Big Ramsey Degrees of Countable Ordinals

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Submitted: Apr 30, 2023; Accepted: TBD; Published: TBD

Abstract

Ramsey's theorem states that for all finite colorings of an infinite set, there exists an infinite homogeneous subset. What if we seek a homogeneous subset that is also order-equivalent to the original set? Let S be a linearly ordered set and $a \in \mathbb{N}$. The big Ramsey degree of a in S, denoted $T(a, S)$, is the least integer t such that, for any finite coloring of the a-subsets of S, there exists $S' \subseteq S$ such that (i) S' is order-equivalent to S, and (ii) if the coloring is restricted to the a-subsets of S' then at most t colors are used.

Mašulović & Šobot (2019) showed that $T(a, \omega + \omega) = 2^a$. From this one can obtain $T(a, \zeta) = 2^a$. We give a direct proof that $T(a, \zeta) = 2^a$.

Mašulović and Šobot (2019) also showed that for all countable ordinals $\alpha < \omega^\omega$, and for all $a \in \mathbb{N}$, $T(a, \alpha)$ is finite. We find exact value of $T(a, \alpha)$ for all ordinals less than ω^ω and all $a \in \mathbb{N}$.

Mathematics Subject Classifications: 05D10, 03E10

*Supported by NSF grant 2150382.

1 Introduction

Definition 1.

1. Let $\mathcal{A} = (A, \preceq_A)$ and $\mathcal{B} = (B, \preceq_B)$ be ordered sets. Then $\mathcal{A}, \mathcal{B}$ are *order-equivalent*, denoted $\mathcal{A} \approx \mathcal{B}$, if there exists an order-preserving bijection $f\colon A \to B$; that is, for all $a_1, a_2 \in A$:

$$a_1 \preceq_A a_2 \iff f(a_1) \preceq_B f(a_2).$$

2. Let $\mathcal{A} = (A, \preceq_A)$ and $\mathcal{B} = (B, \preceq_B)$ be ordered sets. Then the ordered set $\mathcal{A} + \mathcal{B}$ is defined to be $(A \sqcup B, \preceq)$, where $c_1 \preceq c_2$ when either $c_1, c_2 \in A$ and $c_1 \preceq_A c_2$, or if $c_1, c_2 \in B$ and $c_1 \preceq_B c_2$, or if $c_1 \in A$ and $c_2 \in B$. Note that $+$ agrees with the definition of ordinal addition, and is not commutative in general.

3. Let $\mathcal{A} = (A, \preceq_A)$ be an ordered set. Then
   (a) $-A = \{-a\colon a \in A\}$
   (b) $-a \preceq_{-A} -b \iff b \preceq_A a$.
   (c) $-\mathcal{A} = \{-A, \preceq_{-A})$

Throughout this paper we conflate the notation for an ordered set and its underlying set, for example, we could define A as an ordered set and still discuss $\binom{A}{2}$.

Notation 2. We use $\mathbb{N}$ to be the set of natural numbers including 0. Let $a, b \in \mathbb{N}$ and S be an ordered set.

1. $[b]$ is $\{1, \ldots, b\}$. Note that if $b = 0$ then $[b] = \emptyset$. Also note that $|[b]| = b$; this is our main motivation for this notation.

2. $\binom{S}{a}$ is the set of all a-element subsets of S. Often, we index the elements of a subset by their ordering in S.

3. Let $\mathrm{COL}\colon S \to [b]$ and $S' \subseteq S$. Then $\mathrm{COL}(S') = \{\mathrm{COL}(s)\colon s \in S'\}$. Hence $|\mathrm{COL}(S')|$ is the size of the codomain of the restriction of COL to S'.

Definition 3. Let S be an ordered set, $S' \subseteq S$, $a, b, t \in \mathbb{N}$, and $\mathrm{COL}\colon \binom{S}{a} \to [b]$ be a coloring.

1. S' is *homogeneous* if $|\mathrm{COL}\left(\binom{S'}{a}\right)| = 1$. S' is *t-homogeneous* if $|\mathrm{COL}\left(\binom{S'}{a}\right)| \leqslant t$.

2. S' is *S-t-homogeneous* if $|\mathrm{COL}\left(\binom{S'}{a}\right)| \leqslant t$ and $S' \approx S$.

Notation 4.

1. We characterize every ordinal α as the ordered set of all ordinals $\beta < \alpha$, with 0 as the least ordinal.

2. ζ is the ordered set containing the integers, ω is the ordered set containing the naturals, and η is the ordered set containing the rationals under their respective natural orderings.

Definition 5. Let S be an ordered set. For any $a \in \mathbb{N}$, $T(a, S)$ is the least $t \in \mathbb{N}$ such that for all $b \in \mathbb{N}$, for all colorings $\mathrm{COL}\colon \binom{S}{a} \to [b]$, there exists some $S' \subseteq S$ such that S' is S-t-homogeneous. Note that t is independent of b. If no such t exists then we write $T(a, S) = \infty$. $T(a, S)$ is called the *big Ramsey degree* of a in S. The term was first coined by Kechris et al. [6]. In set theoretic notation, it can be written as $S \to (S)^a_{r,T(a,S)}$ and $S \not\to (S)^a_{T(a,S),T(a,S)-1}$ for all $r \in \mathbb{N}$.

Lemma 6. *Definition 5 allows $a = 0$ in $T(a, S)$. For any $b \in \mathbb{N}$, we would then define* $\mathrm{COL}\colon \binom{S}{0} \to [b]$. *Because $\binom{S}{0} = \{\emptyset\}$ for all sets S and $\binom{S'}{0} = \{\emptyset\}$ for all sets $S' \subseteq S$, it is clear that $T(0, S) = 1$ for all sets S.*

This paper focuses on $T(a, \zeta)$ and $T(a, \alpha)$ for ordinals $\alpha < \omega^\omega$. We do not consider $T(a, \eta)$, however, the interested reader should know the following:

Theorem 7.

1. *$T(2, \eta) = 2$. It was established by Sierpiński [13] that $T(2, \eta) \geqslant 2$. Equality was first proven by Galvin, unpublished.*

2. *For all $a \in \mathbb{N}$, $T(a, \eta)$ exists. This was first proven by Laver [9].*

3. *$T(a, \eta)$ is the coefficient of x^{2a+1} in the Taylor series for the tangent function, hence*

$$T(a, \eta) = \frac{B_{2a+1}(-1)^{a+1}(1 - 4^{a+1})}{(2(a+1))!}$$

where B_{2a+1} is the $(2a+1)$th Bernoulli number. This was proven by Devlin [2]. See also Vuksanovic [14] using the work of Halpern & Läuchli [5].

Note 8. The notion of $T(a, S)$ has been defined on structures other than orderings. We give an example. Let $R = (\mathbb{N}, E)$ be the Rado graph. $T(a, R)$ is the least number t such that, for all b, for all colorings $\mathrm{COL} : \binom{\mathbb{N}}{a} \to [b]$, there exists $H \subseteq \mathbb{N}$ where both $|\,\mathrm{COL}\left(\binom{H}{a}\right)| \leqslant t$ and the graph induced by H is isomorphic to R. The numbers $T(a, R)$ are known but complicated; however, $T(2, R) = 2$. See Sauer [12], Laflamme et al. [7], and Larson [8]. See Dobrinen [3] for more references and other examples.

2 Summary of Results

Ramsey's Theorem on $\mathbb{N}$ gives an infinite 1-homogeneous subset of $\mathbb{N}$. Theorem 9 restates Ramsey's Theorem in two equivalent ways.

Theorem 9. *Let $a \in \mathbb{N}$.*

1. $T(a, \omega) = 1$ *for all* $a \in \mathbb{N}$.

2. *Let* $b \geqslant 1$ *and* $\mathrm{COL}\colon \binom{\omega}{a} \to [b]$. *Then there exists some* $H \approx \omega$ *such that*

$$\left|\mathrm{COL}\left(\binom{H}{a}\right)\right| = 1.$$

What happens for other ordered sets? In this paper we do the following.

1. In Section 3 we show that $T(a, \zeta) = 2^a$. This can be obtained by the result due to Mašulović & Šobot [10] that $T(a, \omega + \omega) = 2^a$. We give a more direct proof.

2. In Sections 4, 5, 6, 7, 8, and 9 we construct theorems to eventually determine $T(a, \alpha)$ for all ordinals $\alpha < \omega^\omega$. Mašulović & Šobot [10] previously showed for all ordinals $\alpha \geqslant \omega^\omega$ that $T(a, \alpha) = \infty$. They also showed for ordinals $\alpha < \omega^\omega$ that $T(a, \alpha)$ is finite; however, they did not obtain the exact values of $T(a, \alpha)$.

3 Big Ramsey degrees of ζ

As a warmup we first prove $T(1, \zeta) = 2$ and $T(2, \zeta) = 4$.

Theorem 10. $T(1, \zeta) = 2$.

Proof. Let $b \in \mathbb{N}$.

We first prove $T(1, \zeta) \leqslant 2$. Let $\mathrm{COL}\colon \zeta \to [b]$. Let $\mathrm{COL}'\colon \omega \to [b]^2$ be defined by

$$\mathrm{COL}'(x) = (\mathrm{COL}(-x), \mathrm{COL}(x)).$$

By Theorem 9, there exists an ω-1-homogeneous set H'. Let the color of the homogeneous set be (c_1, c_2). Consider the set $H = -H' + H'$, which is order-equivalent to ζ. Let $h \in H$. If $h \in H'$ then by definition of COL' and 1-homogeneity of H', $\mathrm{COL}(h) = c_1$. Similarly, if $h \in -H'$, $\mathrm{COL}(h) = c_2$. Thus H is ζ-2-homogeneous. Because COL was arbitrary, $T(1, \zeta) \leqslant 2$.

We now prove $T(1, \zeta) \geqslant 2$. Let $\mathrm{COL}\colon \zeta \to [2]$ be the function that colors all nonnegative integers "1" and all negative integers "2". Since the nonnegative integers have no infinitely descending chain and the negative integers have no infinitely ascending chain, there is no ζ-1-homogeneous subset of ζ under COL. Therefore $T(1, \zeta) \geqslant 2$, and with the previous result, $T(1, \zeta) = 2$. □

Theorem 11. $T(2, \zeta) = 4$.

Proof. Let $b \in \mathbb{N}$.

We first prove $T(2, \zeta) \leqslant 4$. Let $\mathrm{COL}\colon \binom{\zeta}{2} \to [b]$. Let $\mathrm{COL}'\colon \binom{\omega}{2} \to [b]^4$ be defined by

$$\mathrm{COL}'(x, y) = (\mathrm{COL}(-x, -y), \mathrm{COL}(-x, y), \mathrm{COL}(x, -y), \mathrm{COL}(x, y)).$$

By Theorem 9 there exists an ω-1-homogeneous set H'. Let the color of the homogeneous set be (c_1, c_2, c_3, c_4). Label H' as $\{h_0 < h_1 < \cdots\}$.

Then consider the set

$$H = \{-h_i \colon i \text{ is even}\} + \{h_i \colon i \text{ is odd}\},$$

which is order-equivalent to ζ. Let $h_i, h_j \in H$. Let $h_i = s_i n_i$ and $h_j = s_j n_j$, where $n_i, n_j \geqslant 0$ and $s_i, s_j \in \{-1, 1\}$. Then by definition of COL$'$, COL(h_i, h_j) is in $\{c_1, c_2, c_3, c_4\}$, depending on s_i and s_j. As an aside, this method only works when n_i and n_j are guaranteed to be distinct, as we forced with our alternation of sign by the parity of index.

