d = 1$, the y_q values are not restricted by any criteria so each of the a variables can be any of the k integers. Therefore $P_a\left(a, \omega^d\right) = k^a$ and $P_p\left(a, \omega^d\right) = 0$ for $p \neq a$. This aligns with the sixth and seventh cases of the result.

Finally, consider $a \geqslant 1, d \geqslant 2$. We prove the final case of our result by showing the process for combining strong colorings described below creates all possible strong colorings of an expression.

For arbitrary integers $a \geqslant 1$, $d \geqslant 2$, $k \geqslant 0$, and $p \geqslant 0$, let $1 \leqslant j \leqslant a$ and $0 \leqslant i \leqslant p-1$ be integers.

We create

$$k\binom{p-1}{i} P_i\left(j, \omega^{d-1}\right) P_{p-1-i}\left(a-j, \omega^d \cdot k\right)$$

strong colorings of size p by combining $P_i\left(j, \omega^{d-1}\right)$ strong colorings with size i and $P_{p-1-i}\left(a-j, \omega^d \cdot k\right)$ strong colorings with size $p-1-i$, with $k\binom{p-1}{i}$ new strong colorings for each pair.

with each strong coloring having j elements equal in their highest-dimension variable with those j elements having a combined size (i.e. count of distinct variables) of i.

Let τ_1 represent one of the $P_i\left(j, \omega^{d-1}\right)$ strong colorings of $\binom{\omega^{d-1}}{j}$ with size i, and τ_2 represent one of the $P_{p-1-i}\left(a-j, \omega^d\right)$ strong colorings of $\binom{\omega^d \cdot k}{a-j}$ with size $p-1-i$.

Then we can combine each τ_1 and τ_2 to form $k\binom{p-1}{i}$ unique new strong colorings of size p: Reindex τ_2, permute the equivalence classes, and insert a leading equivalence class $x_{q_1,d-1} = x_{q_2,d-1} = \cdots = x_{q_j,d-1}$ as in the proof of Lemma **??**. This leads to $\binom{p-1}{i}$ new permutations of the x variables.

Because τ_1 was a strong coloring for $\binom{\omega^{d-1}}{j}$, each of its y values were 0. Now that we are creating a strong coloring for $\binom{\omega^d \cdot k}{a}$, we can choose the y coefficients to be composed of values between 0 and $k-1$. By criterion **??** of Definition **??**, because all elements from τ_1 are bound together in a leading high-dimension equivalence class, they must all have equal y values. This leads to k options for these y values; with the options of permuting the x variables, $k\binom{p-i}{i}$ ways to create a new strong coloring.

For the new strong coloring's y_q values, we assign each element originally from τ_2 with its original y value (likely at a different index due to reindexing). Then, the remaining elements from τ_1 are given all the same y value from one of the k options.

Each strong coloring is unique by the τ_1 and τ_2 used to create it because the process is invertible: we can remove the leading equivalence class and separate the remaining variables into τ_1 and τ_2 by whether their indices were in the leading equivalence class and reindexing. The y values for τ_2 can be found from the strong coloring's y values after reversing the index change, and the y values for τ_1 are all 0.

We claim each strong coloring created by this process has the properties described by Definition **??**: Because the high-dimension equivalence class $x_{q_1,d-1} = x_{q_2,d-1} = \cdots = x_{q_j,d-1}$ was added at the start of the strong coloring, the high-dimension coefficients of each term are smaller than the low-dimension coefficients. Criterion **??** is satisfied by reindexing the variables. Criterion **??** is met by assigning all elements from τ_1 the same y value. The remaining criteria are satisfied because τ_1 and τ_2 satisfied them and their internal orders were preserved in permuting the equivalence classes. Therefore this process does not overcount strong colorings.

