A Pedagogical Approach to Ramsey Multiplicity

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Abstract. It is well known that for all 2-colorings of the edges of K_6 there is a monochromatic triangle. Less well known is that there are two monochromatic triangles. More generally, for all 2-colorings of the edges of K_n there are asymptotically at least $n^3/24$ monochromatic triangles. Another way to state this is that the density of monochromatic triangles is at least $1/4$.

The Ramsey Multiplicity of k is (asymptotically) the greatest α such that for every coloring of K_n the density of monochromatic K_k's is at least α. This concept has been studied for many years. We survey the area and provide proofs that are more complete, more motivated, and use modern notation.

1. INTRODUCTION Throughout this paper we will let $n \in \mathbb{N}$ be a large natural number and $k \in \mathbb{N}$ be a small natural number (i.e. $k \ll n$). We are concerned with coloring the edges of K_n, the complete graph on n vertices. Let $c \in \mathbb{N}$ be the number of colors we use to do this. Our objective will be to find an asymptotic lower bound α on the number of monochromatic copies of K_k in K_n for various values of k, n, and c.

Many, but not all, of the results in this paper are well established; however, some of the proofs in the literature are missing or incomplete. Many of the proofs are not motivated. As such, we present the proofs in a new light, intended to illuminate the problem solving process while still rigorously proving the main results.

In Section 2 we show that for all 2-colorings of the edges of K_6 there are two monochromatic K_3's. In Section 3 we use the ideas of the proof in Section 2 for all 2-colorings of the edges of K_n, showing there are asymptotically at least $\frac{n^3}{24}$ monochromatic K_3's. We view this as saying that $\frac{1}{4}$ of the triangles are monochromatic. Section 4 discusses the best known bounds for c-colorings.

What if we seek monochromatic K_k's? Section 5 introduces the concept of Ramsey Multiplicity, which is the fraction of K_k's that are monochromatic. We discuss some of the early work and give a lower bound on this fraction. In Section 6 we improve this lower bound. We offer two proofs of this improved lower bound. One proof is from the literature. The other is a motivated version of that proof. The motivated version is, in our opinion, easier to generalize to c colors, which we state and prove as our final result.

In Section 7 we state the best known bounds for the Ramsey Multiplicity constants. Finally, In Section 8, we discuss open questions.

2. MONOCHROMATIC TRIANGLES IN ANY 2-COLORING OF THE EDGES OF K_6

Definition 1. We denote the minimum number of monochromatic K_k's in any c-coloring of K_n by $\psi_c(k, n)$.

We give an example by showing that $\psi_2(3, 6) \geq 2$. The proof is from an exposition by Dorwart & Finkbeiner [**4**] based on ideas from Schwenk [**8**].

Figure 1. Mixed Triangles

Theorem 2. $\psi_2(3,6) \geq 2$

Proof. Let $\texttt{COL} : E \longrightarrow \{\text{red, blue}\}$ be an arbitrary 2-coloring of the edges of our graph. Any triangle in our graph will either have 3 red edges, 3 blue edges, or it will be mixed with 2 edges of one color and 1 edge of the other. A mixed triangle would look like the ones in Figure 1.

Let R, B, and M be the sets of red, blue, and mixed triangles respectively. Then

$$|R| + |B| + |M| = \binom{6}{3} = 20$$

We show $|M| \leq 18$ which implies $|R| + |B| \geq 2$.

In each mixed triangle there will be exactly 2 vertices with both a red and blue edge coming out of them.

Definition 3. A *Mix* is an element $(v, \{u, w\}) \in V \times E$ s.t. $v \notin \{u, w\}$ and $COL(v, u) \neq COL(v, w)$. MIX is the set of all *Mixes*.

For example, in our mixed triangles above, the set of *Mixes* is:

$$\{(v_2, \{v_1, v_3\}), (v_3, \{v_1, v_2\}), (v_4, \{v_5, v_6\}), (v_6, \{v_4, v_5\})\}$$

Because there are exactly 2 *Mixes* for each mixed triangle, we see $|\text{MIX}| = 2|M|$. Now we bound $|\text{MIX}|$.

To bound the contribution of a single vertex to MIX, consider the red degree of each vertex in our graph, $d_R(v)$. Since every vertex has degree 5, $d_B(v) = 5 - d_R(v)$.

Case 1: $d_R(v) = 5$. Then v does not have different colored edges coming out of it so it contributes 0 to the size of MIX.

Case 2: $d_R(v) = 4$. Then $d_B(v) = 1$ and there are 4 pairs of edges of different colors coming out of v so this vertex contributes 4 to the size of MIX.

Case 3: $d_R(v) = 3$. Then $d_B(v) = 2$ and there are $3 \cdot 2 = 6$ pairs of edges of different colors coming out of v so this vertex contributes 6 to the size of MIX.

By symmetry we need not consider the cases $d_R(v) < 3$.

So each vertex in our graph will contribute at most 6 to the size of MIX. With 6 vertices in the graph this means

$$|\text{MIX}| \leq 6 \cdot 6 = 36 \Longrightarrow |M| \leq 18 \Longrightarrow |R| + |B| \geq 2$$

∎

3. MONOCHROMATIC TRIANGLES IN ANY 2-COLORING OF THE EDGES OF K_n We rephrase Theorem 2:
For all 2-colorings of K_6 at least $\frac{2}{20} = \frac{1}{10}$ of the triangles are guaranteed to be monochromatic.

What happens if n is large? What should we expect the lower bound on the fraction of monochromatic triangles to be? We give an informal argument, credited to Erdős, for why the answer should be $\frac{1}{4}$ and then prove it formally.

Informal Argument Lower bounds on the Ramsey numbers are often obtained with the probabilistic method where a color is determined by a fair coin flip. Hence we assume that the coloring with the least number of monochromatic triangles is so determined. Color the edges of K_n as follows; for each edge, color it R with probability $\frac{1}{2}$, and (hence) B with probability $\frac{1}{2}$.

To get the density of triangles, pick three vertices at random. There are 8 possible ways to 2-color the edges of a triangle and 2 of those result in a monochromatic triangle. Hence the density of monochromatic triangles is $\frac{1}{4}$.
End of Informal Argument

In the proof of Theorem 2 the red and blue degrees of each vertex played a role in determining the maximum contribution to our set MIX. We saw the maximum contribution occurred when the red and blue degrees were close to equal. It will be useful to formalize this statement when considering coloring the edges of K_n. We leave the proof of this to the reader.