Since COL$(H) \subseteq \{c_1, c_2, c_3, c_4\}$, H is ζ-4-homogeneous. We could have partitioned H' on something other than parity; any two disjoint infinite sets would work. Therefore $T(2, \zeta) \leqslant 4$.

We now prove $T(2, \zeta) \geqslant 4$. Let COL: $\binom{\zeta}{2} \to [4]$ be the coloring

$$\text{COL}(x, y) = \begin{cases} 1 & \text{if } x, y \geqslant 0 \\ 2 & \text{if } x \geqslant 0,\ y < 0, \text{ and } |x| \leqslant |y| \\ 3 & \text{if } x \geqslant 0,\ y < 0, \text{ and } |x| > |y| \\ 4 & \text{if } x < 0,\ y < 0 \end{cases}$$

We leave it to the reader to show there is no ζ-3-homogeneous set. The key idea of the proof is that if we suppose COL does not output some color under some set, then that set cannot be order-equivalent to ζ. Therefore $T(2, \zeta) \geqslant 4$, and with the previous result, $T(2, \zeta) = 4$. □

Theorem 12. *For all $a \in \mathbb{N}$, $T(a, \zeta) = 2^a$.*

Proof. Let $b \in \mathbb{N}$.

We first prove $T(a, \zeta) \leqslant 2^a$. Let COL: $\binom{\zeta}{a} \to [b]$ be an arbitrary coloring. Let COL$'$: $\binom{\omega}{a} \to [b]^{(2^a)}$ be defined by

$$\text{COL}'(x_1, \ldots, x_a) = (\text{COL}(x_1, \ldots, x_a), \text{COL}(-x_1, x_2, \ldots, x_a), \ldots, \text{COL}(-x_1, \ldots, -x_a)).$$

Formally, the tuple contains COL's output on each tuple in the set

$$\{-x_1, x_1\} \times \{-x_2, x_2\} \times \cdots \times \{-x_a, x_a\}.$$

The output of COL$'$ goes through all ways to negate one of the 2^a subsets of the x_i. Note that COL$'$ only depends on the color of elements of $\binom{\zeta}{a}$ where the absolute values of the elements are all different.

By Theorem 9 there exists some homogeneous set H'. Index H' as $\{h_0 < h_1 < \cdots\}$. Then the set

$$H = \{-h_i \colon i \text{ is even}\} + \{h_i \colon i \text{ is odd}\}$$

is ζ-2^a-homogeneous. Because the absolute values of every element in H are all different, each a-subset was considered by COL$'$. Therefore $T(a, \zeta) \leqslant 2^a$.

We now prove $T(a,\zeta) \geqslant 2^a$. We describe a coloring $\text{COL}\colon \binom{\zeta}{a} \to [2]^a$.

Let $\{x_1 < \cdots < x_a\} \in \binom{\zeta}{a}$. Define an ordering $<^*$ as $x <^* y$ if $|x| < |y|$ or both $|x| = |y|$ and $x < y$ (we order by absolute values, and in case of ties, order the negative before the positive).

Let $(i_1, \ldots, i_a)$ be such that

$$|x_{i_1}| <^* |x_{i_2}| <^* \cdots <^* |x_{i_a}|.$$

Let s_{i_j} be 1 if $x_{i_j} \geqslant 0$ and 0 if $x_{i_j} < 0$.

We define $\text{COL}(\{x_1 < \cdots < x_a\})$ as $(s_{i_1}, \ldots, s_{i_a})$.

We use 2^a colors, as there are a elements each either 1 or 0. We leave it to the reader to show that there is no ζ-$(2^a - 1)$-homogeneous set.

□

4 Big Ramsey degrees of finite multiples of ω

As noted in Theorem 9, $T(a,\omega) = 1$. In this and later sections we examine limit ordinals larger than ω. For simplicity in stating results, we save the big Ramsey degrees of successor ordinals such as $\omega+1, \omega+2, \ldots$ as well as some limit ordinals such as $\omega^2+\omega$ for Section 9.

Our first result shows $T(a,\omega\cdot k) = k^a$ for most a, k. This is undefined when $a = k = 0$. Lemma 6 shows that even though $\omega \cdot 0 = 0 = \emptyset$ under ordinal arithmetic, $T(0,\omega\cdot 0) = 1$.

Theorem 13. *For $a, k \in \mathbb{N}$ with at least one of a, k nonzero, $T(a,\omega\cdot k) \leqslant k^a$.*

Proof. Let $a, b, k \in \mathbb{N}$. Let

$$\text{COL}\colon \binom{\omega\cdot k}{a} \to [b]$$

be an arbitrary coloring. Let $\text{COL}'\colon \binom{\omega}{a} \to [b]^{(k^a)}$ be defined by

$$\begin{aligned}\text{COL}'(x_1, x_2, \ldots, x_a) = (&\text{COL}(x_1, \ldots, x_a), \text{COL}(\omega + x_1, x_2, \ldots, x_a), \ldots, \\ &\text{COL}(\omega\cdot(k-1) + x_1, \ldots, \omega\cdot(k-1) + x_a))\end{aligned}$$

where COL' maps a elements of ω to the COL of each of the k^a ways to add one of $\omega \cdot 0$ through $\omega\cdot(k-1)$ with each of the a coordinates. Formally, its output tuple contains COL's assignment of each element of

$$\{x_1, \omega + x_1, \ldots, \omega\cdot(k-1) + x_1\} \times \cdots \times \{x_a, \omega + x_a, \ldots, \omega\cdot(k-1) + x_a\}.$$

Apply Theorem 9 with COL' to find some $G \approx \omega$ such that

$$\left|\text{COL}'\left(\binom{G}{a}\right)\right| = 1.$$

Let the one color in $\text{COL}'(\binom{G}{a})$ be Y. Note that Y is a tuple of length k^a.

Index G as $\{g_0 < g_1 < \cdots\}$ and let

$$\begin{aligned} H = &\{\omega \cdot 0 + g_i \colon i \equiv 0 \mod k\} + \\ &\{\omega \cdot 1 + g_i \colon i \equiv 1 \mod k\} + \cdots + \\ &\{\omega \cdot (k-1) + h_i \colon i \equiv k-1 \mod k\}. \end{aligned}$$

Now $H \approx \omega \cdot k$. Note that the use of modulus was only to ensure each copy of ω within H has different numbers, so this is not the only way to define a useful H. For the case of $k = 3$, we have

$$\begin{aligned} H = \{ \quad & 0, \quad 3, \quad 6, \ldots, \\ & \omega + 1, \quad \omega + 4, \quad \omega + 7, \ldots, \\ & \omega \cdot 2 + 2, \omega \cdot 2 + 5, \omega \cdot 2 + 8, \ldots\}. \end{aligned}$$

Then

$$\left| \mathrm{COL}\left(\binom{H}{a}\right) \right| \leqslant k^a :$$

for any selection of a elements from H, its color was considered in COL' so it must be one of the k^a colors in Y. □

We now prove that $T(a, \omega \cdot k)$ is bounded below by k^a by providing an example coloring. The coloring is inspired by Theorem 13, although we use it to prove a different bound. This duality will become more clear in later sections.

Theorem 14. *For $a, k \in \mathbb{N}$, $T(a, \omega \cdot k) \geqslant k^a$.*

Proof. We give a k^a-coloring of $\binom{\omega \cdot k}{a}$ that has no $(k^a - 1)$-homogeneous $H \approx \omega \cdot k$. We represent $\omega \cdot k$ as

$$\omega \cdot k \approx X_1 + \cdots + X_k$$

where each $X_i \approx \omega$. If an element $x \in \omega \cdot k$ is the kth element of X_i, we represent it as the ordered pair (i, x).

Before defining the coloring in general we give an example with $a = 5$ and $k = 200$. We define the color of the element

$$e = \{(3, 12), (50, 2), (110, 12), (110, 7777), (117, 3)\}$$

as follows. Order the ordered pairs by their second coordinates. If two elements have the same second coordinates, order by their first coordinate. We have

$$((50, 2), (117, 3), (3, 12), (110, 12), (110, 7777)).$$

We define the color of the element as the sequence of first coordinates, so

$$\mathrm{COL}(e) = (50, 117, 3, 110, 110).$$

Since the set of possible colors is the set of all 5-tuples of numbers $\{0, \ldots, 199\}$, there are 200^5 possible colors.

In general, for any $e = \{(i_1, x_1), \ldots, (i_a, x_a)\}$, order the ordered pairs by their second coordinates, break ties with their first coordinates, and then COL(e) is the sequence of first coordinates after ordering.

Notice that the number of colors is the number of a-tuples where each number is in $\{1, \ldots, k\}$. Hence there are k^a colors. We leave it to the reader to show that there can be no $(\omega \cdot k)$-$(k^a - 1)$-homogeneous H. The key idea of the proof, much like the previous lower bounds in this paper, is supposing that one of the k^a colors is not output by COL on some set, and using that to show that the set cannot be order-equivalent to $\omega \cdot k$. □

Theorem 15. *For $a, k \in \mathbb{N}$, $T(a, \omega \cdot k) = k^a$.*

Proof. By Theorem 13, $T(a, \omega \cdot k) \leqslant k^a$. By Theorem 14, $T(a, \omega \cdot k) \geqslant k^a$. The result follows. □

5 A big Ramsey degree of ω^2

This section provides a concrete example involving ordinals greater than ω. We use ω^2, which is the set of all ordinals $\omega \cdot a + b$ with $a, b \in \omega$. ω^2 is order-equivalent to the ordered-set-concatenation of countably infinite copies of ω, as visualized below:

$$\begin{array}{rrrrl} 0, & 1, & 2, & 3, & \ldots \\ \omega + 0, & \omega + 1, & \omega + 2, & \omega + 3, & \ldots \\ \omega \cdot 2 + 0, & \omega \cdot 2 + 1, & \omega \cdot 2 + 2, & \omega \cdot 2 + 3, & \ldots \\ \omega \cdot 3 + 0, & \omega \cdot 3 + 1, & \omega \cdot 3 + 2, & \omega \cdot 3 + 3, & \ldots \\ \vdots & & & & \end{array}$$

Note that although we typeset ω^2 as a grid, it is totally linearly ordered with $0 < 1 < \ldots < \omega < \omega + 1 < \ldots$ and so on.

The method used to prove $T(1, \omega^2) = 1$ is similar to those already seen in this paper. We begin with $T(2, \omega^2)$ before proving the general case.

Theorem 16. $T(2, \omega^2) = 4$.

Proof. We first prove $T\left(2, \omega^2\right) \leqslant 4$.

Let $b \in \mathbb{N}$. Let

$$\text{COL}\colon \binom{\omega^2}{2} \to [b]$$

be an arbitrary coloring. We define four functions f_1, f_2, f_3, f_4 from domain $\binom{\omega}{4}$ to codomain $\binom{\omega^2}{2}$ and then use them to define a coloring from $\binom{\omega}{4}$ to $[b]^4$. In what follows, we index variables as $x_1 < x_2 < x_3 < x_4$.