We also claim that every strong coloring of $\binom{\omega^d \cdot k}{a}$ is counted by this process: each can be mapped to some τ_1 and τ_2 that create it by a similar argument to proving that the process creates unique strong colorings. Every strong coloring of $\binom{\omega^d \cdot k}{a}$ must have a leading equivalence class of $x_{q_1,d-1} = x_{q_2,d-1} = \cdots = x_{q_j,d-1}$ to satisfy Definition **??** (the equivalence class might only contain one variable); taking only the variables $x_{q,n}$ with indices appearing in that equivalence class (but not those variables in the equivalence class itself) with all-zero y values forms τ_1, a strong coloring for $\binom{\omega^{d-1}}{j}$. The variables with q indices not in the equivalence class with their y values form τ_2, a strong coloring for $\binom{\omega^d \cdot k}{a-j}$. The original strong coloring of $\binom{\omega^d \cdot k}{a}$ was counted by interleaving τ_1 with τ_2 and inserting the leading equivalence class of $x_{q_1,d-1} = x_{q_2,d-1} = \cdots = x_{q_j,d-1}$. Therefore the final case of the result holds. □

8.1 $T(a, \omega^d \cdot k) \leqslant P(a, \omega^d \cdot k)$

Lemma 25. *For nonnegative integers a, d, k and $N \approx \omega$, there exists some $H \subseteq \omega^d \cdot k$, $H \approx \omega^d \cdot k$ where for all $e \in \binom{H}{a}$, e satisfies a strong coloring of $\binom{\omega^d \cdot k}{a}$ and each coefficient in e is contained in N.*

Proof. Because $N \approx \omega$, we can index it $x_1, x_2, x_3, \ldots$ with $x_1 < x_2 < x_3 < \cdots$.

If $d = 0$, $H = \{x_1, x_2, \ldots, x_k\}$ suffices.

If $d \geqslant 1$, we can first apply Lemma **??** with N to attain some $H' \approx \omega^{d+1}$ with the listed properties. Then, let H be the first k copies of ω^d within H': formally,

$$H = \{\omega^d y + \omega^{d-1} x_{d-1} + \ldots + \omega^1 x_1 + x_0 \in H' \mid y < k\}.$$

Because the edges of H' satisfied criterion **??** of Definition **??** at dimension $n = d + 1$, the edges of H satisfy criterion **??**. The remaining criteria are satisfied directly because H' satisfied them. □

Theorem 26. *For integers $a, d, k \geqslant 0$, $T\left(a, \omega^d\right) \leqslant P\left(a, \omega^d\right)$.*

Proof. Let $E = \binom{\omega^d \cdot k}{a}$ and

$$\mathrm{COL}\colon E \to [b]$$

be an arbitrary coloring of E for some nonnegative integer b.

Enumerate the strong colorings of E from τ_1 to $\tau_{P(a,\omega^d \cdot k)}$. The maximum size of any strong coloring of E is $a \cdot d$. For each τ_i, let

$$f_i\colon \binom{\omega}{a \cdot d} \to E$$

where if τ_i has size p, f_i maps X to the unique $e \in E$ where e satisfies τ_i and the p equivalence classes of e are made up of the p least elements of X. For example, one strong coloring of $\binom{\omega^2 \cdot 2}{2}$ is

$$y_1 = 0,\ y_2 = 1,\ x_{11} < x_{21} < x_{20} < x_{10}.$$

The corresponding f_i would be $f_i\colon \binom{\omega}{4} \to \binom{\omega^2 \cdot 2}{2}$ with

$$f_i(x_1, x_2, x_3, x_4) = \mathrm{COL}(\{\omega^2 \cdot 0 + \omega \cdot x_1 + x_4, \omega^2 \cdot 1 + \omega \cdot x_2 + x_3\})$$

where $x_1 < x_2 < x_3 < x_4$. Note that the y_q values are used directly in of the definition of f_i – for the strong coloring with identical x permutation but $y_0 = 1$ and $y_1 = 0$, the y_q coefficients would be swapped.

Then, define $\mathrm{COL}'\colon \binom{\omega}{a\cdot d} \to [b]^{P(a,\omega^d\cdot k)}$ with

$$\mathrm{COL}'(X) = (f_1(X), f_2(X), \ldots, f_{P(a,\omega^d\cdot k)}(X))$$

and apply Theorem **??** to find some $N \approx \omega$ where

$$\mathrm{COL}'\left(\binom{N}{a\cdot d}\right)$$

expresses only one tuple Y containing $|Y| = P(a, \omega^d \cdot k)$ colors.