Lemma 4. *Let $x, y \in \mathbb{N}$. Then the maximum value of xy is achieved w.r.t. the constraint $x + y = d$ for some fixed $d \in \mathbb{N}$ when $x = \left\lfloor \frac{d}{2} \right\rfloor$ (or when $y = \left\lfloor \frac{d}{2} \right\rfloor$).*

We now prove a theorem about 2-coloring the edges of K_n. This result was first proved by Goodman [**7**] and later a simpler proof was given by Schwenk [**8**] using the method of Theorem 2. Both authors demonstrate that the lower bound from Theorem 5 is tight by construction. Here is a presentation of the proof by Schwenk, though we have modernized the notation of the original paper.

We split the theorem into two theorems: a lower bound and an upper bound.

Theorem 5. *For $n \geq 6$ a natural number,*

$$\psi_2(3, n) \geq \begin{cases} \frac{n^3}{24} - \frac{n^2}{4} + \frac{n}{3}, & n \equiv 0 \pmod 2 \\ \frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24}, & n \equiv 1 \pmod 4 \\ \frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24} + \frac{1}{2}, & n \equiv 3 \pmod 4 \end{cases}$$

Proof. We proceed in a manner analogous to the previous example, constructing the sets R, B, M, and MIX, noting that $|\text{MIX}| = 2|M|$. To bound $|\text{MIX}|$ above we consider the maximum number of mixed triangles.

Case 1: $n \equiv 0 \mod 2$

The degree of each vertex, $n - 1$, is odd and therefore from Lemma 4, the maximum contribution of a given vertex to MIX is $\frac{n}{2} \cdot \frac{n-2}{2}$. So

$$|M| = \frac{|\text{MIX}|}{2} \leq \frac{n^3 - 2n^2}{8}$$
$$\implies |R| + |B| \geq \binom{n}{3} - \frac{n^3 - 2n^2}{8}$$

$$= \frac{n^3}{24} - \frac{n^2}{4} + \frac{n}{3}$$

Case 2: $n \equiv 1 \mod 4$

Each vertex has degree $n-1$, which is an even number divisible by 4. Using Lemma 4, this means we have a maximum contribution of $\frac{(n-1)^2}{4}$ to MIX from each vertex. Therefore

$$\begin{aligned} |M| = \frac{|\text{MIX}|}{2} &\leq n\frac{(n-1)^2}{8} \\ \implies |R| + |B| &\geq \binom{n}{3} - n\frac{(n-1)^2}{8} \\ &= \frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24} \end{aligned}$$

Case 3: $n \equiv 3 \mod 4$

Since n is odd and $\frac{n-1}{2}$ is odd, our previous calculation of $|\text{MIX}|$ yields an odd number. Because $|M|$ must be a whole number, MIX can't be odd and we take $|M| = \left\lfloor \frac{|\text{MIX}|}{2} \right\rfloor = \frac{|\text{MIX}|-1}{2}$.

$$\begin{aligned} |M| = \frac{|\text{MIX}|-1}{2} &\leq n\frac{(n-1)^2}{8} - \frac{1}{2} \\ \implies |R| + |B| &\geq \binom{n}{3} - n\frac{(n-1)^2}{8} + \frac{1}{2} \\ &= \frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24} + \frac{1}{2} \end{aligned}$$

■

We now give the upper bound.

Theorem 6. *For $n \geq 6$ a natural number,*

$$\psi_2(3,n) \leq \begin{cases} \frac{n^3}{24} - \frac{n^2}{4} + \frac{n}{3}, & n \equiv 0 \pmod 2 \\ \frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24}, & n \equiv 1 \pmod 4 \\ \frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24} + \frac{1}{2}, & n \equiv 3 \pmod 4 \end{cases}$$

Proof. We consider two cases depending on the parity of n.

Case 1: n even

Partition the vertices into two sets of the same size, $A = \{a_1, \ldots, a_d\}$ and $B = \{b_1, \ldots, b_d\}$ where $d = \frac{n}{2}$. Color all the edges between vertices in the same set red and color all edges between vertices in different sets blue. Figure 2 shows the construction.

We claim there are no blue triangles in this construction. Every set of 3 vertices must have either 2 vertices in A or 2 vertices in B and in both cases we have a red edge. Therefore we concern ourselves with counting the number of red triangles in the graph.

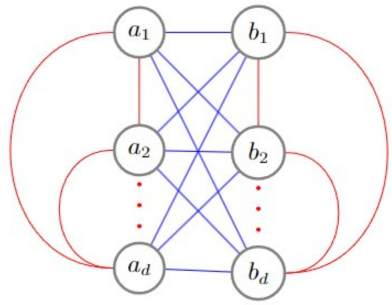

Figure 2. Edge Coloring of K_n, n Even, with Minimum Number of Monochromatic Triangles

$|A| = |B| = \frac{n}{2}$. In each set there are thus $\binom{n/2}{3}$ red triangles.

$$2 \cdot \binom{\frac{n}{2}}{3} = \frac{n(n-2)(n-4)}{24} = \frac{n^3}{24} - \frac{n^2}{4} + \frac{n}{3}$$

Case 2: n odd

We do the case where $n \equiv 1 \pmod 4$; however, we indicate what changes in the construction if $n \equiv 3 \pmod 4$.

Label one vertex in the graph v, which will have different behavior than the others. The remaining $n-1$ vertices we divide into two sets of the same size, $A = \{a_1, \ldots, a_d\}$ and $B = \{b_1, \ldots, b_d\}$ where $d = \frac{n-1}{2}$. Color all the edges between vertices in the same set red as before, but we also color red the edges in the following sets:

1. $\{(a_i, v) : 1 \leq i \leq \frac{n-1}{4}\}$ (If $n \equiv 3 \pmod 4$ then the upper bound is $\frac{n-3}{4}$.)
2. $\{(b_j, v) : 1 \leq j \leq \frac{n-1}{4}\}$ (If $n \equiv 3 \pmod 4$ then the upper bound is $\frac{n-3}{4}$.)
3. $\{(a_i, b_j) : i = j \text{ and } i, j \geq \frac{n+3}{4}\}$ (If $n \equiv 3 \pmod 4$ then the lower bound is $\frac{n+1}{4}$.)

We color the remaining edges blue. Figure 3 shows the construction when $n \equiv 1 \pmod 4$.