$f_1 \colon \binom{\omega}{4} \to \binom{\omega^2}{2}$ is defined by

$$f_1(x_1, x_2, x_3, x_4) = \{\omega \cdot x_1 + x_2, \omega \cdot x_3 + x_4\}.$$

$f_2 \colon \binom{\omega}{4} \to \binom{\omega^2}{2}$ is defined by

$$f_2(x_1, x_2, x_3, x_4) = \{\omega \cdot x_1 + x_3, \omega \cdot x_2 + x_4\}.$$

$f_3 \colon \binom{\omega}{4} \to \binom{\omega^2}{2}$ is defined by

$$f_3(x_1, x_2, x_3, x_4) = \{\omega \cdot x_1 + x_4, \omega \cdot x_2 + x_3\}.$$

$f_4 \colon \binom{\omega}{4} \to \binom{\omega^2}{2}$ is defined by

$$f_4(x_1, x_2, x_3, x_4) = \{\omega \cdot x_1 + x_2, \omega \cdot x_1 + x_3\}.$$

$\mathrm{COL}' \colon \binom{\omega}{4} \to [b]^4$ is defined by

$$\mathrm{COL}'(X) = (\mathrm{COL}(f_1(X)), \mathrm{COL}(f_2(X)), \mathrm{COL}(f_3(X)), \mathrm{COL}(f_4(X))).$$

Apply Theorem 9 on COL' to find some $G \approx \omega$ where $|\,\mathrm{COL}'\left(\binom{G}{4}\right)| = 1$. Enumerate G as $G = \{x_0, x_1, \ldots\}$ with $x_0 < x_1 < \cdots$. Let

$$\begin{aligned} H = \quad & \{\omega \cdot x_1 + x_2, \omega \cdot x_1 + x_6, \omega \cdot x_1 + x_{10}, \ldots\} \\ + \; & \{\omega \cdot x_3 + x_4, \omega \cdot x_3 + x_{12}, \omega \cdot x_3 + x_{20}, \ldots\} \\ + \; & \{\omega \cdot x_5 + x_8, \omega \cdot x_5 + x_{24}, \omega \cdot x_5 + x_{40}, \ldots\} \\ & \vdots \end{aligned}$$

Formally,

$$H = X_1 + X_2 + \cdots$$

where

$$X_i = \{\omega \cdot x_{2i-1} + x_j \colon j = 2^i + k2^{i+1}, k \in \mathbb{N}\}.$$

Note that each of the infinite sets' coefficients are disjoint.

Then $H \approx \omega^2$, as it is the concatenation of countably infinite sets order-equivalent to ω. Note that for every $\omega \cdot x_i + x_j \in H$ we have $x_i < x_j$.

For any edge $\{\omega \cdot y_1 + y_2, \omega \cdot y_3 + y_4\} \in \binom{H}{2}$ with $\omega \cdot y_1 + y_2 < \omega \cdot y_3 + y_4$, either $y_1 \neq y_3$ or $y_1 = y_3$.

- If $y_1 \neq y_3$, then $y_1 < y_3$ by the ordering of the two elements and $y_2 \neq y_4$ by the construction of H. We also have $y_1 < y_2$, $y_1 < y_4$, and $y_3 < y_4$ by the construction of H. Then either $y_1 < y_2 < y_3 < y_4$, $y_1 < y_3 < y_2 < y_4$, or $y_1 < y_3 < y_4 < y_2$. In each of the three cases, $f_1(y_1, y_2, y_3, y_4) \in Y$, $f_2(y_1, y_3, y_2, y_4) \in Y$, and $f_3(y_1, y_3, y_4, y_2) \in Y$ respectively so $\mathrm{COL}(\{\omega \cdot y_1 + y_2, \omega \cdot y_3 + y_4\}) \in Y$.

- If $y_1 = y_3$, then $y_2 < y_4$ by the ordering of the elements and so $y_1 = y_3 < y_2 < y_4$ by the construction of H. Because $f_4(y_1, y_2, y_4, 1) \in Y$ (note that f_4's output does not depend on its 4th argument), $\mathrm{COL}(\{\omega \cdot y_1 + y_2, \omega \cdot y_3 + y_4\}) \in Y$.

In all cases, $\mathrm{COL}(\{\omega \cdot y_1 + y_2, \omega \cdot y_3 + y_4\}) \in Y$ so

$$\mathrm{COL}\left(\binom{H}{2}\right) \subseteq Y$$

with $|Y| = 4$. Because $H \approx \omega^2$ and COL was arbitrary, $T(2, \omega^2) \leqslant 4$.

We now prove $T(2, \omega^2) \geqslant 4$. Let $\mathrm{COL}\colon \binom{\omega^2}{2} \to [4]$ with

$$\mathrm{COL}(\omega \cdot x_1 + x_2, \omega \cdot x_3 + x_4) = \begin{cases} 1 & x_1 < x_2 < x_3 < x_4 \\ 2 & x_1 < x_3 < x_2 < x_4 \\ 3 & x_1 < x_3 < x_4 < x_2 \\ 4 & \text{otherwise} \end{cases}$$

where $\omega \cdot x_1 + x_2 < \omega \cdot x_3 + x_4$. Let $H \subseteq \omega^2$ be any set such that $H \approx \omega^2$. We show that $|\,\mathrm{COL}(\binom{H}{2}))| = 4$. Because $H \approx \omega^2$, we have $H = H_1 + H_2 + \cdots$ where each $H_i \approx \omega$. Let y_i and z_{ij} be such that every element of H_i is of the form $\omega \cdot y_i + z_{ij}$. Note that

$$y_1 < y_2 < \cdots$$

and for all i,

$$z_{i1} < z_{i2} < \cdots .$$

We now show each color is output by COL on H.

- Color 1: Starting with any y_i, find some z_{ij} where $z_{ij} > y_i$. This is guaranteed, as y_i is finite and the z_{ij} are infinitely increasing. Then find some y_k where $y_k > z_{ij}$; again guaranteed because z_{ij} is finite and the y_i are infinitely increasing. Finally, find a $z_{k\ell} > y_k$ guaranteed by similar means. Then

  $$\mathrm{COL}(\omega \cdot y_i + z_{ij}, \omega \cdot y_k + z_{k\ell}) = 1.$$

- Colors 2 and 3 are guaranteed to exist by a similar argument similar to color 1's argument.

- Color 4. The case of $x_1 = x_3 < x_2 < x_4$ falls into the category of *otherwise*. With this in mind, we can start with some y_i, and then find a $z_{ij} > y_i$. Then, we only need to find a $z_{i\ell} > z_{ij}$; this is guaranteed because z_{ij} is finite. Then

  $$\mathrm{COL}(\omega \cdot y_i + z_{ij}, \omega \cdot y_i + z_{i\ell}) = 4.$$

While it was simple to find edges that output each of these four colors, the proof that $T(2, \omega^2) \leqslant 4$ shows that a coloring with more than four colors can never guarantee that more than four colors are output on an order-equivalent subset.

□

6 Coloring Rules

We introduce a concept called *coloring rules* (hereafter CRs) to prove general results about $T(a, \omega^d)$. The concept behind CRs is built on the ideas of Blass et al. [1]. We motivate the concept by examining at the proof of Theorem 16.

The proof of Theorem 16 used four functions f_1, f_2, f_3, and f_4. These functions were chosen to cover H in a way where the color of every edge in H was output by one of f_1, f_2, f_3, or f_4. We note a function that was *not* used:

$f\colon \binom{\omega}{4} \to \binom{\omega^2}{2}$ defined by

$$f(x_1, x_2, x_3, x_4) = \mathrm{COL}(\{\omega \cdot x_1 + x_3, \omega \cdot x_2 + x_3\}).$$

We didn't use f in the lower bound proof because f didn't cover *any* edges in H: we constructed H in a way where distinct copies of ω had distinct finite coefficients. Since $x_1 \neq x_2$, the elements $\omega \cdot x_1 + x_3$ and $\omega \cdot x_2 + x_3$ couldn't both be from H no matter the values of x_1, x_2, and x_3.

We could have designed H differently to require more than 4 functions to cover it, but that would have weakened the upper bound result of Theorem 16. We define a notion of colorings that only f_1, f_2, f_3, and f_4 qualify. We also show how to count these colorings, and how these colorings are linked to big Ramsey degrees.

Definition 17. We now define CRs (coloring rules) rigorously. We impose a structure on edges and list criteria that CRs must satisfy. For integers $a, d, k \geqslant 0$, an edge $e = \{p_1, \ldots, p_a\}$ in $\binom{\omega^d \cdot k}{a}$ consists of a points which are elements of $\omega^d \cdot k$. Each element p_q is equal to

$$\omega^d \cdot y_q + \omega^{d-1} \cdot x_{q,d-1} + \omega^{d-2} \cdot x_{q,d-2} + \cdots + \omega^1 \cdot x_{q,1} + x_{q,0},$$

where $x_{q,n} \geqslant 0$ and $0 \leqslant y_q < k$.

Thus, any edge e is defined by the a values of the y_q's and the $a \cdot d$ values of the $x_{q,n}$'s.

A *CR (coloring rule)* on $\binom{\omega^d \cdot k}{a}$ is a pair $(\mathcal{Y}, \preceq_{\mathcal{X}})$ of constraints on these values y_q and $x_{q,n}$ satisfying certain properties we enumerate below. $\mathcal{Y}$ is an assignment of the values for the y_q; formally, it is a map $\mathcal{Y} : [a] \to \{0, \ldots, k-1\}$ from indices of the y_q to the possible values for the y_q. Having $\mathcal{Y}(q) = v$ means we constrain $y_q = v$. $\preceq_{\mathcal{X}}$ is a total preorder (each pair of elements are compared, and $\preceq_{\mathcal{X}}$ is reflexive and transitive) on the indices of the $x_{q,n}$. Having $(q_1, n_1) \preceq_{\mathcal{X}} (q_2, n_2)$ means we constrain $x_{q_1,n_1} \leqslant x_{q_2,n_2}$.

For some $\mathcal{Y}$, we often denote it by an ordered list of clauses:

$$y_1 = \mathcal{Y}(1), y_2 = \mathcal{Y}(2), \ldots, y_a = \mathcal{Y}(a).$$

For some preorder $\preceq_{\mathcal{X}}$, we ocften denote it by some permutation of the characters $x_{1,0}, \ldots, x_{a,d-1}$ interspersed by either $<$ or $=$. We write $x_{q_1,n_1} < x_{q_2,n_2}$ to mean both $(q_1, n_1) \preceq_{\mathcal{X}} (q_2, n_2)$ and $(q_2, n_2) \not\preceq_{\mathcal{X}} (q_1, n_1)$ and we write $x_{q_1,n_1} = x_{q_2,n_2}$ to mean both $(q_1, n_1) \preceq_{\mathcal{X}} (q_2, n_2)$ and $(q_2, n_2) \preceq_{\mathcal{X}} (q_1, n_1)$. Note that $x_a = x_b < x_c$ and $x_b = x_a < x_c$ are two representations of the same preorder, so this notation is not unique. One example of such a representation is

$$x_{11} = x_{21} < x_{10} < x_{20}.$$

To be a CR the following must hold:

1. If $d \geqslant 1$, $x_{i0} < x_{j0}$ for all $i < j$. Otherwise when $d = 0$, $y_i < y_j$ for all $i < j$. (The element indices are ordered by their lowest-exponent variable.)