Apply Lemma **??** to find some $H \approx \omega^d \cdot k$ with the properties listed in Lemma **??**. Now we claim

$$\mathrm{COL}\left(\binom{H}{a}\right)$$

expresses at most $P(a, \omega^d \cdot k)$ colors.

By Lemma **??**, each element $e \in \binom{H}{a}$ satisfies a strong coloring of E. Then for some arbitrary edge e, let e satisfy τ_i with size $p \leqslant a \cdot d$. Then take the p unique values in e, and if necessary, insert any new larger nonnegative integers from N to form a set of $a \cdot d$ values; denote this $X \in \binom{N}{a\cdot d}$. $\mathrm{COL}'(X) = Y$ so by the definition of COL', $\mathrm{COL}(e) \in Y$. Because $|Y| = P(a, \omega^d \cdot k)$, $T(a, \omega^d \cdot k) \leqslant P(a, \omega^d \cdot k)$. □

8.2 $T(a, \omega^d \cdot k) \geqslant P(a, \omega^d \cdot k)$

Theorem 27. *For nonnegative integers a, d, k, $T\left(a, \omega^d \cdot k\right) \geqslant P\left(a, \omega^d \cdot k\right)$.*

Proof. If $P(a, \omega^d \cdot k) = 0$, this is satisfied vacuously because $T(a, \omega^d \cdot k) \geqslant 0$. Suppose $P(a, \omega^d \cdot k) \geqslant 1$. Let $E = \binom{\omega^d\cdot k}{a}$. Note that all strong colorings of E are disjoint from each other. That is, for any edge $e \in E$, if e satisfies τ', then it does not satisfy any nonequivalent strong coloring of E. This is because if e were to satisfy two strong colorings τ_1 and τ_2, then τ_1 and τ_2 must share the same y_q values, equivalence classes and order, so the strong colorings must be equivalent. Therefore, we can index them $\tau_1 \ldots \tau_{P(a,\omega^d)}$ and construct a coloring $\mathrm{COL}\colon E \to [P(a, \omega^d \cdot k)]$ with

$$\mathrm{COL}(e) = \begin{cases} i & e \text{ satisfies } \tau_i \\ 1 & \text{otherwise} \end{cases}$$

Similar to Example ??, our coloring has two ways to output color 1, both through satisfaction of τ_1 and through the catch-all case. The part that forces color 1 to be present in all order-equivalent subsets is the satisfaction of τ_1.

For arbitrary $H \approx \omega^d \cdot k$ and τ, we find y_q and z_{qn} variables where

$$\{\omega^d y_1 + \omega^{d-1} z_{1,d-1} + \cdots + \omega^1 z_{1,1} + z_{1,0}, \ldots, \omega^d y_a + \omega^{d-1} z_{a,d-1} + \cdots + \omega^1 z_{a,1} + z_{a,0}\}$$

satisfies τ.

Given an arbitrary $H \approx \omega^d \cdot k$ and τ, we first separate H into k ordered sets by the leading coefficient, each order-equivalent to ω^d.

Then, if there are equivalence classes in τ, using the process formally described in the proof of Theorem ??, we consider the leading equivalence class of τ. By criterion ?? of Definition ??, all variables in that equivalence class must come from same set order-equivalent to ω^d. We assign a finite value to that equivalence class, and move to the next class with a potentially different y value, using the assigned finite value as a lower bound for the next one. We can repeat this process to find z_{qn} that satisfy every strong coloring of E for arbitrary $H \approx \omega^d$. Then, we can assign the y_q variables directly as the y variables in τ.

If there are no equivalence classes in τ (it has size $p = 0$), we can simply assign the y_q variables directly as the y variables in τ.

Therefore for all $H \approx \omega^d$, $|\,\mathrm{COL}\left(\binom{H}{a}\right)| \geqslant P(a, \omega^d \cdot k)$ so $T(a, \omega^d \cdot k) \geqslant P\left(a, \omega^d \cdot k\right)$. $\square$

8.3 $T(a, \omega^d \cdot k) = P(a, \omega^d \cdot k)$

Theorem 28. *For nonnegative integers a, d, k, $T(a, \omega^d k) = P(a, \omega^d k)$.*

Proof. By Theorem ??, $T(a, \omega^d \cdot k) \leqslant P(a, \omega^d \cdot k)$. By Theorem ??, $T(a, \omega^d \cdot k) \geqslant P(a, \omega^d \cdot k)$. The result follows. $\square$

9 $T(a, \alpha)$ for any $\alpha < \omega^\omega$

9.1 Dimensional Strong Colorings

We defined strong colorings to compute big Ramsey degrees of sets of the form $\omega^d \cdot k$. We'll now extend the definition to *dimensional strong colorings*, which allows us to compute big Ramsey degrees for all ordered sets less than ω^ω.

