The Hales-Jewitt Theorem

0.1 Robyn and Bob Play Tic-Tac-Toe While Drunk

Robyn and Bob enjoy playing tic-tac-toe. In the spirit of Ramsey Theory, (1) they are using R and B instead of X and O, they think of the game as coloring points in the grid. As is well known, if they both play perfectly, the game will be a tie. Clearly, if they both play drunk, the game can be a tie. We make this statement more rigorous and change it a little.

1. Let $A = \{(x, y) : 1 \le x, y \le 3\}$.
2. A move in Tic-Tac-Toe is to take a point of A that is not colored and give it your color.

Normally if a player colors any of the following 3-point-sets all the same then they win:

$(1, 1), (1, 2), (1, 3)$
$(1, 1), (2, 1), (3, 1)$
$(1, 1), (2, 2), (3, 3)$
$(2, 1), (2, 2), (2, 3)$
$(3, 1), (2, 2), (1, 3)$ (This one is different! We'll see why soon.)
$(3, 1), (2, 2), (1, 3)$
$(1, 2), (2, 2), (3, 2)$
$(1, 3), (2, 3), (3, 3)$

We will consider a slightly different game. For 7 of these we get that, for each position, the number either stays constant or goes up. For example in
$(1, 1), (1, 2), (1, 3)$
the number in the first coordinate is constant and the number in the second coordinate goes up. There is one exception. For
$(3, 1), (2, 2), (1, 3)$
the number in the first coordinate goes up and the number in the second coordinate goes down. We *will not* count this as a line.

Def 0.1.1 Let $n, t \in \mathbb{N}$.

1. $C_n^t = \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in [t]\}$. Note that C_2^3 is the board for the usual tic-tac-toe game. We call n *the dimension* and t *the size.*

2. A *line* in C_n^t is a sequence of t points $p_1, \ldots, p_t \in C_n^t$ such that, for all $1 \le i \le n$ either

   (a) $p_1(i) = \cdots = p_t(i)$, or

   (b) $p_1(i) = 1$, $p_2(i) = 2$, $\ldots$, $p_t(i) = t$.

3. *2-player Tic-Tac-Toe on* C_n^t is the following game. Robyn and Bob alternate picking uncolored points and coloring them. Robyn (Bob) uses color R (B). Robyn goes first. The first player to get a monochromatic line in their color wins.

4. Henceforth we will use the term *mono line.*

5. Let $c \in \mathbb{N}$. *c-player Tic-Tac-Toe on* C_n^t is the following game. Players $A_1, \ldots, A_c$ go in the order $A_1, \ldots, A_c, A_1, \ldots, A_c, \ldots$ picking uncolored points and coloring them. A_i uses color i. The first player to get a mono line in their color wins.

Now back to Robyn and Bob. Clearly there is a way they could play so that the game is a tie. We want to find them a board C_n^t so that no matter how badly they play *someone* wins. We will phrase this as

for which n, t is it the case that, for all 2-colorings χ of C_n^t, there exists a mono line.

If the size of the board is 2, this is easy.

Exercise 1 Show that for all 2-colorings χ of C_2^2 there is a mono line.

What if the size of the board is 3? Alas, there is a coloring where nobody wins.

Theorem 0.1.2 *Let $t \ge 3$. There exists a 2-coloring of C_2^t that has no mono lines. The 2-coloring will have half R and half B (or* $\mathrm{B} - 1$ *if t is odd) so the coloring could arise in a drunk game.*

Proof sketch:

We do an example for $t = 4$ which will show how to do this for any even t.

R	R	B	B
B	B	R	R
R	R	B	B
B	B	R	R

We leave the case of t odd to the reader.

∎

So what to do? What if we have Robyn and Bob play on a cube? Or a hypercubes? We will show that if Robyn and Bob play on an 8-dimensional hypercube with sides of length 3 then someone must win.

Theorem 0.1.3 *For all 2-colorings of C_8^3 there exists a mono line.*

Proof:

Let χ be a 2-coloring of C_8^3. The colors are R and B. Assume, by way of contradiction, that there are no mono lines.

We look points of C_6^3 and append 2 more coordinates, either 11, 12, or 22. For example, we associate to 221311 the set

$$A = \{22131111, 22131112, 22131122\}.$$

Since A has 3 points and χ maps to 2 colors, 2 of the points must be colored the same. The lack of mono lines will force another points color. For example If $\chi(22131111) = R$ and $\chi(22131122) = R$ then $\chi(22131112) = B$.

More generally, for every $abcdef \in C_6^3$ at least one of the following 6 scenarios holds.

1. $\chi(abcdef11) = \chi(abcdef12) = R$ so $\chi(abcdef13) = B$.
2. $\chi(abcdef11) = \chi(abcdef22) = R$ so $\chi(abcdef33) = B$.
3. $\chi(abcdef12) = \chi(abcdef22) = R$ so $\chi(abcdef32) = B$.
4. $\chi(abcdef11) = \chi(abcdef12) = B$ so $\chi(abcdef13) = R$.
5. $\chi(abcdef11) = \chi(abcdef22) = B$ so $\chi(abcdef33) = R$.
6. $\chi(abcdef12) = \chi(abcdef22) = B$ so $\chi(abcdef32) = R$.

Let $\chi' : C_6^3 \to [6]$ map each point of C_6^3 to the smallest index in the above list that describes a scenario for that point.

Look at the following 7 points of C_6^3:

$$B = \{111111, 111112, 111122, 111222, 112222, 122222, 222222\}$$

Since B has 7 elements and χ' maps to 6 colors, two of the points map to the same color, so the same scenario. We do an indicative example and leave the general proof to the reader.

Suppose for instance 111122 and 122222 fall into class (3), thus

1. $\chi(11112212) = \chi(11112222) = R$ and $\chi(11112232) = B$.
2. $\chi(12222212) = \chi(12222222) = R$ and $\chi(12222232) = B$.

Since
$\chi(11112232) = \chi(12222232) = B$ and there are no mono lines, $\chi(13332232) = R$.

But then

$$\chi(11112212) = \chi(12222222) = \chi(13332232) = R.$$

This contradicts that there are no mono lines.

▮

Theorem **??** is not optimal. Hindman & Tressler [**?**] proved the following.

Theorem 0.1.4

For all 2-colorings of C_4^3 there exists a mono line.

2. *There exists a 2-coloring of C_3^3 without a mono line. (The coloring has 14 R's and 13 B's so it could arise in a drunk game.)*

0.2 $A_1, \ldots, A_c$ Play Tic-Tac-Toe While Drunk

What if there are c players? Now the question is

for which n, t is it the case that, for all c-colorings χ of C_n^t, there exists a mono line.

The case of $t = 2$ is easy.

Theorem 0.2.1 *Let $c \in \mathbb{N}$, $c \geq 2$. For all c-colorings $\chi : C_c^2 \rightarrow [c]$ there exists a mono line.*

Proof: Consider the points
R $\cdots$ R (n R's and 0 B's)
R $\cdots$ RB ($n-1$ R's and 1 B's)
R $\cdots$ RBB ($n-2$ R's and 2 B's)
$\vdots$
B $\cdots$ B (0 R's and n B's)

Since χ uses c colors and there are $c+1$ points, two of the points have the same color. Those two points form a mono line. ∎

BILL TO BILL: WHAT HAPPENS WITH $\chi : C_{c-1}^2 \rightarrow [c]$.