2. $y_i \neq y_j \implies x_{in} \neq x_{jn}$ for all n. (Elements that have different y values have all different x values.)

3. $x_{qa} < x_{qb}$ for all $a > b$. (The high-exponent variables of each element are strictly less than the low-exponent variables.)

4. $x_{ia} = x_{jb} \implies a = b$. (Only variables with the same exponent can be equal.)

5. $x_{in} \neq x_{jn} \implies x_{i,n-1} \neq x_{j,n-1}$ for all $n > 0$. (Elements that differ in a high-exponent variable differ in all lower-exponent variables.)

An example of a CR for $\binom{\omega^2}{2}$ is

$$y_1 = 0,\ y_2 = 0,\ x_{11} = x_{21} < x_{10} < x_{20}.$$

Note that because $k = 1$ in the example, we must have $y_q = 0$ for every q.

Two CRs $(\mathcal{Y}, \preceq_{\mathcal{X}})$ and $(\mathcal{Y}', \preceq'_{\mathcal{X}})$ are equivalent if and only if $\mathcal{Y} = \mathcal{Y}'$ and $\preceq_{\mathcal{X}} = \preceq'_{\mathcal{X}}$.

Definition 18. The *size* of a CR is how many equivalence classes its x_{qn} form under $\preceq_X$: for example, $x_{11} = x_{21} < x_{10} < x_{20}$ would have size $p = 3$ regardless of of y_q. Clearly a CR's size p can be no larger than $a \cdot d$, how many x variables $\binom{\omega^d}{a}$ has.

Definition 19.

1. $P_p\left(a, \omega^d \cdot k\right)$ is the number of CRs with size p there are for $\binom{\omega^d \cdot k}{a}$.

2. $P\left(a, \omega^d \cdot k\right)$ is the total number of CRs there are for $\binom{\omega^d \cdot k}{a}$ of any size. It can be calculated as
$$\sum_{p=0}^{a \cdot d} P_p(a, \omega^d \cdot k).$$

We will show $T(a, \omega^d \cdot k) = P(a, \omega^d \cdot k)$.

Definition 20. An edge

$$\{\omega^d \cdot y_q + \omega^{d-1} \cdot x_{q,d-1} + \omega^{d-2} \cdot x_{q,d-2} + \cdots + \omega^1 \cdot x_{q,1} + x_{q,0} \colon 1 \leqslant q \leqslant a\}$$

satisfies some CR if $\mathcal{Y}(q) = y_q$ for every $1 \leqslant q \leqslant a$ and if $(q_1, n_1) \preceq_{\mathcal{X}} (q_2, n_2) \iff x_{q_1,n_1} \leqslant x_{q_2,n_2}$. Note that some edges might not satisfy any CRs.

7 Big Ramsey degrees of ω^d

This section is devoted to the case where $k = 1$ in $\binom{\omega^d \cdot k}{a}$. When $k = 1$, each y_q in a CR is forced to be 0. Then all y_q values are the same, so criterion 2 of Definition 17 is always satisfied. In this section, our proofs focus only on how $\preceq_X$ permutes the x_{qn} variables. In this section, when values for y_q are not specified, they are assumed to be all 0 by default.

This section's aim is to show equality between big Ramsey degrees and numbers of CRs. We start with a recurrence that counts CRs.

Lemma 21. *For integers $a, d \geqslant 0$,*

$$P_p\left(a, \omega^d\right) = \begin{cases} 0 & d = 0 \wedge a \geqslant 2 \\ 1 & a = 0 \wedge p = 0 \\ 0 & a = 0 \wedge p \geqslant 1 \\ 1 & d = 0 \wedge a = 1 \wedge p = 0 \\ 0 & d = 0 \wedge a = 1 \wedge p \geqslant 1 \\ 1 & d = 1 \wedge a \geqslant 1 \wedge a = p \\ 0 & d = 1 \wedge a \geqslant 1 \wedge a \neq p \\ \sum_{j=1}^{a} \sum_{i=0}^{p-1} \binom{p-1}{i} P_i\left(j, \omega^{d-1}\right) P_{p-1-i}\left(a-j, \omega^d\right) & d \geqslant 2 \wedge a \geqslant 1 \end{cases}$$

Proof. First suppose $a \geqslant 2$ and $d = 0$. As argued at the beginning of this section, $y_q = 0$ for all q. But by criterion 1 of Definition 17, since $d = 0$ we need $y_1 < y_2$, so no CRs are possible regardless of size p. This proves the first case of the result.

Suppose $a = 0$. Since there are no y_q variables, and since $a \cdot d = 0$, there are no x variables to permute. Therefore the criteria are vacuously satisfied. There is only one CR and it has size $p = 0$. This proves the second and third cases of the result.

When both $d = 0$ and $a \leqslant 1$, criterion 1 of Definition 17 is vacuously satisfied with either no y_q or $y_1 = 0$. Again, because $a \cdot d = 0$, there are no x_{qn} variables to permute. Thus there is only one CR with size $p = 0$, which proves the fourth and fifth cases of the result.

Now suppose $a \geqslant 1$ and $d = 1$. To ensure criterion 1 of Definition 17, each of the a variables $x_{q,0}$ can only form one CR $x_{1,0} < x_{2,0} < \ldots < x_{a,0}$ with size a so $P_a\left(a, \omega^d\right) = 1$ and $P_p\left(a, \omega^d\right) = 0$ for $p \neq a$. This proves the sixth and seventh cases of the result.

Finally, consider $a \geqslant 1, d \geqslant 2$. We prove this final case by showing that the process described below creates all possible CRs of an expression.

For arbitrary integers $a \geqslant 1$, $d \geqslant 2$, and $p \geqslant 0$, let $1 \leqslant j \leqslant a$ and $0 \leqslant i \leqslant p-1$ be integers. As we proceed through the process, we work with an example of $a = 4$, $d = 5$, $p = 13$, $j = 2$, and $i = 5$.

We create

$$\binom{p-1}{i} P_i\left(j, \omega^{d-1}\right) P_{p-1-i}\left(a-j, \omega^d\right)$$

CRs of size p by combining $P_i\left(j,\omega^{d-1}\right)$ CRs with size i and $P_{p-1-i}\left(a-j,\omega^{d}\right)$ CRs with size $p-1-i$, with $\binom{p-1}{i}$ CRs derived from each pair.

Let τ_1 represent one of the $P_i\left(j,\omega^{d-1}\right)$ CRs of $\binom{\omega^{d-1}}{j}$ with size i, and τ_2 represent one of the $P_{p-1-i}\left(a-j,\omega^{d}\right)$ CRs of $\binom{\omega^{d}}{a-j}$ with size $p-1-i$. In our example, let

$$\tau_1\colon x_{1,3}=x_{2,3}<x_{1,2}=x_{2,2}<x_{1,1}=x_{2,1}<x_{1,0}<x_{2,0}$$
$$\tau_2\colon x_{1,4}=x_{2,4}<x_{1,3}=x_{2,3}<x_{1,2}=x_{2,2}<x_{2,1}<x_{1,1}<x_{1,0}<x_{2,0}.$$

Then we can combine each τ_1 and τ_2 to form $\binom{p-1}{i}$ unique new CRs of size p: Reindex each variable $x_{q,n}$ as τ_2 to $x_{q+j,n}$, and permute the equivalence classes of the CRs together, preserving each CR's original ordering of its own equivalence classes: there are $\binom{p-1}{i}$ ways to do this. In our example, after reindexing τ_2 we have

$$\tau_1\colon x_{1,3}=x_{2,3}<x_{1,2}=x_{2,2}<x_{1,1}=x_{2,1}<x_{1,0}<x_{2,0}$$
$$\tau_2\colon x_{3,4}=x_{4,4}<x_{3,3}=x_{4,3}<x_{3,2}=x_{4,2}<x_{4,1}<x_{3,1}<x_{3,0}<x_{4,0}$$

and one of the $\binom{12}{5}$ permutations is

$$x_{3,4}=x_{4,4}<x_{1,3}=x_{2,3}<x_{1,2}=x_{2,2}<x_{3,3}=x_{4,3}<x_{3,2}=x_{4,2}$$
$$<x_{1,1}=x_{2,1}<x_{4,1}<x_{1,0}<x_{3,1}<x_{3,0}<x_{2,0}<x_{4,0}.$$

This new CR likely breaks criterion 1 of Definition 17; for each $1\leqslant q\leqslant a$, reindex each $x_{q,n}$ according to where $x_{q,0}$ is in the ordering of all $x_{i,0}$. In our example, we have $x_{1,0}<x_{3,0}<x_{2,0}<x_{4,0}$; after swapping indices 2 and 3 to enforce criterion 1 we have

$$x_{2,4}=x_{4,4}<x_{1,3}=x_{3,3}<x_{1,2}=x_{3,2}<x_{2,3}=x_{4,3}<x_{2,2}=x_{4,2}$$
$$<x_{1,1}=x_{3,1}<x_{4,1}<x_{1,0}<x_{2,1}<x_{2,0}<x_{3,0}<x_{4,0}.$$

There are now $d\cdot(a-j)+(d-1)\cdot j=a\cdot d-j$ variables in the CR. There are j variables of the form $x_{q_i,d-1}$ for $1\leqslant i\leqslant j$ that are not in the CR yet; insert one equivalence class $x_{q_1,d-1}=x_{q_2,d-1}=\cdots=x_{q_j,d-1}$ at the front of the new CR, bringing its size to p. We insert $x_{1,4}=x_{3,4}$ in our example to get

$$x_{1,4}=x_{3,4}<x_{2,4}=x_{4,4}<x_{1,3}=x_{3,3}<x_{1,2}=x_{3,2}<x_{2,3}=x_{4,3}<x_{2,2}=x_{4,2}$$
$$<x_{1,1}=x_{3,1}<x_{4,1}<x_{1,0}<x_{2,1}<x_{2,0}<x_{3,0}<x_{4,0}.$$

Each CR is unique by the τ_1 and τ_2 used to create it because the process is invertible: we can remove the leading equivalence class and separate the remaining variables into τ_1 and τ_2 by whether their indices were in the leading equivalence class and reindexing. Here, τ_1 corresponds to indices 1 and 3 (not including the leading equivalence class) and is **bolded**, and τ_2 corresponds to indices 2 and 4 and is underlined.

$$x_{1,4}=x_{3,4}<\underline{x_{2,4}=x_{4,4}}<\mathbf{x_{1,3}=x_{3,3}}<\mathbf{x_{1,2}=x_{3,2}}<\underline{x_{2,3}=x_{4,3}}<\underline{x_{2,2}=x_{4,2}}$$
$$<\mathbf{x_{1,1}=x_{3,1}}<\underline{x_{4,1}}<\mathbf{x_{1,0}}<\underline{x_{2,1}}<\underline{x_{2,0}}<\mathbf{x_{3,0}}<\underline{x_{4,0}}.$$

We now show that each CR created by this process has the properties described by Definition 17. Because the high-exponent equivalence class $x_{q_1,d-1} = x_{q_2,d-1} = \cdots = x_{q_j,d-1}$ was added at the start of the CR, the high-exponent coefficients of each term are smaller than the low-exponent coefficients. Criterion 1 is satisfied by reindexing the variables. The remaining criteria are satisfied because τ_1 and τ_2 satisfied them and their internal orders were preserved in permuting the equivalence classes. Therefore this process does not overcount CRs.