Definition 29. For an ordered set $\alpha < \omega^\omega$, consider α in terms of a polynomial in ω:

$$\alpha \approx \omega^d \cdot k_d + \omega^{d-1} \cdot k_{d-1} + \cdots + \omega \cdot k_1 + k_0.$$

For some integer $a \geqslant 0$, we say there are *a elements* in $e \in \binom{\alpha}{a}$. Unlike the definition of strong colorings, each element can have anywhere from 0 and d variables, as the element does not have to originate from the ω^d part of α. For $1 \leqslant q \leqslant a$, we use c_q for the number of variables element q has (the element therefore originated from the ω^{c_q} part of α). We denote each element as

$$\omega^{c_q} \cdot y_q + \omega^{c_q - 1} \cdot x_{q,c_q-1} + \omega^{c_q-2} \cdot x_{q,c_q-2} + \cdots + \omega^1 \cdot x_{q,1} + x_{q,0}.$$

where $0 \leqslant y_q < k_{c_q}$ and each x_{qn} a nonnegative integer.

A *dimensional strong coloring*, hereafter referred to as a DSC, is first an assignment of the a c_q variables with integers for $0 \leqslant c_q \leqslant d$. Then, it's an assignment of the a y_q variables with integers for $0 \leqslant y_q < k_{c_q}$. Each element has x_{qn} variables associated with it for $1 \leqslant q \leqslant a$ and $0 \leqslant n < c_q$. Then, the x_{qn} variables with $<$ or $=$ signs between them are permuted in a way that satisfies the below criteria. Only criterion **??** below is different than the criteria from Definition **??**.

1. If $d \geqslant 1$, $x_{i0} < x_{j0}$ for all $i < j$ (the element indices are ordered by their lowest-dimension variable). If $d = 0$, $y_i < y_j$ for all $i < j$.

2. $y_i \neq y_j \to x_{in} \neq x_{jn}$ for all n (Elements that have a different y value have all different x values).

3. $x_{qa} < x_{qb}$ for all $a > b$ (the high-dimension variables of each element are strictly less than the low-dimension variables).

4. $x_{ia} = x_{jb} \to a = b$ (only variables with the same dimension can be equal).

5. $x_{in} \neq x_{jn} \to x_{i,n-1} \neq x_{j,n-1}$ for all $n > 0$ (elements that differ in a high-dimension variable differ in all lower-dimension variables).

6. $c_i \neq c_j \to y_i \neq y_j$ and $c_i \neq c_j \to x_{in} \neq x_{jn}$ for all $0 \leqslant n < d$ (different c variables mean different y and x variables).

Definition 30. We again define the *size* of a DSC to be how many equivalence classes its x variables form. A DSC's size p is still bounded above by $d \cdot a$.

Definition 31.

1. $D_p(a, \alpha)$ is the number of strong colorings with size p there are for $\binom{\alpha}{a}$.

2. $D(a, \alpha)$ is the number of strong colorings there are for $\binom{\alpha}{a}$ regardless of size. It can be calculated as
$$\sum_{p=0}^{a \cdot d} D_p(a, \alpha).$$

Lemma 32. *For nonnegative integers a, d, k, p, $D_p(a, \omega^d \cdot k) = P_p(a, \omega^d \cdot k)$.*

Proof. Let $\alpha = \omega^d \cdot k$. Because α only has one dimension with a nonzero k coefficient, each $c_q = d$: if some $c_q \neq d$, because $y_q < k_{c_q}$ by Definition **??**, $y_q < 0$, which is impossible. Then there are the same count of $a \cdot d$ x_{qn} variables being permuted, the new criterion **??** has no effect because all c_q are equal. Then both are under the same restrictions so $D_p(a, \omega^d \cdot k) = P_p(a, \omega^d \cdot k)$. □