We now count the monochromatic triangles in this construction. In sets A and B we have

$$2\binom{(n-1)/2}{3} = \frac{(n-1)(n-3)(n-5)}{24} = \frac{n^3}{24} - \frac{3n^2}{8} + \frac{23n}{24} - \frac{15}{24}$$

red triangles. Through vertex v we have

$$2\binom{(n-1)/4}{2} = \frac{(n-1)(n-5)}{16} = \frac{n^2}{16} - \frac{3n}{8} + \frac{5}{16}$$

red triangles formed in the upper half of our diagram, and we have

$$\frac{n-1}{4} \cdot \left(\frac{n-1}{4} - 1\right) = \frac{(n-1)(n-5)}{16} = \frac{n^2}{16} - \frac{3n}{8} + \frac{5}{16}$$

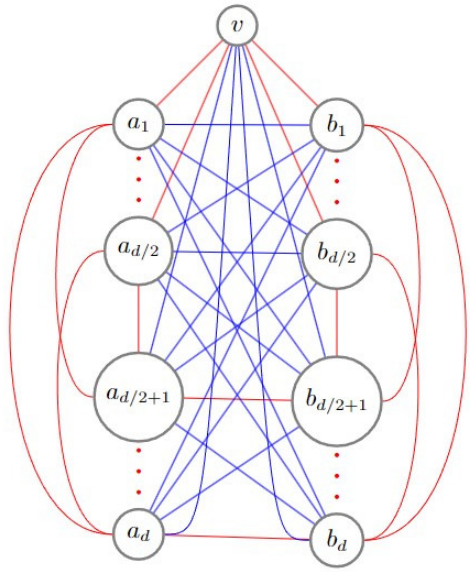

Figure 3. Edge Coloring of K_n, $n \equiv 1 \pmod 4$, with Minimum Number of Monochromatic Triangles

blue triangles formed in the lower half of our diagram. Hence the number of monochromatic triangles that pass through v is

$$\frac{n^2}{8} - \frac{3n}{4} + \frac{5}{8}.$$

Therefore the total number of monochromatic triangles is

$$\frac{n^3}{24} - \frac{3n^2}{8} + \frac{23n}{24} - \frac{15}{24} + \frac{n^2}{8} - \frac{3n}{4} + \frac{5}{8} = \frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24}$$

■

The construction we provide in Theorem 6 has the property that approximately half the edges are of each color. Erdős [**5**] conjectured this type of graph was the worst case for all colorings, which was true for $k = 3$. Is it true for general k? Alas, no. Thomason [**9**] proved this was not the case for $k \geq 4$.

4. WHAT ABOUT MORE COLORS? With monochromatic triangles in 2-colorings completely solved using elementary techniques, one could ask if we can extrapolate these ideas to more colors. Cummings et al. [**3**] gives an asymptotic bound for $c = 3$, but for $c > 3$ the best known upper and lower bounds are far from tight. In this Section we state and provide an interpretation of the best known bounds for these problems. We begin by defining the Ramsey numbers.

Definition 7. $\mathrm{R}_c(k)$ is the smallest $n \in \mathbb{N}$ such that, for any c-coloring of the edges of K_n, there is a monochromatic K_k.

We make use of the following bounds on $\mathrm{R}_c(3)$.

Theorem 8.

1. *(Exoo et al. [**10**])* $\mathrm{R}_c(3) \geq (3.199\ldots)^c \geq 2^{\Omega(c)}$.
2. *(Xiaodong et al. [**11**])* $\mathrm{R}_c(3) \leq 2.55c! \leq 2^{O(c\log c)}$.

Theorem 8 combined with the work of Fox [**6**] and Cummings et al. [**3**] gives the following as the best known bounds for $c \geq 3$.

Theorem 9. *For large n:*

1. *(Fox [**6**])*
   (a) *For any c-coloring of the edges of K_n the proportion of triangles that are monochromatic is $\geq \binom{\mathrm{R}_c(3)}{3}^{-1}$. By Theorem 8.2 this proportion is $\geq \frac{1}{2^{O(c\log c)}}$.*
   (b) *There is a c-coloring of the edges of K_n such that the proportion of triangles that are monochromatic is $\leq (\mathrm{R}_{c-1}(3)-1)^{1-c}$. By Theorem 8.1 this proportion is $\leq \frac{1}{2^{\Omega((c-1)^2)}}$.*
2. *(Cummings et al. [**3**])*
   (a) *For any 3-coloring of the edges of K_n, the proportion of triangles that are monochromatic is $\geq \frac{1}{25}$.*
   (b) *There is a 3-coloring of the edges of K_n such that the proportion of triangles that are monochromatic is $\leq \frac{1}{25}$.*

5. RAMSEY MULTIPLICITY

Definition 10. The **Ramsey Multiplicity** given by:

$$\mathrm{RM}_c(k) = \lim_{n\to\infty} \frac{\psi_c(k,n)}{\binom{n}{k}}$$

represents the minimum density of monochromatic subgraphs of size k in any c-coloring of K_n as n gets large.

Example 11.

1. By Theorems 5 and 6, $\mathrm{RM}_2(3) = \frac{1}{4}$.
2. By Theorem 9.2, $\mathrm{RM}_3(3) = \frac{1}{25}$.
3. By Theorem 9.1, for fixed c,

$$\frac{1}{2^{O(c\log c)}} \leq \mathrm{RM}_c(3) \leq \frac{1}{2^{\Omega((c-1)^2)}}.$$

That these limits exist for all c, k was first claimed by Erdős [**5**] without proof. Other authors have quoted it; however, to our knowledge, a proof has never been written down. We do so.

Lemma 12. $\forall c, k, \mathrm{RM}_c(k)$ *exists and is finite.*

Proof. $\mathrm{RM}_c(k) \leq 1, \forall c, k$ is clear. Now we must argue that the sequence

$$\frac{\psi_c(k,n)}{\binom{n}{k}}$$

is non-decreasing in n.