We next show that every CR of $\binom{\omega^d}{a}$ is counted by this process. Each can be mapped to some τ_1 and τ_2 that create it by a similar argument to proving that the process creates unique CRs. Every CR of $\binom{\omega^d}{a}$ must have a leading equivalence class of $x_{q_1,d-1} = x_{q_2,d-1} = \cdots = x_{q_j,d-1}$ to satisfy Definition 17 (the equivalence class might only contain one variable); taking only the variables $x_{q,n}$ with indices appearing in that equivalence class (but not those variables in the equivalence class itself) forms τ_1, a CR for $\binom{\omega^{d-1}}{j}$. The variables with q indices not in the equivalence class form τ_2, a CR for $\binom{\omega^d}{a-j}$. The original CR of $\binom{\omega^d}{a}$ is counted by interleaving τ_1 with τ_2 and inserting the leading equivalence class of $x_{q_1,d-1} = x_{q_2,d-1} = \cdots = x_{q_j,d-1}$. Therefore the final case of the result holds. □

For more information, see the OEIS [11], where $P(a, \omega^2)$ is sequence A000311 and $P(2, \omega^d)$ is A079309. Other sequences such as $P(a, \omega^3)$ are not contained in the OEIS at the time this paper was produced, although we can compute them. Values for small a, d are tabulated in the appendix; see Table 1.

7.1 $T(a, \omega^d) \leqslant P(a, \omega^d)$

We use the following lemma to show that CRs bound big Ramsey degrees from above.

Lemma 22. *For integers $a, d \geqslant 0$ and $G \approx \omega$, there exists some $H \subseteq \omega^d$ with $H \approx \omega^d$ where for all $e \in \binom{H}{a}$, e satisfies a CR of $\binom{\omega^d}{a}$ and each coefficient in e is contained in G.*

Proof. Because $G \approx \omega$, we can index it $x_0, x_1, x_2, \ldots$ with $x_0 < x_1 < x_2 < \cdots$. We proceed by induction on d. When $d = 0$, $\omega^0 \approx 1$ and so $H = \{x_0\}$ suffices. When $d = 1$, $\omega^1 \approx \omega$ so $H = G$ suffices.

For $d \geqslant 2$, partition $G \setminus \{x_0\}$ into infinite sets order-equivalent to ω:

$$\begin{aligned} X_0 &= \{x_1, x_3, x_5, \ldots\} \\ X_1 &= \{x_2, x_6, x_{10}, \ldots\} \\ X_2 &= \{x_4, x_{12}, x_{20}, \ldots\} \\ X_3 &= \{x_8, x_{24}, x_{40}, \ldots\} \\ &\vdots \end{aligned}$$

Formally,

$$X_i = \{x_j \colon j = 2^i + k2^{i+1}, k \in \mathbb{N}\}.$$

Apply the inductive hypothesis on X_i for all $i \geqslant 1$, yielding $S_i \approx \omega^{d-1}$ for all $i \geqslant 1$. Then for all $i \geqslant 1$, for all $e \in \binom{S_i}{a}$, e satisfies a CR of $\binom{\omega^{d-1}}{a}$ and each coefficient of e is contained in X_i. For all i, let $S_i = \{y_{i,0}, y_{i,1}, \ldots\}$. Then let

$$\begin{aligned} H = \quad & \{\omega^{d-1}x_1 + y_{1,0}, \omega^{d-1}x_1 + y_{1,1}, \omega^{d-1}x_1 + y_{1,2}, \ldots\} \\ & + \{\omega^{d-1}x_3 + y_{2,0}, \omega^{d-1}x_3 + y_{2,1}, \omega^{d-1}x_3 + y_{2,2}, \ldots\} \\ & + \{\omega^{d-1}x_5 + y_{3,0}, \omega^{d-1}x_5 + y_{3,1}, \omega^{d-1}x_5 + y_{3,2}, \ldots\} \\ & + \ \cdots . \end{aligned}$$

Then H is the concatenation of ω ordered sets, each order-equivalent to ω^{d-1}. Hence $H \approx \omega^d$. For any edge e in $\binom{S}{a}$, index its variables to satisfy criterion 1 of Definition 17 (this is possible because all low-exponent coefficients are distinct in H). Criterion 3 is satisfied inductively for variables with exponents lower than $d-1$. Because $\min X_i = x_{2^i}$ for all i and $2i-1 < 2^i$ for all integers $i \geqslant 1$, $x_{2i-1} < x$ for all $x \in X_i$. So criterion 3 is satisfied by e. Because X_0 is disjoint from all X_i with $i \geqslant 1$, criterion 4 is satisfied for variables with exponent $d-1$ and by induction, it is satisfied for lower exponents. Because X_i is disjoint with X_j for all $i \neq j$, elements that differ in variables with exponent $d-1$ differ in all lower-exponent variables. The induction with the previous statement satisfies criterion 5. Therefore e satisfies a CR of $\binom{\omega^d}{a}$. The coefficients in e are contained in G by the construction of H from G. □

Theorem 23. *For integers $a, d \geqslant 0$, $T\left(a, \omega^d\right) \leqslant P\left(a, \omega^d\right)$.*

Proof. Let $E = \binom{\omega^d}{a}$ and

$$\mathrm{COL}\colon E \to [b]$$

be an arbitrary coloring of E for some $b \in \mathbb{N}$.

Enumerate the CRs of E as τ_1 to $\tau_{P(a,\omega^d)}$. The maximum size of any CR of E is $a \cdot d$. For each τ_i, let

$$f_i\colon \binom{\omega}{a \cdot d} \to E$$

where if τ_i has size p, f_i maps X to the unique $e \in E$ where e satisfies τ_i and the p equivalence classes of e are made up of the p least elements of X. For example, one CR of $\binom{\omega^2}{2}$ is

$$x_{11} = x_{21} < x_{10} < x_{20}.$$

The corresponding f_i would be $f_i\colon \binom{\omega}{4} \to \binom{\omega^2}{2}$ with

$$f_i(x_1, x_2, x_3, x_4) = \mathrm{COL}(\{\omega \cdot x_1 + x_2, \omega \cdot x_1 + x_3\})$$

where $x_1 < x_2 < x_3 < x_4$. Note that f_i does not depend on x_4 – this is because the example CR has size 3, but the largest CR for $\binom{\omega^2}{2}$ has size 4.

Then, define $\mathrm{COL}'\colon \binom{\omega}{a\cdot d} \to [b]^{P(a,\omega^d)}$ with

$$\mathrm{COL}'(X) = (f_1(X), f_2(X), \ldots, f_{P(a,\omega^d)}(X))$$

and apply Theorem 9 to find some $G \approx \omega$ where

$$\left|\mathrm{COL}'\left(\binom{G}{a\cdot d}\right)\right| = 1.$$

Let the one color in $\mathrm{COL}'(\binom{G}{a\cdot d})$ be Y. Note that Y is a tuple of $P(a,\omega^d)$ colors.

Apply Lemma 22 to find some $H \approx \omega^d$ with the properties listed in Lemma 22. Now we claim

$$\left|\mathrm{COL}\left(\binom{H}{a}\right)\right| \leqslant P(a,\omega^d).$$

By Lemma 22, each element $e \in \binom{H}{a}$ satisfies a CR of E. Then for any arbitrary edge e, let e satisfy τ_i with size $p \leqslant a\cdot d$. Then take the p unique values in e, and if necessary, insert any new larger nonnegative integers from G to form a set of $a\cdot d$ values; denote this $X \in \binom{G}{a\cdot d}$. $\mathrm{COL}'(X) = Y$ so by the definition of COL', $\mathrm{COL}(e) \in Y$. Because $|Y| = P(a,\omega^d)$, $T(a,\omega^d) \leqslant P(a,\omega^d)$. □

7.2 $T(a,\omega^d) \geqslant P(a,\omega^d)$

Theorem 24. *For integers $a,d \geqslant 0$, $T\left(a,\omega^d\right) \geqslant P\left(a,\omega^d\right)$.*

Proof. If $P(a,\omega^d) = 0$, this is satisfied vacuously because $T(a,\omega^d) \geqslant 0$. Now suppose $P(a,\omega^d) \geqslant 1$. Let $E = \binom{\omega^d}{a}$. Note that all CRs of E are disjoint from each other. That is, for any edge $e \in E$, if e satisfies τ', then it does not satisfy any nonequivalent CR of E. This is because if e were to satisfy two CRs τ_1 and τ_2, then τ_1 and τ_2 must share the same equivalence classes and order, so the CRs must be equivalent. Therefore, we can index them $\tau_1 \ldots \tau_{P(a,\omega^d)}$ and construct a coloring $\mathrm{COL}\colon E \to [P(a,\omega^d)]$ with

$$\mathrm{COL}(e) = \begin{cases} i & e \text{ satisfies } \tau_i \\ 1 & \text{otherwise} \end{cases}$$

Similar to the proof of Theorem 16, our coloring has two ways to output color 1, both through satisfaction of τ_1 and through the catch-all case. The part that forces color 1 to be present in all order-equivalent subsets is the satisfaction of τ_1.

We now prove there is no ω^d-$(P(a,\omega^d)-1)$-homogeneous set. For all $H \approx \omega^d$ and for every CR τ of E, we find some $e \in \binom{H}{a}$ that satisfies τ. For arbitrary $H \approx \omega^d$ and τ, we find z_{qn} where

$$\{\omega^{d-1}z_{1,d-1} + \cdots + \omega^1 z_{1,1} + z_{1,0}, \ldots, \omega^{d-1}z_{a,d-1} + \cdots + \omega^1 z_{a,1} + z_{a,0}\}$$

satisfies τ.

We do this by assigning values to each z_{qn} according to where the equivalence class that contains x_{qn} is found in τ, moving left to right in τ's permutation. By criterion 3 of Definition 17, each z_{qn} is assigned before $z_{q,n-1}$. As we do this, we ensure that if the leftmost unassigned value in τ is z_{qn}, then

$$\{\omega^{d-1}z_{q,d-1}+\cdots+\omega^{n+1}z_{q,n+1}+\omega^n c_n+\omega^{n-1}c_{n-1}+\cdots+\omega^1 c_1+c_0 \mid c_i\in\omega\}\cap H\approx\omega^{n+1}.$$

By criterion 3 of Definition 17, the leftmost variable in τ must be $x_{q,d-1}$. Before any values are assigned, it is clear that

$$\{\omega^{d-1}c_{d-1}+\cdots+\omega^1 c_1+c_0 \mid c_i\in\omega\}=\omega^d,$$

and because $H\subseteq\omega^d$, $\omega^d\cap H=H\approx\omega^d$.

By criterion 4 of Definition 17, all variables in an equivalence class must have the same exponent d. Let the leftmost equivalence class in τ be $x_{q_1,n}=x_{q_2,n}=\cdots=x_{q_m,n}$. By criterion 3, each $x_{q_i,\ell}$ for $1\leqslant i\leqslant m$ and $\ell>n$ appeared to the left of this equivalence class and has already been assigned a value, and by criterion 5 the values for each exponent are equal: for all $\ell>n$ and $1\leqslant i\leqslant m$, $z_{q_i,\ell}=z_{q_1,\ell}$.