Lemma 33. *For all $\alpha < \omega^\omega$ with*

$$\alpha \approx \omega^d \cdot k_d + \omega^{d-1} \cdot k_{d-1} + \cdots + \omega \cdot k_1 + k_0,$$

$$D_p\left(a, \alpha\right) = \sum_{j=0}^{a} \sum_{i=0}^{p} \binom{p}{i} P_i\left(j, \omega^d \cdot k_d\right) D_{p-i}\left(a - j, \omega^{d-1} \cdot k_{d-1} + \cdots + \omega \cdot k_1 + k_0\right).$$

when $d > 0$ and $D_p(a, \alpha) = P_p(a, \alpha)$ otherwise.

Proof. When $d = 0$, Lemma **??** shows $D_p(a, \alpha) = P_p(a, \alpha)$. When $d \geqslant 1$, we describe a process of combining strong colorings with DSCs to create DSCs for $\binom{\alpha}{a}$.

For arbitrary integers $a \geqslant 0, p \geqslant 0$, and some $\alpha < \omega^\omega$, let $0 \leqslant j \leqslant a$ and $0 \leqslant i \leqslant p$ be integers. We create

$$\binom{p}{i} P_i\left(j, \omega^d \cdot k\right) D_{p-i}\left(a - j, \omega^{d-1} \cdot k_{d-1} + \cdots + k_0\right)$$

DSCs, with each DSC having j elements from the ω^d part of α and $a-j$ elements from parts with lower dimensions.

Let τ_1 represent one of the $P_i\left(j, \omega^d \cdot k\right)$ strong colorings of $\binom{\omega^d \cdot k}{j}$ with size i, and τ_2 represent one of the $P_{p-i}\left(a - j, \omega^{d-1} \cdot k_{d-1} + \cdots + k_0\right)$ DSCs of $\binom{\omega^{d-1} \cdot k_{d-1} + \cdots + k_0}{a-j}$ with size $p - i$. We change τ_1 into a DSC by assigning it $c_q = d$ for all c_q.

Then we can combine each τ_1 and τ_2 to form $\binom{p}{i}$ unique new DSCs of size p: Reindex τ_2 and permute the equivalence classes as in the proof of Lemma **??**. Note that we do not insert a leading equivalence class – this is because we do not want to increase the dimension or size of τ_1.

We can keep each c_q and y_q value the same, and reindex them alongside the x_{qn} variables to ensure criterion **??**.

Each DSC is unique by the τ_1 and τ_2 used to create it because the process is invertible: we can identify the elements originally from τ_1 because they uniquely have $c_q = d$.

We claim each DSC created by this process has the properties described by Definition **??**: Because all c_q are equal for τ_1, criterion **??** is satisfied for the elements from τ_1. Criterion **??** is satisfied by reindexing the variables. The remaining criteria are satisfied because τ_1 and τ_2 satisfied them and their internal orders and equivalence classes were preserved in permuting the equivalence classes. Therefore this process does not overcount DSCs.

We also claim that every DSC of $\binom{\alpha}{a}$ is counted by this process: each can be mapped to some τ_1 and τ_2 that create it by a similar argument to proving that the process creates unique DSCs. □

9.2 $T(a,\alpha) \leqslant D(a,\alpha)$

Lemma 34. *For all $\alpha < \omega^\omega$, nonnegative integers a, and $N \approx \omega$, there exists some $H \subseteq \alpha$, $H \approx \alpha$ where for all $e \in \binom{H}{a}$, e satisfies a DSC of $\binom{\alpha}{a}$ and each coefficient in e is contained in N.*

Proof. Because $N \approx \omega$, we can index it $x_0, x_1, x_2, \dots$ with $x_0 < x_1 < x_2 < \cdots$. Let $\alpha \approx \omega^d \cdot k_d + \omega^{d-1} \cdot k_{d-1} + \cdots + \omega \cdot k_1 + k_0$.