Consider a c-coloring of the graph $G = K_{n+1}$. Let G_v be the subgraph of G given by removing the vertex v (and its associated edges). Thus G_v is a c-coloring of K_n and as such it will have at least $\psi_c(k, n)$ monochromatic K_k's. This is true for any subgraph G_v and therefore

$$\texttt{number of monochromatic } K_k\texttt{'s in } G_v \geq \psi_c(k, n)$$

There are $n + 1$ choices for G_v and as such there are $(n + 1) \cdot \psi_c(k, n)$ total monochromatic K_k's that can be counted in the associated subgraphs. We are counting some of these K_k multiple times. How many times are we counting each? Each monochromatic K_k only appears once in a particular choice of G_v and the number of choices for v for which this K_k appears is $n + 1 - k$, choosing any vertex from our set of $n + 1$ which is not part of the K_k. Therefore:

$$\psi_c(k, n + 1) \geq \frac{n + 1}{n + 1 - k}\psi_c(k, n)$$

Thus,

$$\begin{aligned} \frac{\psi_c(k, n+1)}{\binom{n+1}{k}} \Big/ \frac{\psi_c(k, n)}{\binom{n}{k}} &= \frac{(n + 1 - k)\psi_c(k, n + 1)}{(n + 1)\psi_c(k, n)} \\ &\geq \frac{(n + 1)\psi_c(k, n)}{(n + 1)\psi_c(k, n)} = 1 \end{aligned}$$

So our sequence is non-decreasing in n and upper bounded by 1 which means the limit exists $\forall c, k$. ■

We now turn our attention to finding lower bounds on these constants for different values of c and k. First we must state a well-known bound on the Ramsey numbers.

Theorem 13. $\mathrm{R}_2(k) \leq 4^k$.

Theorem 13 is folklore, and while there are better upper bounds (see Conlon [**1**]), we don't need to make use of them. With this result we can derive a loose bound on $\mathrm{RM}_2(k)$. The proof of the following Theorem is due to Erdős [**5**]. We present it using modern notation and supply some of the missing details.

Theorem 14. $\mathrm{RM}_2(k) \geq \left(\frac{1}{4}\right)^{k^2}$

Proof. Let $R = \mathrm{R}_2(k)$. Let $\mathcal{A} = \{A_1, \cdots, A_{\binom{n}{R}}\}$ be an enumeration of all R-subsets of $[n]$. We will iterate the following process to find a lower bound on the number of monochromatic K_k's while $\mathcal{A}$ is not empty:

1. Choose $A_i \in \mathcal{A}$.
2. There is a monochromatic K_k in A_i, which increases our count. Call it C.
3. Remove from $\mathcal{A}$ every A_i containing C.

Every iteration produces a distinct monochromatic K_k and we are removing at most $\binom{n-k}{R-k}$ elements from $\mathcal{A}$. Hence:

$$\psi_2(k, n) \geq \frac{\binom{n}{R}}{\binom{n-k}{R-k}} = \frac{n!}{R!(n-R)!} \cdot \frac{(n-R)!(R-k)!}{(n-k)!} = \frac{n!}{(n-k)!} \cdot \frac{(R-k)!}{R!}$$

Now we take the lower bound $\mathrm{RM}_2(3)$ by utilizing Theorem 13:

$$\mathrm{RM}_2(k) = \lim_{n\to\infty} \frac{\psi_2(k,n)}{\binom{n}{k}} \geq \frac{\frac{n!}{(n-k)!}\cdot\frac{(R-k)!}{R!}}{\binom{n}{k}} = \frac{k!(R-k)!}{R!}$$
$$= \frac{1}{R}\cdot\frac{2}{R-1}\cdots\frac{k}{R-k+1} \geq \frac{1}{R^k} \geq \frac{1}{(4^k)^k} = \frac{1}{4^{k^2}}$$

■

To get a sense of how this lower bound compares to known values, we utilize $\mathrm{RM}_2(3) = \frac{1}{4}$ which we computed earlier. Theorem 14 gives $\mathrm{RM}_2(3) \geq \frac{1}{4^{3^2}} = \frac{1}{262144}$, which is a significant disparity.

6. COMPARING TWO PROOFS OF A TIGHTER BOUND In this section we give two proofs of an improvement to Theorem 14. They are both the same proof and due to Conlon [**2**]. The first one is essentially what Conlon presented. It is straightforward but unmotivated and hard to generalize to c colors. The second one is motivated and from it one can see how to generalize it to c colors.

Notation 15. For the remaining Theorems we will use the notation $O_{a,b}(f(a,b,n))$ to denote that the function f has a coefficient of the highest-order term which depends on a and b.

As an example, in Theorem 17 we write $O_{a,b}(n^{a-1})$ to suggest the highest order term is $f(a,b)n^{a-1}$ where $f(a,b)$ is some function of a and b (e.g. $f(a,b) = 2^{ab}$).

Conlon's Proof

Lemma 16. *Let $n \gg d$ and $0 < x < 1$. Then*

$$\binom{x(n-1)}{d} \approx x^d\binom{n}{d}.$$

Proof. See Appendix A. ■

We want a better lower bound on $\mathrm{RM}_2(k)$ than was shown in Theorem 14. We only care about what happens when n is large. However, in the proof of Theorem 17 we need to formally bound n below for the induction to proceed, hence the condition in the Theorem statement $n \geq 3^{a+b}$.

Theorem 17. *Let $a, b \geq 1$ be natural numbers, and let $n \geq 3^{a+b}$. Then in any red/blue-coloring of the edges of K_n there are at least:*

$$2^{-a(b-2)-\binom{a+1}{2}}\binom{n}{a} - O_{a,b}(n^{a-1})$$

red K_a's OR at least:

$$2^{-b(a-2)-\binom{b+1}{2}}\binom{n}{b} - O_{a,b}(n^{b-1})$$

blue K_b's.

Proof. We induct on $a+b$.
Base Case: $a+b=2$.

$$2^{-1(1-2)-\binom{2}{2}}\binom{n}{1}=n$$

and since we can think of every vertex being a monochromatic K_1 in either color, the statement holds.

Induction Hypothesis: The theorem is true for all (a_0, b_0, n_0) such that $a_0+b_0<a+b$ and $n_0 \geq 3^{a_0+b_0}$.

Induction Step: We show the theorem is true for (a,b,n) with $n \geq 3^{a+b}$.

Assume we are given a red/blue coloring of the edges of K_n.

For each vertex v_i there is a color C_i s.t. there are at least $\frac{n-1}{2}$ neighbors to which it is connected by C_i. Either there are at least $\frac{n}{2}$ vertices associated with red or there are at least $\frac{n}{2}$ vertices associated with blue. Suppose WLOG these vertices are red and call them $\{v_1, \cdots, v_{\frac{n}{2}}\}$. For all i, let V_i be the respective red neighbors of v_i.