By our previous steps,

$$\{\omega^{d-1}z_{q_1,d-1}+\cdots+\omega^{n+1}z_{j,q_1,n+1}+\omega^n c_n+\omega^{n-1}c_{n-1}+\cdots+\omega^1 c_1+c_0 \mid c_i\in\omega\}\cap H\approx\omega^{n+1}.$$

Then there exists some value z' where

$$\{\omega^{d-1}z_{q_1,d-1}+\cdots+\omega^{n+1}z_{q_1,n+1}+\omega^n z'+\omega^{n-1}c_{n-1}+\cdots+\omega^1 c_1+c_0 \mid c_i\in\omega\}\cap H\approx\omega^{n},$$

where z' is greater than all previously assigned (and therefore finite) z_{qn} values. Then for $1\leqslant i\leqslant m$, assign $z_{q_i,n}$ to be z'.

We can repeat this process to find z_{qn} for each CR of E for arbitrary $H\approx\omega^d$. Therefore for all $H\approx\omega^d$, $|\operatorname{COL}(\binom{S}{a})|\geqslant P(a,\omega^d)$ so $T(a,\omega^d)\geqslant P\left(a,\omega^d\right)$. □

7.3 $T(a,\omega^d)=P(a,\omega^d)$

Theorem 25. *For all $a,d\in\mathbb{N}$, $T(a,\omega^d)=P(a,\omega^d)$.*

Proof. By Theorem 23, $T(a,\omega^d)\leqslant P(a,\omega^d)$. By Theorem 24, $T(a,\omega^d)\geqslant P(a,\omega^d)$. The result follows. □

8 Big Ramsey degrees of $\omega^d\cdot k$

We now use the theory we developed for the case $k=1$ to prove results for arbitrary k. We first extend the recurrence from Lemma 21.

Lemma 26. *For integers $a, d, k \geqslant 0$,*

$$P_p\left(a, \omega^d \cdot k\right) = \begin{cases} 0 & d = 0 \wedge a > k \\ 1 & a = 0 \wedge p = 0 \\ 0 & a = 0 \wedge p \geqslant 1 \\ \binom{k}{a} & d = 0 \wedge 1 \leqslant a \leqslant k \wedge p = 0 \\ 0 & d = 0 \wedge 1 \leqslant a \leqslant k \wedge p \geqslant 1 \\ k^a & d = 1 \wedge a \geqslant 1 \wedge a = p \\ 0 & d = 1 \wedge a \geqslant 1 \wedge a \neq p \\ k \sum_{j=1}^{a} \sum_{i=0}^{p-1} \binom{p-1}{i} P_i\left(j, \omega^{d-1}\right) P_{p-1-i}\left(a-j, \omega^d \cdot k\right) & d \geqslant 2 \wedge a \geqslant 1 \end{cases}$$

Proof. First suppose $a > k$ and $d = 0$. Since $0 \leqslant y_q < k$ for all y_q, there are at most k unique values for the y_q. But by criterion 1 of Definition 17, since $d = 0$ we need a unique values of y_q, so no CRs are possible regardless of size p. This proves the first case of the result.

Suppose $a = 0$. Then there can be no y_q, and since $a \cdot d = 0$, there can be no x_{qn}. Thus all criteria are vacuously satisfied. Because there are no y_q or x_{qn}, there is only one CR, and it has size $p = 0$. This proves the second and third cases of the result.

When both $d = 0$ and $a \leqslant k$, criterion 1 of Definition 17 can be satisfied with the assignments to the a variables y_q being any permutation of a unique values out of k possible integer values. This leads to $\binom{k}{a}$ feasible combinations. Again, because $a \cdot d = 0$, there are no variables x_{qn} to permute so there are $\binom{k}{a}$ empty CRs with size $p = 0$, which proves the fourth and fifth cases of the result.

Now suppose $a \geqslant 1$ and $d = 1$. To ensure criteria 1 of Definition 17, each of the a variables $x_{q,0}$ can only form one permutation $x_{1,0} < x_{2,0} < \ldots < x_{a,0}$ with size a. Because all x_{qn} are distinct and $d = 1$, the values y_q are not restricted by any criteria so each of the a variables can be any of the k integers. Therefore $P_a\left(a, \omega^d \cdot k\right) = k^a$ and $P_p\left(a, \omega^d \cdot k\right) = 0$ for $p \neq a$. This proves the sixth and seventh cases of the result.

Finally, consider $a \geqslant 1, d \geqslant 2$. We prove the final case of our result by showing the process for combining CRs described below creates all possible CRs of an expression.

For arbitrary integers $a \geqslant 1$, $d \geqslant 2$, $k \geqslant 0$, and $p \geqslant 0$, let $1 \leqslant j \leqslant a$ and $0 \leqslant i \leqslant p-1$ be integers.

We create

$$k\binom{p-1}{i} P_i\left(j, \omega^{d-1}\right) P_{p-1-i}\left(a-j, \omega^d \cdot k\right)$$

CRs of size p by combining $P_i\left(j, \omega^{d-1}\right)$ CRs with size i and $P_{p-1-i}\left(a-j, \omega^d \cdot k\right)$ CRs with size $p-1-i$, with $k\binom{p-1}{i}$ new CRs for each pair.

Let τ_1 represent one of the $P_i\left(j, \omega^{d-1}\right)$ CRs of $\binom{\omega^{d-1}}{j}$ with size i, and τ_2 represent one of the $P_{p-1-i}\left(a-j, \omega^d \cdot k\right)$ CRs of $\binom{\omega^d \cdot k}{a-j}$ with size $p-1-i$.

Then we can combine each τ_1 and τ_2 to form $k\binom{p-1}{i}$ unique new CRs of size p: Reindex τ_2, permute the equivalence classes, and insert a leading equivalence class $x_{q_1,d-1} = x_{q_2,d-1} = \cdots = x_{q_j,d-1}$ as in the proof of Lemma 21. This leads to $\binom{p-1}{i}$ new permutations of the x variables.

Because τ_1 was a CR for $\binom{\omega^{d-1}}{j}$, each of its y_q had value 0. Now that we are creating a CR for $\binom{\omega^d \cdot k}{a}$, we can choose the y coefficients to be composed of values between 0 and $k-1$. By criterion 2 of Definition 17, because all elements from τ_1 are bound together in a leading high-exponent equivalence class, they must all have equal y values. This leads to k options for these y values; with the options of permuting the x variables, $k\binom{p-i}{i}$ ways to create a new CR.

For the new CR's values for y_q, we assign each element originally from τ_2 with its original y value (likely at a different index due to reindexing). Then, the remaining elements from τ_1 are given all the same y value from one of the k options.

Each CR is unique by the τ_1 and τ_2 used to create it because the process is invertible: we can remove the leading equivalence class and separate the remaining variables into τ_1 and τ_2 by whether their indices were in the leading equivalence class and reindexing. The y values for τ_2 can be found from the CR's y values after reversing the index change, and the y values for τ_1 are all 0.

We claim each CR created by this process has the properties described by Definition 17: Because the high-exponent equivalence class $x_{q_1,d-1} = x_{q_2,d-1} = \cdots = x_{q_j,d-1}$ was added at the start of the CR, the high-exponent coefficients of each term are smaller than the low-exponent coefficients. Criterion 1 is satisfied by reindexing the variables. Criterion 2 is met by assigning all elements from τ_1 the same y value. The remaining criteria are satisfied because τ_1 and τ_2 satisfied them and their internal orders were preserved in permuting the equivalence classes. Therefore this process does not overcount CRs.

We also claim that every CR of $\binom{\omega^d \cdot k}{a}$ is counted by this process: each can be mapped to some τ_1 and τ_2 that create it by a similar argument to proving that the process creates unique CRs. Every CR of $\binom{\omega^d \cdot k}{a}$ must have a leading equivalence class of $x_{q_1,d-1} = x_{q_2,d-1} = \cdots = x_{q_j,d-1}$ to satisfy Definition 17 (the equivalence class might only contain one variable); taking only the variables $x_{q,n}$ with indices appearing in that equivalence class (but not those variables in the equivalence class itself) with all-zero y values forms τ_1, a CR for $\binom{\omega^{d-1}}{j}$. The variables with q indices not in the equivalence class with their y values form τ_2, a CR for $\binom{\omega^d \cdot k}{a-j}$. The original CR of $\binom{\omega^d \cdot k}{a}$ was counted by interleaving τ_1 with τ_2 and inserting the leading equivalence class of $x_{q_1,d-1} = x_{q_2,d-1} = \cdots = x_{q_j,d-1}$. Therefore the final case of the result holds. □

8.1 $T(a, \omega^d \cdot k) \leqslant P(a, \omega^d \cdot k)$

Lemma 27. *For nonnegative integers a, d, k and $G \approx \omega$, there exists some $H \subseteq \omega^d \cdot k$, $H \approx \omega^d \cdot k$ where for all $e \in \binom{H}{a}$, e satisfies a CR of $\binom{\omega^d \cdot k}{a}$ and each coefficient in e is contained in G.*

Proof. Because $G \approx \omega$, we can index it $x_1, x_2, x_3, \ldots$ with $x_1 < x_2 < x_3 < \cdots$.

If $d = 0$, $H = \{x_1, x_2, \ldots, x_k\}$ suffices.

If $d \geqslant 1$, we can first apply Lemma 22 with G to attain some $H' \approx \omega^{d+1}$ with the listed properties. Then, let H be the first k copies of ω^d within H': formally,

$$H = \{\omega^d \cdot y + \omega^{d-1} \cdot x_{d-1} + \ldots + \omega^1 \cdot x_1 + x_0 \in H' \mid y < k\}.$$

Because the edges of H' satisfied criterion 5 of Definition 17 at exponent $n = d + 1$, the edges of H satisfy criterion 2. The remaining criteria are satisfied because H' satisfied them. □

Theorem 28. *For integers $a, d, k \geqslant 0$, $T\left(a, \omega^d \cdot k\right) \leqslant P\left(a, \omega^d \cdot k\right)$.*

Proof. Let $E = \binom{\omega^d \cdot k}{a}$ and

$$\mathrm{COL}\colon E \to [b]$$

be an arbitrary coloring of E for some $b \in \mathbb{N}$.

Enumerate the CRs of E from τ_1 to $\tau_{P(a,\omega^d \cdot k)}$. The maximum size of any CR of E is $a \cdot d$. For each τ_i, let

$$f_i\colon \binom{\omega}{a \cdot d} \to E$$

where if τ_i has size p, f_i maps X to the unique $e \in E$ where e satisfies τ_i and the p equivalence classes of e are made up of the p least elements of X. For example, one CR of $\binom{\omega^2 \cdot 2}{2}$ is

$$y_1 = 0,\ y_2 = 1,\ x_{11} < x_{21} < x_{20} < x_{10}.$$

The corresponding f_i would be $f_i\colon \binom{\omega}{4} \to \binom{\omega^2 \cdot 2}{2}$ with

$$f_i(x_1, x_2, x_3, x_4) = \mathrm{COL}(\{\omega^2 \cdot 0 + \omega \cdot x_1 + x_4, \omega^2 \cdot 1 + \omega \cdot x_2 + x_3\})$$

where $x_1 < x_2 < x_3 < x_4$. Note that the values of the y_q's are used directly in of the definition of f_i – for the CR with identical orderings on x_{qn} except $y_0 = 1$ and $y_1 = 0$, the coefficients y_q would be swapped.