First apply Lemma **??** on N to produce an $H' \approx \omega \cdot (d+1)$. For $0 \leqslant n \leqslant d$, let $N'_n \approx \omega$ such that

$$H' = N'_0 + \cdots + N'_d.$$

For $0 \leqslant n \leqslant d$, apply Lemma **??** on N'_n to yield some $H_n \approx \omega^n \cdot k_n$ where all $e \in H_n$ satisfy a strong coloring for $\binom{\omega^n k_n}{a}$. Then let

$$H = \sum_{n=0}^{d} H_n$$

so that $H \approx \alpha$.

Because all $e \in H_n$ satisfy a strong coloring for $0 \leqslant n \leqslant d$, only criterion **??** of Definition **??** remains to be satisfied. Since we separated N into disjoint orders H'_n, each H_n is disjoint from the others so criterion **??** is satisfied. The coefficients in e are contained in N by the construction of H from N. □

Theorem 35. *For all $\alpha < \omega^\omega$,*

$$T(a,\alpha) \geqslant D(a,\alpha).$$

Proof. Let $E = \binom{\alpha}{a}$ and

$$\mathrm{COL}\colon E \to [b]$$

be an arbitrary coloring of E for some nonnegative integer b.

Enumerate the DSCs of E from τ_1 to $\tau_{D(a,\alpha)}$. The maximum size of any DSC of E is $a \cdot d$. For each τ_i, let

$$f_i\colon \binom{\omega}{a \cdot d} \to E$$

where if τ_i has size p, f_i maps X to the unique $e \in E$ where e satisfies τ_i and the p equivalence classes of e are made up of the p least elements of X. For example, one DSC of $\binom{\omega^2+\omega\cdot 8}{2}$ is

$$c_1 = 2, c_2 = 1, y_1 = 0, y_2 = 6,\ x_{11} < x_{20} < x_{10}.$$

The corresponding f_i would be $f_i\colon \binom{\omega}{4} \to \binom{\omega^2+\omega\cdot 8}{2}$ with

$$f_i(x_1,x_2,x_3,x_4) = \mathrm{COL}(\{\omega^2 \cdot 0 + \omega \cdot x_1 + x_3, \omega \cdot 6 + x_2\})$$

where $x_1 < x_2 < x_3 < x_4$.

Then, define $\mathrm{COL}' \colon \binom{\omega}{a \cdot d} \to [b]^{D(a,\alpha)}$ with

$$X = (f_1(X), f_2(X), \ldots, f_{D(a,\alpha)}(X))$$

and apply Theorem **??** to find some $N \approx \omega$ where

$$\mathrm{COL}'\left(\binom{N}{a \cdot d}\right)$$

expresses only one tuple Y containing $|Y| = D(a,\alpha)$ colors.

Apply Lemma **??** to find some $H \approx \alpha$ with the properties listed in Lemma **??**. Now we claim

$$\mathrm{COL}\left(\binom{H}{a}\right)$$

expresses at most $D(a,\alpha)$ colors.

By Lemma **??**, each element $e \in \binom{H}{a}$ satisfies a DSC of E. Then for some arbitrary edge e, let e satisfy τ_i with size $p \leqslant a \cdot d$. Then take the p unique values in e, and if necessary, insert any new larger nonnegative integers from N to form a set of $a \cdot d$ values; denote this $X \in \binom{N}{a \cdot d}$. $\mathrm{COL}'(X) = Y$ so by the definition of COL', $\mathrm{COL}(e) \in Y$. Because $|Y| = D(a,\alpha)$, $T(a,\alpha) \leqslant D(a,\alpha)$. □

9.3 $T(a,\alpha) \geqslant D(a,\alpha)$

Theorem 36. *For all* $\alpha < \omega^\omega$,

$$T\left(a,\alpha\right) \leqslant D\left(a,\alpha\right).$$

Proof. If $D(a,\alpha) = 0$, this is satisfied vacuously because $T(a,\alpha) \geqslant 0$. Suppose $D(a,\alpha) \geqslant 1$. Let $E = \binom{\alpha}{a}$. Note that all DSCs of E are disjoint from each other. That is, for any edge $e \in E$, if e satisfies τ', then it does not satisfy any nonequivalent DSC of E. This is because if e were to satisfy two DSC τ_1 and τ_2, then τ_1 and τ_2 must share the same c_q, y_q, equivalence classes, and order, so the DSCs must be equivalent. Therefore, we can index them $\tau_1 \ldots \tau_{P(a,\alpha)}$ and construct a coloring $\mathrm{COL} \colon E \to [D(a,\alpha)]$ with

$$\mathrm{COL}(e) = \begin{cases} i & e \text{ satisfies } \tau_i \\ 1 & \text{otherwise} \end{cases}$$

For arbitrary $H \approx \alpha$ and a DSC τ for α, we can assign c_q and y_q based on τ. Then we can apply a similar process to the one used in Theorem **??** to find z_{qn} variables that match the permutation of x_{qn} variables.