We apply the induction hypothesis to the subgraph induced by each V_i separately. Each subgraph has at least $\frac{n-1}{2}$ vertices and by the induction hypothesis we know:

$$\begin{aligned} n \geq 3^{a+b} &\implies n-1 \geq 3^{a+b}-1 \\ &\implies \frac{n-1}{2} \geq \frac{3^{a+b}-1}{2} \geq 3^{a+b-1} \end{aligned}$$

So we can apply the induction hypothesis to the triple $\left(a-1, b, \frac{n-1}{2}\right)$.

For each V_i, our induction hypothesis tells us there are either:

$$2^{-(a-1)(b-2)-\binom{a}{2}}\binom{\frac{n-1}{2}}{a-1}-O_{a,b}((n/2)^{a-2})$$

red K_{a-1}'s OR at least:

$$2^{-(b)(a-3)-\binom{b+1}{2}}\binom{\frac{n-1}{2}}{b}-O_{a,b}((n/2)^{b-1})$$

blue K_b's.

First suppose there exists some i for which the latter case holds for V_i. Then the number of blue K_b's is at least:

$$\begin{aligned} &2^{-(b)(a-3)-\binom{b+1}{2}}\binom{\frac{n-1}{2}}{b}-O_{a,b}((n/2)^{b-1}) \\ &=2^{-(b)(a-3)-\binom{b+1}{2}}2^{-b}\binom{n}{b}-O_{a,b}(n^{b-1}) \quad \texttt{(Lemma 16)} \\ &=2^{-(b)(a-2)-\binom{b+1}{2}}\binom{n}{b}-O_{a,b}(n^{b-1}) \end{aligned}$$

completing the proof. Therefore we assume for all i, V_i instead has at least:

$$2^{-(a-1)(b-2)-\binom{a}{2}}\binom{\frac{n-1}{2}}{a-1} - O_{a,b}((n/2)^{a-2})$$

red K_{a-1}'s. By our assumption, since each V_i is connected to v_i by red edges, each of these forms a red K_a. It is possible we have counted each of these a times (once per each vertex), and there are at least $\frac{n}{2}$ of them so in total we have at least:

$$\begin{aligned}
&\frac{1}{a}\cdot\frac{n}{2}\left(2^{-(a-1)(b-2)-\binom{a}{2}}\binom{\frac{n-1}{2}}{a-1} - O_{a,b}((n/2)^{a-2})\right)\\
&= \frac{n}{2a}\left(2^{-(a-1)(b-2)-\binom{a}{2}}2^{1-a}\binom{n}{a-1} - O_{a,b}(n^{a-2})\right) \quad \texttt{(Lemma 16)}\\
&= 2^{-(a-1)(b-2)-\binom{a}{2}}2^{-a}\binom{n}{a} - O_{a,b}(n^{a-1})\\
&= 2^{-(a-1)(b-2)-\binom{a+1}{2}}\binom{n}{a} - O_{a,b}(n^{a-1})\\
&> 2^{-(a)(b-2)-\binom{a+1}{2}}\binom{n}{a} - O_{a,b}(n^{a-1})
\end{aligned}$$

red K_a's. ∎

Corollary 18. $\mathrm{RM}_2(k) \geq \left(\frac{1}{2\sqrt{2}}\right)^{k^2(1-o(1))}$

Proof. In Theorem 17, let $a = b = k$. Then

$$\begin{aligned}
\psi_2(k,n) &\geq 2^{-k(k-2)-\binom{k+1}{2}}\binom{n}{k} - O_k(n^{k-1})\\
&= 2^{-\frac{3}{2}k^2+\frac{3}{2}k}\binom{n}{k} - O_k(n^{k-1})\\
&= \left(\frac{1}{2\sqrt{2}}\right)^{k^2-k}\binom{n}{k} - O_k(n^{k-1})
\end{aligned}$$

So

$$\mathrm{RM}_2(k) = \lim_{n\to\infty}\frac{\psi_2(k,n)}{\binom{n}{k}} \geq \left(\frac{1}{2\sqrt{2}}\right)^{k^2(1-o(1))}$$

∎

Another Version of the Proof Theorem 17 is succinct and the proof is easy to follow. A curious reader may wonder where the leading coefficients came from; we must postpone this exploration until our next result. As a comparison with the bound we computed for $\mathrm{RM}_2(3)$ earlier, this result gives us:

$$\mathrm{RM}_2(3) \geq \left(\frac{1}{2\sqrt{2}}\right)^{3^2} \approx \frac{1}{11585}$$

We now step back and consider the problem in generality. Due to the definition of Ramsey Multiplicity, we want to construct a function whose leading term is a product of some coefficient with $\binom{n}{k}$. If we structure the Theorem in a similar way, this function can depend on both a and b to represent different sized subgraphs for each color. The lower order terms are of no consequence as we plan to take a limit. With this in mind, we state and prove a similar result to Theorem 17 as a second method of obtaining the bound in Corollary 18.

Rather than simply state the Theorem in its entirety upfront, we methodically proceed from our general statement and derive relations on our functions. Then, under certain intuitive, relaxed conditions, we use these relations to provide a recurrence for our functions. Finally we solve the recurrence and realize the same bound from Corollary 18.

Theorem 19. *Let $a, b \geq 1$ be natural numbers, and $n \geq 3^{a+b}$. Let $T = T(a,b)$ and $U = U(a,b)$ be functions (determined later). Then in any red/blue-coloring of the edges of K_n there are at least:*

$$T\binom{n}{a} - O_{a,b}(n^{a-1})$$

red K_a's OR at least:

$$U\binom{n}{b} - O_{a,b}(n^{b-1})$$

blue K_b's.

Proof. We wish to prove this by induction, in a similar manner to the previous Theorem, but because we will later determine T and U, we only end up with a set of relations for these functions in order for our inductive proof to work.

Again we induct on $a + b$.

Base Case: $a + b = 2$. Note that if $a = 1$ or $b = 1$ we have n monochromatic K_a's or K_b's respectively. So we can simply set:

$$T(a,1) = T(1,b) = U(a,1) = U(1,b) = 1, \forall a, b \tag{1}$$

Induction Hypothesis: The theorem is true for all (a_0, b_0, n_0) such that $a_0 + b_0 < a + b$ and $n_0 \geq 3^{a_0+b_0}$.

Induction Step: We show the theorem is true for (a, b, n) with $n \geq 3^{a+b}$.

Assume we are given a red/blue coloring of the edges of K_n.