Then, define $\mathrm{COL}'\colon \binom{\omega}{a \cdot d} \to [b]^{P(a,\omega^d \cdot k)}$ with

$$\mathrm{COL}'(X) = (f_1(X), f_2(X), \ldots, f_{P(a,\omega^d \cdot k)}(X))$$

and apply Theorem 9 to find some $G \approx \omega$ where

$$\left|\mathrm{COL}'\left(\binom{G}{a \cdot d}\right)\right| = 1$$

Let Y be the one color in $\mathrm{COL}'(\binom{G}{a \cdot d})$. Note that Y is a tuple of $P(a, \omega^d \cdot k)$ colors.

Apply Lemma 27 to find some $H \approx \omega^d \cdot k$ with the properties listed in Lemma 27. Now we claim

$$\left|\mathrm{COL}\left(\binom{H}{a}\right)\right| \leqslant P(a, \omega^d \cdot k).$$

By Lemma 27, each element $e \in \binom{H}{a}$ satisfies a CR of E. Then for any arbitrary edge e, let e satisfy τ_i with size $p \leqslant a \cdot d$. Then take the p unique values in e, and if necessary, insert any new larger nonnegative integers from G to form a set of $a \cdot d$ values; denote this $X \in \binom{G}{a \cdot d}$. $\mathrm{COL}'(X) = Y$ so by the definition of COL', $\mathrm{COL}(e) \in Y$. Because $|Y| = P(a, \omega^d \cdot k)$, $T(a, \omega^d \cdot k) \leqslant P(a, \omega^d \cdot k)$. □

8.2 $T(a, \omega^d \cdot k) \geqslant P(a, \omega^d \cdot k)$

Theorem 29. *For $a, d, k \in \mathbb{N}$, $T\left(a, \omega^d \cdot k\right) \geqslant P\left(a, \omega^d \cdot k\right)$.*

Proof. If $P(a, \omega^d \cdot k) = 0$, this is satisfied vacuously because $T(a, \omega^d \cdot k) \geqslant 0$. Suppose $P(a, \omega^d \cdot k) \geqslant 1$. Let $E = \binom{\omega^d \cdot k}{a}$. Note that all CRs of E are disjoint from each other. That is, for any edge $e \in E$, if e satisfies τ', then it does not satisfy any nonequivalent CR of E. This is because if e were to satisfy two CRs τ_1 and τ_2, then τ_1 and τ_2 must share the same y_q values, equivalence classes and order, so the CRs must be equivalent. Therefore, we can index them $\tau_1 \ldots \tau_{P(a,\omega^d)}$ and construct a coloring $\mathrm{COL}\colon E \to [P(a, \omega^d \cdot k)]$ with

$$\mathrm{COL}(e) = \begin{cases} i & e \text{ satisfies } \tau_i \\ 1 & \text{otherwise} \end{cases}$$

Similar to Theorem 16, our coloring has two ways to output color 1, both through satisfaction of τ_1 and through the catch-all case. The part that forces color 1 to be present in all order-equivalent subsets is the satisfaction of τ_1. For arbitrary $H \approx \omega^d \cdot k$ and τ, we find variables y_q and z_{qn} where

$$\{\omega^d y_1 + \omega^{d-1} z_{1,d-1} + \cdots + \omega^1 z_{1,1} + z_{1,0}, \ldots, \omega^d y_a + \omega^{d-1} z_{a,d-1} + \cdots + \omega^1 z_{a,1} + z_{a,0}\}$$

satisfies τ.

For any $H \approx \omega^d \cdot k$ and τ, we first separate H into k ordered sets by the leading coefficient, each order-equivalent to ω^d.

Then, if there are equivalence classes in τ, using the process formally described in the proof of Theorem 24, we consider the leading equivalence class of τ. By criterion 2 of Definition 17, all variables in that equivalence class must come from same set order-equivalent to ω^d. We assign a finite value to that equivalence class, and move to the next class with a potentially different y value, using the assigned finite value as a lower bound for the next one. We can repeat this process to find z_{qn} that satisfy every CR of E for arbitrary $H \approx \omega^d$. Then, we can assign the y_q variables directly as the y variables in τ.

If there are no equivalence classes in τ (it has size $p = 0$), we can simply assign the variables y_q directly according to τ.

Therefore for all $H \approx \omega^d$, $|\mathrm{COL}\left(\binom{H}{a}\right)| \geqslant P(a, \omega^d \cdot k)$ so $T(a, \omega^d \cdot k) \geqslant P\left(a, \omega^d \cdot k\right)$. □

8.3 $T(a, \omega^d \cdot k) = P(a, \omega^d \cdot k)$

Theorem 30. *For $a, d, k \in \mathbb{N}$, $T(a, \omega^d \cdot k) = P(a, \omega^d \cdot k)$.*

Proof. By Theorem 28, $T(a, \omega^d{\cdot}k) \leqslant P(a, \omega^d{\cdot}k)$. By Theorem 29, $T(a, \omega^d{\cdot}k) \geqslant P(a, \omega^d{\cdot}k)$. The result follows. □

9 Big Ramsey degrees of ordinals less than ω^ω

9.1 General Coloring Rules

We defined CRs (coloring rules) to compute big Ramsey degrees of ordinals of the form $\omega^d \cdot k$. We'll now extend the definition to *GCRs* (general coloring rules), which allows us to compute big Ramsey degrees for all ordinals less than ω^ω.

Definition 31. We now define GCRs rigorously. Much like the definition of CRs, we impose a structure on edges, and then list criteria that GCRs must satisfy. Consider some ordinal $\alpha < \omega^\omega$:

$$\alpha \approx \omega^d \cdot k_d + \omega^{d-1} \cdot k_{d-1} + \cdots + \omega \cdot k_1 + k_0.$$

Then α is the addition of $d+1$ ordinals each of the form $\omega^n \cdot k_n$. For any element β of α, there is some least q such that $b \in \omega^q \cdot k_q$. In this case, we write β *originated* from the ω^q part of α.

For an integer $a \geqslant 0$, there are a elements in $e \in \binom{\alpha}{a}$. Unlike the definition of CRs, each element can have anywhere from 0 to d variables x_{qn}, depending on which of the $d+1$ ordered sets the element originated from. For $1 \leqslant q \leqslant a$, we use c_q for the number of variables x_{qn} element q has (the element therefore originated from the ω^{c_q} part of α). We denote each element as

$$\omega^{c_q} \cdot y_q + \omega^{c_q-1} \cdot x_{q,c_q-1} + \omega^{c_q-2} \cdot x_{q,c_q-2} + \cdots + \omega^1 \cdot x_{q,1} + x_{q,0}.$$

where $0 \leqslant y_q < k_{c_q}$ and $0 \leqslant x_{qn}$.

A *general coloring rule*, hereafter referred to as a GCR, is a triple $(\mathcal{C}, \mathcal{Y}, \preceq_\mathcal{X})$ of constraints on the c_q, y_q, and $x_{q,n}$.

$\mathcal{C}$ is a map $\mathcal{C} : [a] \to \{0, \ldots, d\}$ from indices of the c_q to the possible values for the c_q. Similarly, $\mathcal{Y}$ is a map $\mathcal{Y} : [a] \to \{0, \ldots, k_{c_q}\}$ from indices of the y_q to the possible values for the y_q. $\preceq_\mathcal{X}$ is a total preorder on the indices of the $x_{q,n}$. We will continue to use the same notation to represent the preorder.

GCRs must fulfill the following criteria (only criterion 6 below is different from its corresponding criterion in Definition 17):

1. If $d \geqslant 1$, $x_{i0} < x_{j0}$ for all $i < j$. Otherwise when $d = 0$, $y_i < y_j$ for all $i < j$. (The element indices are ordered by their lowest-exponent variable.)

2. $y_i \neq y_j \implies x_{in} \neq x_{jn}$ for all n. (Elements that have a different y value have all different x values.)

3. $x_{qa} < x_{qb}$ for all $a > b$. (The high-exponent variables of each element are strictly less than the low-exponent variables.)

4. $x_{ia} = x_{jb} \implies a = b$. (Only variables with the same exponent can be equal.)

5. $x_{in} \neq x_{jn} \implies x_{i,n-1} \neq x_{j,n-1}$ for all $n > 0$. (Elements that differ in a high-exponent variable differ in all lower-exponent variables.)

6. $c_i \neq c_j \implies (y_i \neq y_j$ and $x_{in} \neq x_{jn})$ for all $0 \leqslant n < d$. (Different c variables mean different y and x variables.)

For an edge

$$e = \{\omega^{c_q} \cdot y_q + \omega^{c_q - 1} \cdot x_{q,c_q-1} + \omega^{c_q-2} \cdot x_{q,c_q-2} + \cdots + \omega^1 \cdot x_{q,1} + x_{q,0} \colon 1 \leqslant q \leqslant a\},$$

e *satisfies* the GCR $(\mathcal{C}, \mathcal{Y}, \preceq_{\mathcal{X}})$ if $c_q = \mathcal{C}(q)$ and $y_q = \mathcal{Y}(q)$ for every $1 \leqslant q \leqslant k_{c_q}$, and if $(q_1, n_1) \preceq_{\mathcal{X}} (q_2, n_2) \iff x_{q_1,n_1} \leqslant x_{q_2,n_2}$ for all $1 \leqslant q_1, q_2 \leqslant k_{c_q}$ and $0 \leqslant n_1, n_2 < d$.

Definition 32. We again define the *size* of a GCR to be how many equivalence classes its x variables form. A GCR's size p is still bounded above by $d \cdot a$.

Definition 33.

1. $S_p(a, \alpha)$ is the number of GCRs with size p there are for $\binom{\alpha}{a}$.

2. $S(a, \alpha)$ is the total number of GCRs there are for $\binom{\alpha}{a}$ regardless of size. It can be calculated as
$$\sum_{p=0}^{a \cdot d} S_p(a, \alpha).$$

   We will show $T(a, \alpha) = S(a, \alpha)$.

Lemma 34. *For $a, d, k, p \in \mathbb{N}$, $S_p(a, \omega^d \cdot k) = P_p(a, \omega^d \cdot k)$.*

Proof. Let $\alpha = \omega^d \cdot k$. Suppose some $c_q \neq d$. Then $y_q < k_{c_q}$ by Definition 31, so $y_q < 0$, which is impossible. Thus $c_q = d$. Then there are the same count of $a \cdot d$ variables x_{qn} being permuted, the new criterion 6 has no effect because all c_q are equal. Hence both are under the same restrictions so $S_p(a, \omega^d \cdot k) = P_p(a, \omega^d \cdot k)$. □

Lemma 35. *For all $\alpha < \omega^\omega$ with*

$$\alpha \approx \omega^d \cdot k_d + \omega^{d-1} \cdot k_{d-1} + \cdots + \omega \cdot k_1 + k_0 \text{ with } k_d \neq 0 \text{ and } d > 0,$$
$$S_p(a, \alpha) = \sum_{j=0}^{a} \sum_{i=0}^{p} \binom{p}{i} P_i\left(j, \omega^d \cdot k_d\right) S_{p-i}\left(a - j, \omega^{d-1} \cdot k_{d-1} + \cdots + \omega \cdot k_1 + k_0\right).$$

When the conditions $k_d \neq 0$ and $d > 0$ cannot be satisfied, then $\alpha < \omega$ and $S_p(a, k_0) = P_p(a, k_0)$.