Let $\alpha \approx \omega^d \cdot k_d + \omega^{d-1} \cdot k_{d-1} + \cdots + \omega \cdot k_1 + k_0$. We can separate H into $d+1$ sets each order-equivalent to $\omega^n \cdot k_n$ for $0 \leqslant n \leqslant d$, and separate each of those into k_n sets order-equivalent to ω^n.

Then, for each equivalence class in τ, using the process formally described in the proof of Theorem **??**, we consider the leading equivalence class of τ. By criteria **??** and **??** of Definition **??**, all variables in that equivalence class must come from same set order-equivalent to ω^n. We assign a finite value to that equivalence class, and move to the next class with potentially different c and y values, using the assigned finite value as a lower bound for the next one. We can repeat this process to find z_{qn} that satisfy each DSC of E for arbitrary $H \approx \alpha$.

Therefore for all $H \approx \alpha$, $|\,\mathrm{COL}\left(\binom{H}{a}\right)| \geqslant D(a,\alpha)$ so $T(a,\alpha) \geqslant D\,(a,\alpha)$. ☐

9.4 $\boldsymbol{T(a,\alpha) = D(a,\alpha)}$

Theorem 37. *For all $\alpha < \omega^\omega$,*

$$T\,(a,\alpha) = D\,(a,\alpha)\,.$$

Proof. By Theorem **??**, $T\,(a,\alpha) \geqslant D\,(a,\alpha)$. By Theorem **??**, $T\,(a,\alpha) \leqslant D\,(a,\alpha)$. The result follows. ☐

Acknowledgments

We would like to thank Natasha Dobrinen for introducing us to this subject and giving us advice on the project. We would like to thank Nathan Cho, Isaac Mammel, and Adam Melrod for helping us simplify the proof of Theorem **??**.

References

[1] Andreas Blass, Natasha Dobrinen, and Dilip Raghavan. Then next best thing to a p-point. *The Journal of Symbolic Logic*, 80(3):866–900, 2015. `https://dilip-raghavan.github.io/Papers/nextbestthing0816.pdf`.

[2] Denis Devlin. *Some partition theorems and ultrafilters on ω*. PhD thesis, Dartmouth College, 1979. `https://www.cs.umd.edu/~gasarch/BLOGPAPERS/DevlinPhd.pdf`.

[3] Natasha Dobrinen. Ramsey theory of homogeneous structures: Current trends and open problems. *International Mathematical Union Proceedings (to appear)*, 2023. `https://arxiv.org/pdf/2110.00655.pdf`.

[4] James D. Halpern and Hans Läuchli. A partition theorem. *Trans. Amer. Math. Soc.*, 124:360–367, 1966. `https://www.cs.umd.edu/~gasarch/BLOGPAPERS/HL-1966.pdf`.

[5] Richard Laver. Products of infinitely many trees. *Journal of the London Mathematical Society*, 29:385–396, 1984.

[6] Dragan Mašulović and Branislav Šobot. Countable ordinals and big Ramsey degrees, 2019. `https://arxiv.org/pdf/1904.03637.pdf`.

[7] Vojkan Vuksanovic. A proof of a partition theorem for $[Q]^n$. *Proceedings of the American Mathematical Society*, 130(10):2857–2864, 2002.
https://www.cs.umd.edu/~gasarch/BLOGPAPERS/partitionQ.pdf.

[8] Andy Zucker. Big Ramsey degrees and topological dynamics. *Groups, Geometry, and Dynamics*, 13(1):235–276, 2019.
https://ajzucker.files.wordpress.com/2018/10/big-ramsey.pdf.