Every vertex v is connected to either $\frac{n-1}{2}$ vertices by blue edges or $\frac{n-1}{2}$ vertices by red edges. Additionally, either the first case occurs $\frac{n}{2}$ times or the second case occurs $\frac{n}{2}$ times. We will first work through the case where we have $\frac{n}{2}$ vertices $\{v_i\}$ each connected to $\frac{n-1}{2}$ vertices (respectively V_i) by blue edges and the remaining cases follow similarly.

We apply our induction hypothesis to the subgraph induced by each V_i separately, similar to the process in Theorem 17. Our induction hypothesis says for each V_i we must have at least:

$$T(a,b-1)\binom{\frac{n-1}{2}}{a} - O_{a,b}(n^{a-1})$$

red K_a's OR at least:

$$U(a,b-1)\binom{\frac{n-1}{2}}{b-1} - O_{a,b}(n^{b-2})$$

blue K_{b-1}'s.

If the first case occurs for any of the vertices, then (making use of Lemma 16) we have at least:

$$T(a,b-1)\binom{\frac{n-1}{2}}{a} - O_{a,b}(n^{a-1}) = \left(\frac{1}{2}\right)^a T(a,b-1)\binom{n}{a} - O_{a,b}(n^{a-1})$$

red K_a's. To complete our proof from here we would need the following relation on T:

$$\left(\frac{1}{2}\right)^a T(a,b-1) \geq T(a,b) \tag{2}$$

If this case does not occur for any of the vertices, then for each vertex set V_i our Inductive Hypothesis says we have at least:

$$U(a,b-1)\binom{\frac{n-1}{2}}{b-1} - O_{a,b}(n^{b-2})$$

blue K_{b-1}'s. By our assumption, since each V_i is connected to v_i by blue edges, each of these forms a blue K_b. It is possible we have counted each of these b times (once per each vertex of the K_b), and there are at least $\frac{n}{2}$ of them so in total we have at least:

$$\begin{aligned}
&\frac{1}{b}\cdot\frac{n}{2}\left(U(a,b-1)\binom{\frac{n-1}{2}}{b-1} - O_{a,b}(n^{b-2})\right)\\
&= \frac{1}{b}\cdot\frac{n}{2}\left(U(a,b-1)\left(\frac{1}{2}\right)^{b-1}\binom{n}{b-1} - O_{a,b}(n^{b-2})\right)\\
&= \left(\frac{1}{2}\right)^b U(a,b-1)\binom{n}{b} - O_{a,b}(n^{b-1})
\end{aligned}$$

blue K_b's. To complete our proof from here we would need the following relation on U:

$$\left(\frac{1}{2}\right)^b U(a,b-1) \geq U(a,b) \tag{3}$$

If we worked through the other cases, we would similarly obtain the following relations on T and U:

$$\left(\frac{1}{2}\right)^a T(a,b-1) \geq T(a,b)$$
$$\left(\frac{1}{2}\right)^b U(a,b-1) \geq U(a,b)$$
$$\left(\frac{1}{2}\right)^a T(a-1,b) \geq T(a,b)$$
$$\left(\frac{1}{2}\right)^b U(a-1,b) \geq U(a,b)$$
$$\left(\frac{1}{2}\right)^a T(a-1,b) \geq T(a,b)$$
$$\left(\frac{1}{2}\right)^b U(a-1,b) \geq U(a,b)$$

Note the redundancy in some of these equations. This is actually due to an arbitrary, although understandable choice we made earlier in the proof. We will revisit this observation shortly.

For clarity we will now list our required relations on T and U to complete the proof.

$$T(a,1) = U(1,b) = 1, \forall a,b \tag{4}$$
$$\left(\frac{1}{2}\right)^a T(a,b-1) \geq T(a,b) \tag{5}$$
$$\left(\frac{1}{2}\right)^b U(a,b-1) \geq U(a,b) \tag{6}$$
$$\left(\frac{1}{2}\right)^a T(a-1,b) \geq T(a,b) \tag{7}$$
$$\left(\frac{1}{2}\right)^b U(a-1,b) \geq U(a,b) \tag{8}$$

■

Note 20. Early on in the induction step of the proof of Theorem 19 we said *Every vertex v is connected to either $\frac{n-1}{2}$ vertices* $\cdots$. This is true, but our choice of the fraction $\frac{1}{2}$ may not have been optimal. Indeed it is not! Conlon shows that better results can be obtained by having the fraction depend on a and b. We offer some intuition as to why this might be true. Consider the scenario where b is much larger than a, say $b = 100a$. In this scenario we could imagine the worst-case graph may be one with significantly more blue edges than red edges, and could thus imagine tweaking our proof to account for this possibility. Still, our choice of $\frac{1}{2}$ yields a non-trivial bound with a simple proof that still contains most of the ideas of the original proof.

In Section 7 we state a result of Conlon who provides a numerical solution to the recurrence given in Equations 4 - 8, although the methods used are beyond the scope of this paper. Now we proceed to show how to obtain the same bound referenced earlier using these relations.

Corollary 21. $\mathrm{RM}_2(k) \geq \left(\frac{1}{2\sqrt{2}}\right)^{k^2(1-o(1))}$

Proof. We wish to maximize the quantity $T(k,k)$, noting that the relations are symmetric in T and U when $a = b = k$.

To do this, imagine a lattice of points $\{1 \leq a \leq k\} \times \{1 \leq b \leq k\}$. To reach the point (k,k) we must begin along an edge and sequentially take steps, increasing by one our 1^{st} or 2^{nd} coordinate. Each of these steps comes at a multiplicative cost of 2^{-a} or 2^{-b} in our respective coordinates. Importantly, because $T(a,b)$ must be smaller than the product of each step, we need to minimize the quantity over all possible step sequences. Let us formalize this process with a definition:

Definition 22. Let $[x] \times [y]$ be an integer lattice of points. A **path** P in this lattice to (x,y) is defined as a sequence of points $\{a_i, b_i\}_{i=0,\cdots,t}$ which satisfies the following properties:

1. $a_0 = 1$ or $b_0 = 1$
2. $a_1 \neq 1$ and $b_1 \neq 1$
3. $(a_i, b_i) = (a_{i-1}+1, b_{i-1})$ or $(a_i, b_i) = (a_{i-1}, b_{i-1}+1)$
4. $(a_t, b_t) = (x, y)$

We will denote by $\mathcal{P}_{x,y}$ the set of all paths to (x,y).