Proof. When k_0 is the only nonzero k term, Lemma 34 shows $S_p(a,\alpha) = P_p(a,\alpha)$. When $d \geqslant 1$, we describe a process of combining CRs with GCRs to create GCRs for $\binom{\alpha}{a}$.

For arbitrary integers $a \geqslant 0, p \geqslant 0$, and some $\alpha < \omega^\omega$, let $0 \leqslant j \leqslant a$ and $0 \leqslant i \leqslant p$ be integers. We create

$$\binom{p}{i} P_i\left(j, \omega^d \cdot k_d\right) S_{p-i}\left(a-j, \omega^{d-1} \cdot k_{d-1} + \cdots + k_0\right)$$

GCRs, with each GCR having j elements from the $\omega^d \cdot k_d$ part of α and $a-j$ elements from parts with lower exponents.

Let τ_1 represent one of the $P_i\left(j, \omega^d \cdot k_d\right)$ CRs of $\binom{\omega^d \cdot k_d}{j}$ with size i, and τ_2 represent one of the $S_{p-i}\left(a-j, \omega^{d-1} \cdot k_{d-1} + \cdots + k_0\right)$ GCRs of $\binom{\omega^{d-1} \cdot k_{d-1} + \cdots + k_0}{a-j}$ with size $p-i$. We change τ_1 into a GCR by assigning it $c_q = d$ for all c_q.

Then we can combine each τ_1 and τ_2 to form $\binom{p}{i}$ unique new GCRs of size p: Reindex τ_2 and permute the equivalence classes as in the proof of Lemma 21. Note that we do not insert a leading equivalence class – this is because we do not need to increase the exponent or size of τ_1.

We can keep each c_q and y_q value the same, and reindex them alongside the x_{qn} variables to ensure criterion 1.

Each GCR is unique by the τ_1 and τ_2 used to create it because the process is invertible: we can identify the elements originally from τ_1 because they uniquely have $c_q = d$.

We claim each GCR created by this process has the properties described by Definition 31: Because all c_q are equal for τ_1, criterion 6 is satisfied for the elements from τ_1. Criterion 1 is satisfied by reindexing the variables. The remaining criteria are satisfied because τ_1 and τ_2 satisfied them and their internal orders and equivalence classes were preserved in permuting the equivalence classes. Therefore this process does not overcount GCRs.

We also claim that every GCR of $\binom{\alpha}{a}$ is counted by this process: each can be mapped to some τ_1 and τ_2 that create it by a similar argument to proving that the process creates unique GCRs. □

9.2 $T(a,\alpha) \leqslant S(a,\alpha)$

Lemma 36. *For all $\alpha < \omega^\omega$, $a \in \mathbb{N}$, and $G \approx \omega$, there exists some $H \subseteq \alpha$, $H \approx \alpha$ where for all $e \in \binom{H}{a}$, e satisfies a GCR of $\binom{\alpha}{a}$ and each coefficient in e is contained in G.*

Proof. Because $G \approx \omega$, we can index it $x_0, x_1, x_2, \ldots$ with $x_0 < x_1 < x_2 < \cdots$. Let $\alpha \approx \omega^d \cdot k_d + \omega^{d-1} \cdot k_{d-1} + \cdots + \omega \cdot k_1 + k_0$.

First, apply Lemma 27 on G to produce an $H' \approx \omega \cdot (d+1)$. For $0 \leqslant n \leqslant d$, let $G'_n \approx \omega$ such that

$$H' = G'_0 + \cdots + G'_d.$$

For $0 \leqslant n \leqslant d$, apply Lemma 27 on G'_n to yield some $H_n \approx \omega^n \cdot k_n$ where all $e \in H_n$ satisfy a CR for $\binom{\omega^n k_n}{a}$. Then let

$$H = \sum_{n=0}^{d} H_n$$

so that $H \approx \alpha$.

Because all $e \in H_n$ satisfy a CR for $0 \leqslant n \leqslant d$, only criterion 6 of Definition 31 remains to be satisfied. Since we separated G into disjoint orders H'_n, each H_n is disjoint from the others so criterion 6 is satisfied. The coefficients in e are contained in G by the construction of H from G. □

Theorem 37. *For all $\alpha < \omega^\omega$,*

$$T(a, \alpha) \geqslant S(a, \alpha).$$

Proof. Let $E = \binom{\alpha}{a}$ and

$$\mathrm{COL}\colon E \to [b]$$

be an arbitrary coloring of E for some $b \in \mathbb{N}$.

Enumerate the GCRs of E from τ_1 to $\tau_{S(a,\alpha)}$. The maximum size of any GCR of E is $a \cdot d$. For each τ_i, let

$$f_i\colon \binom{\omega}{a \cdot d} \to E$$

where if τ_i has size p, f_i maps X to the unique $e \in E$ where e satisfies τ_i and the p equivalence classes of e are made up of the p least elements of X. For example, one GCR of $\binom{\omega^2+\omega\cdot 8}{2}$ is

$$c_1 = 2, c_2 = 1, y_1 = 0, y_2 = 6,\ x_{11} < x_{20} < x_{10}.$$

The corresponding f_i would be $f_i\colon \binom{\omega}{4} \to \binom{\omega^2+\omega\cdot 8}{2}$ with

$$f_i(x_1, x_2, x_3, x_4) = \mathrm{COL}(\{\omega^2 \cdot 0 + \omega \cdot x_1 + x_3, \omega \cdot 6 + x_2\})$$

where $x_1 < x_2 < x_3 < x_4$.

Then, define $\mathrm{COL}'\colon \binom{\omega}{a \cdot d} \to [b]^{S(a,\alpha)}$ with

$$X = (f_1(X), f_2(X), \ldots, f_{S(a,\alpha)}(X))$$

and apply Theorem 9 to find some $G \approx \omega$ where

$$\left|\mathrm{COL}'\left(\binom{G}{a \cdot d}\right)\right| = 1.$$

Let Y be the one color in $\mathrm{COL}'(\binom{G}{a\cdot d})$. Note that Y is a tuple of $S(a, \alpha)$ colors.

Apply Lemma 36 to find some $H \approx \alpha$ with the properties listed in Lemma 36. Now we claim

$$\left|\mathrm{COL}\left(\binom{H}{a}\right)\right| \leqslant S(a, \alpha)$$

By Lemma 36, each element $e \in \binom{H}{a}$ satisfies a GCR of E. Then for any edge e, let e satisfy τ_i with size $p \leqslant a \cdot d$. Then take the p unique values in e, and if necessary, insert any new larger nonnegative integers from G to form a set of $a \cdot d$ values; denote this $X \in \binom{G}{a \cdot d}$. $\mathrm{COL}'(X) = Y$ so by the definition of COL', $\mathrm{COL}(e) \in Y$. Because $|Y| = S(a, \alpha)$, $T(a, \alpha) \leqslant S(a, \alpha)$. □

9.3 $T(a, \alpha) \geqslant S(a, \alpha)$

Theorem 38. *For all $\alpha < \omega^\omega$,*

$$T(a, \alpha) \leqslant S(a, \alpha).$$

Proof. If $S(a, \alpha) = 0$, this is satisfied vacuously because $T(a, \alpha) \geqslant 0$. Suppose $S(a, \alpha) \geqslant 1$. Let $E = \binom{\alpha}{a}$. Note that all GCRs of E are disjoint from each other. That is, for any edge $e \in E$, if e satisfies τ', then it does not satisfy any nonequivalent GCR of E. This is because if e were to satisfy two GCR τ_1 and τ_2, then τ_1 and τ_2 must share the same c_q, y_q, equivalence classes, and order, so the GCRs must be equivalent. Therefore, we can index them $\tau_1 \dots \tau_{S(a,\alpha)}$ and construct a coloring $\mathrm{COL}\colon E \to [S(a, \alpha)]$ with

$$\mathrm{COL}(e) = \begin{cases} i & e \text{ satisfies } \tau_i \\ 1 & \text{otherwise} \end{cases}$$

For arbitrary $H \approx \alpha$ and a GCR τ for α, we can assign c_q and y_q based on τ. Then we can apply a similar process to the one used in Theorem 29 to find z_{qn} variables that match the permutation of x_{qn} variables.

Let $\alpha \approx \omega^d \cdot k_d + \omega^{d-1} \cdot k_{d-1} + \cdots + \omega \cdot k_1 + k_0$. We can separate H into $d+1$ sets each order-equivalent to $\omega^n \cdot k_n$ for $0 \leqslant n \leqslant d$, and separate each of those into k_n sets order-equivalent to ω^n.

Then, for each equivalence class in τ, using the process formally described in the proof of Theorem 24, we consider the leading equivalence class of τ. By criteria 2 and 6 of Definition 31, all variables in that equivalence class must come from same set order-equivalent to ω^n. We assign a finite value to that equivalence class, and move to the next class with potentially different c and y values, using the assigned finite value as a lower bound for the next one. We can repeat this process to find z_{qn} that satisfy each GCR of E for arbitrary $H \approx \alpha$.

Therefore for all $H \approx \alpha$, $\left|\mathrm{COL}\left(\binom{H}{a}\right)\right| \geqslant S(a, \alpha)$ so $T(a, \alpha) \geqslant S(a, \alpha)$. □

9.4 $T(a, \alpha) = S(a, \alpha)$

Theorem 39. *For all $\alpha < \omega^\omega$,*

$$T(a, \alpha) = S(a, \alpha).$$

Proof. By Theorem 37, $T(a, \alpha) \geqslant S(a, \alpha)$. By Theorem 38, $T(a, \alpha) \leqslant S(a, \alpha)$. The result follows. □

10 Open Problems

The original motivation for this paper was pedagogicial (see the open problems column by Dobrinen & Gasarch [4]). We sought easier proofs of results in the literature. For the case of $T(a, \zeta)$ we succeeded, as the proof we give is easy in that it uses Ramsey's Theorem on ω to do most of the work. For the case of $T(a, \alpha)$ where $\alpha < \omega^\omega$ our proof is more accessible than literature, and gives exact bounds, but cannot really be called *easy*.

With this in mind, the following open problems remain:

1. Find an easier proof that $T(a, \alpha) < \infty$. If the easier proof does not give exact bounds, that is fine.
2. Find an easier proof of the exact values for $T(a, \alpha)$.
3. Find an easy proof that for all $\alpha \geqslant \omega^\omega$, and all $a \geqslant 2$, $T(a, \omega^\omega)$ does not exist.
4. For $k \geqslant 3$, find combinatorial interpretations for the sequences $T(a, \omega^k)$.
5. Find an easier proof of $T(a, \eta) < \infty$. If the easier proof does not give exact bounds, that is fine.
6. Find an easier proof of the exact values for $T(a, \eta)$.

Acknowledgments

We would like to thank Natasha Dobrinen for introducing us to this subject, giving us advice on the project, and helpful comments on the final draft. We would like to thank Nathan Cho, Isaac Mammel, and Adam Melrod for helping us simplify the proof of Theorem 12.

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Appendix

Table 1 shows $T(a, \omega^d)$ for small a, d.

	a					
d	0	1	2	3	4	5
0	1	1	1	1	1	1
1	1	1	1	1	1	1
2	1	1	4	26	236	2752
3	1	1	14	509	35839	4154652
4	1	1	49	10340	5941404	7244337796
5	1	1	175	222244	1081112575	14372713082763

Table 1: Big Ramsey degrees of ω^d.