Using this definition and notation we can succinctly describe our optimization problem as:

$$T(k,k) = \min_{P \in \mathcal{P}_{k,k}} \Pi_{i=1}^{t} 2^{-a_i}$$

To minimize this product, we wish to maximize the exponents $\{a_i\}$ as quickly as possible and this can be done with the path (2, 1), (2, 2), (3, 2), (4, 2), $\cdots$, (k, 2), (k, 3), $\cdots$, (k, k).

$$\begin{aligned} T(k,k) &= \min_{P \in \mathcal{P}_{k,k}} \Pi_{i=1}^{t} 2^{-a_i} \\ &= 2^{-(2+\sum_{i=2}^{k} i + \sum_{i=2}^{k} k)} \\ &= 2^{-(2-1+(k(k+1)/2)+k(k-1))} \\ &= 2^{-(\frac{3}{2}k^2 - \frac{1}{2}k+1)} \\ &= \left(\frac{1}{2\sqrt{2}}\right)^{k^2(1-o(1))} \end{aligned}$$

■

Conlon states the process used to obtain the bounds for 2-colorings is also effective for c-colorings, but does not explore this in his original work. We now offer our main result, a generalization of the previous Theorems for c-colorings. The technique used to solve the problem will be exactly as presented in the proof of Theorem 19 and the subsequent Corollary 21.

Theorem 23. *Let $c \geq 2$ represent the number of colors, $\boldsymbol{m} = \{m_1, \cdots, m_c\}$ with $m_i \geq 1\ \forall i$, and $n \geq (c+1)^{\sum m_i}$. Let $\{U_i = U_i(\boldsymbol{m})\}_{i=1}^{c}$ be functions (determined later). Then in any c-coloring of the edges of K_n there are at least:*

$$U_1 \binom{n}{m_1} - O_{\boldsymbol{m}}(n^{m_1 - 1})$$

monochromatic K_{m_1}'s OR at least:

$$\vdots$$

OR at least:

$$U_c \binom{n}{m_c} - O_{\boldsymbol{m}}(n^{m_c - 1})$$

monochromatic K_{m_c}'s.

Proof. We prove this by induction on $\sum_i m_i$.
Base Case: Any $m_i = 1$. Then we have n monochromatic K_{m_i}'s. In this case we may set $U = 1$ if there is a 1 in any coordinate of the vector $\mathbf{m}$.

Induction Hypothesis: The theorem is true for all $(\mathbf{m}_0, n_0)$ such that $\sum m_{0_i} < \sum m_i$ and $n_0 \geq (c+1)^{\sum m_{0_i}}$.

Induction Step: We show the theorem is true for $(\mathbf{m}, n)$ with $n \geq (c+1)^{\sum m_i}$.

Assume we are given a c-coloring of the edges of K_n.

For each vertex v there must be a color C_i for which there are at least $\frac{n-1}{c}$ edges of this color connecting v to other vertices. Across all vertices, choose the color C_i which occurs the most often, which results in a set $\{v_i\}$ of size at least $\frac{n}{c}$. WLOG suppose this color is C_1. Let V_i be the set of vertices connected to each v_i by color C_1.

We apply the induction hypothesis to the subgraph induced by each V_i separately. Each subgraph has at least $\frac{n-1}{c}$ vertices and by the induction hypothesis we know:

$$\begin{aligned} n \geq (c+1)^{\sum m_i} &\implies n - 1 \geq (c+1)^{\sum m_i} - 1 \\ &\implies \frac{n-1}{c} \geq \frac{(c+1)^{\sum m_i} - 1}{c} \geq (c+1)^{(\sum m_i) - 1} \end{aligned}$$

So we can apply our induction hypothesis to the tuple $\left(m_1 - 1, m_2, \cdots, m_c, \frac{n-1}{c}\right)$. By our inductive hypothesis this means there are at least:

$$U_1(m_1 - 1, m_2, \cdots, m_c) \binom{\frac{n-1}{c}}{m_1 - 1} - O_{\mathbf{m}}(n^{m_1 - 2})$$

monochromatic K_{m_1}'s of color C_1 OR at least:

$$U_2(m_1 - 1, m_2, \cdots, m_c)\binom{\frac{n-1}{c}}{m_2} - O_{\mathbf{m}}(n^{m_2-1})$$

monochromatic K_{m_2}'s of color C_2 OR at least:

$$U_3(m_1 - 1, m_2, \cdots, m_c)\binom{\frac{n-1}{c}}{m_3} - O_{\mathbf{m}}(n^{m_3-1})$$

monochromatic K_{m_3}'s of color C_3 OR at least:

$$\vdots$$

$$U_c(m_1 - 1, m_2, \cdots, m_c)\binom{\frac{n-1}{c}}{m_c} - O_{\mathbf{m}}(n^{m_c-1})$$

monochromatic K_{m_c}'s of color C_c.

For $2 \leq j \leq c$, our argument proceeds in the following way.

$$U_j(m_1 - 1, m_2, \cdots, m_c)\binom{\frac{n-1}{c}}{m_j} - O_{\mathbf{m}}(n^{m_j-1}) = \left(\frac{1}{c}\right)^{m_j} U_j(m_1 - 1, m_2, \cdots, m_c)\binom{n}{m_j} - O_{\mathbf{m}}(n^{m_j-1})$$

$$\implies \left(\frac{1}{c}\right)^{m_j} U_j(m_1 - 1, m_2, \cdots, m_c) \geq U_j(m_1, m_2, \cdots, m_c)$$

If none of these cases happens, since all of our vertices are connected to each of their corresponding vertex sets by the same color, C_1, our Inductive Hypothesis states we have at least:

$$U_1(m_1 - 1, m_2, \cdots, m_c)\binom{\frac{n-1}{c}}{m_1 - 1} - O_{\mathbf{m}}(n^{m_1-2})$$

monochromatic K_{m_1}'s. There are $\frac{n}{c}$ vertex sets, but each monochromatic K_{m_1} may be overcounted m_1 times and thus:

$$\frac{n}{c} \cdot \frac{1}{m_1}\left(U_1(m_1 - 1, m_2, \cdots, m_c)\binom{\frac{n-1}{c}}{m_1 - 1} - O_{\mathbf{m}}(n^{m_1-2})\right)$$

$$= \left(\frac{1}{c}\right)^{m_1} U_1(m_1 - 1, m_2, \cdots, m_c)\binom{n}{m_1} - O_{\mathbf{m}}(n^{m_1-1})$$

$$\implies \left(\frac{1}{c}\right)^{m_1} U_1(m_1 - 1, m_2, \cdots, m_c) \geq U_1(m_1, m_2, \cdots, m_c)$$

The relations we desire on $\{U\}$ must hold for each color and each coordinate of our vector $\mathbf{m}$, so we are left with the following set of relations that must be satisfied for our Theorem to hold:

$$U(m_{0_1},\cdots,m_{0_i}=1,\cdots,m_{0_c})=1, 1\le i\le c, 1\le m_{0_j}\le m_j, \forall i,j \tag{9}$$

$$c^{-\max_i\{m_{0_i}\}}U(m_{0_1},\cdots,m_{0_i}-1,\cdots,m_{0_c})\ge U(m_{0_1},\cdots,m_{0_c}), \forall i \tag{10}$$

■

Corollary 24. $\mathrm{RM}_c(k)\ge\left(\left(\frac{1}{c}\right)^{c-\frac{1}{2}}\right)^{k^2(1-o(1))}$

Proof. Similar to our construction of the function T in the proof of Corollary 21 we wish to find paths, now in a c-dimensional lattice. The minimizing path to reach $(k,\cdots,k)$ can be done with the path (2, $\cdots$, 2, 1), (2, $\cdots$, 2, 2), (3, $\cdots$, 2, 2), (4, $\cdots$, 2, 2), $\cdots$, (k, $\cdots$, 2, 2), (k, 3, $\cdots$, 2, 2), $\cdots$, (k, k, 2, $\cdots$, 2, 2), $\cdots$, (k, k, $\cdots$, k, 2), $\cdots$, (k, $\cdots$, k). Thus

$$\begin{aligned}
U(k,\cdots,k) &= \min_{P\in\mathcal{P}_{k,\cdots,k}} \Pi_{i=1}^{t} c^{-a_i}\\
&= c^{-(2+\sum_{i=2}^{k} i+(c-1)\sum_{i=2}^{k} k)}\\
&= c^{-((2-1+k(k+1)/2)+(c-1)k(k-1))}\\
&= c^{-((c-\frac{1}{2})k^2-(c-\frac{3}{2})k+1)}\\
&= \left(\left(\frac{1}{c}\right)^{c-\frac{1}{2}}\right)^{k^2(1-o(1))}
\end{aligned}$$

■

7. TIGHTER BOUNDS Conlon provides an analytic approximation to the solution of the recurrence given in Equations 4 - 8. Keep in mind that our arbitrary choice of the fraction $\frac{1}{2}$ was not optimal and their work utilizes a more general recurrence.

Theorem 25. *Let $t_\epsilon(x)$ be a function with $t_\epsilon(0)=\epsilon$ and satisfying the differential equation:*

$$t'_\epsilon(x)=\log t_\epsilon(x)\frac{t_\epsilon(x)(1-t_\epsilon(x)}{x-(1-x)t_\epsilon(x)}$$

Let $L=\lim_{\epsilon\to 0} t_\epsilon(1)$ and $C=(L(1-L))^{-1/2}$, then

$$\mathrm{RM}_2(k)\ge C^{-k^2(1-o(1)}$$

A numeric approximation yields the value $C\approx 2.18$. This is the best known bound and we therefore have the following result:

$$\mathrm{RM}_2(3)\ge\left(\frac{1}{2.18}\right)^{3^2}\approx\frac{1}{1112}$$

Their analysis is simple enough to follow, but goes beyond the scope of this paper. From a quick glance it is not immediately obvious to us how their process could be generalized for c-colorings, but that is an area for further research.

8. OPEN PROBLEMS

Open 26.

1. *The results on $\psi_2(3,n)$ are obtained with completely elementary techniques. Can this be done for $\psi_2(4,n)$? $\psi_3(3,n)$?, $\psi_2(5,n)$?*
2. *Obtain an easier proof of Theorem 25. One litmus test is if the proof easily generalizes to c colors.*

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A. APPENDIX Proof of Lemma 16

Proof. We wish to show for n large, $d \ll n$, and $0 < x < 1$ fixed:

$$\binom{x(n-1)}{d} = x^d \binom{n}{d} - O(n^{d-1})$$

We carefully keep track of the sign of the lower order term for completeness, though this is not strictly necessary for our earlier results because we are taking the limit as $n \longrightarrow \infty$.

$$\binom{x(n-1)}{d} = \frac{x(n-1) \cdot (x(n-1)-1) \cdots (x(n-1)-d+1)}{d!}$$

$$
\begin{aligned}
&= \frac{(x(n-1))^d - \left[\sum_{i=1}^{d-1} i\right](x(n-1))^{d-1}}{d!} + O(n^{d-2}) \\
&= x^d \frac{(n-1)^d}{d!} - x^{d-1} \frac{\left[\sum_{i=1}^{d-1} i\right](n-1)^{d-1}}{d!} + O(n^{d-2}) \\
&= x^d \frac{n^d}{d!} - x^d \frac{(d-1)(n-1)^{d-1}}{d!} - x^{d-1} \frac{\left[\sum_{i=1}^{d-1} i\right](n-1)^{d-1}}{d!} + O(n^{d-2}) \\
&= x^d \binom{n}{d} - x^d \binom{n}{d} + x^d \frac{n^d}{d!} - x^d \frac{(d-1)(n-1)^{d-1}}{d!} - x^{d-1} \frac{\left[\sum_{i=1}^{d-1} i\right](n-1)^{d-1}}{d!} + O(n^{d-2}) \\
&= x^d \binom{n}{d} - x^d \frac{n^d}{d!} + x^d \frac{\left[\sum_{i=1}^{d-1} i\right] n^{d-1}}{d!} + x^d \frac{n^d}{d!} \\
&\qquad - x^d \frac{(d-1)(n-1)^{d-1}}{d!} - x^{d-1} \frac{\left[\sum_{i=1}^{d-1} i\right](n-1)^{d-1}}{d!} + O(n^{d-2}) \\
&= x^d \binom{n}{d} + x^d \frac{\left[\sum_{i=1}^{d-1} i\right] n^{d-1}}{d!} - x^d \frac{(d-1)(n-1)^{d-1}}{d!} - x^{d-1} \frac{\left[\sum_{i=1}^{d-1} i\right](n-1)^{d-1}}{d!} + O(n^{d-2})
\end{aligned}
$$

Comparing the coefficients of n^{d-1} in the second and fourth terms of this last equation, since $x < 1$ we see that of the fourth term is larger in absolute value and thus:

$$\binom{x(n-1)}{d} = x^d \binom{n}{d} - O(n^{d-1})$$

■