Regular Graph Properties
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1 Introduction

Consider the following question:

Is the set of Hamiltonian graphs regular?

To pose this question properly we need to specify (a) which strings represent graphs, (b) what do to if a string that does not represent a graph is input.

We represent graphs with an adjacency matrix except that all elements of the lower triangle (below the diagonaly) are X. We still refer to this object as an *adjancey matrix* even though it is not standard. We now proceed formally.

Def 1.1 All strings are over the alphabet $\{0, 1, X, \$\}$. Let x be a string of the form $\$x_1\$x_2\$\cdots\$x_n\$$ where the following happen:

1. The ith element of the x_j is denoted x_{ij}.
2. For $1 \le i < j \le n$, $x_{ij} \in \{0, 1\}$.
3. For all $1 \le i \le n$, $x_{ii} = 0$ (no self loops).
4. For all $1 \le i < j \le n$, $x_{ji} = X$.

Any string of the form above is interpreted as the adjacency matrix of a graph.

We identify a graph G with the string that is its adjacency matrix, as above. Hence we will say things like *Run DFA M on G.*

Example 1.2 The graph K_4 is the string

$$\$0111\$1011\$1101\$1110\$$$

We will never express a graph that way. We will instead write down the matrix. In this case

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ X & X & 1 & 1 \\ X & X & X & 1 \\ X & X & X & 0 \end{pmatrix}$$

Def 1.3 Let $\mathcal{G}$ be a set of graphs.

1. $\mathcal{G}$ is a *graph property* if $\mathcal{G}$ satisfies the following: for all pairs of graphs, (G_1, G_2), if G_1 and G_2 are isomorphic then either $G_1, G_2 \in \mathcal{G}$ or $G_1, G_2 \notin \mathcal{G}$.

2. A graph property $\mathcal{G}$ is *regular* if there exists a DFA M such that the following hold:

   (a) If $G \in \mathcal{G}$ then $M(G)$ accepts.

   (b) If $G \notin \mathcal{G}$ then $M(G)$ rejects.

   (c) If w is a string that does not represent an adjacency matrix then we have *no condition* on what $M(w)$ is.

2 Graph Properties that are Regular

Def 2.1

1. If $w \in \{0,1\}^*$ then $\#_1(w)$ is the number of 1's in w.

2. Let $A \subseteq \mathsf{N}$. A is *regular* if the following set is regular:

$$\{w \in \{0,1\}^* : \#_1(w) \in A\}$$

   is regular.

Theorem 2.2 *Let $A \subseteq \mathsf{N}$ be regular.*

1. *The following graph property is regular:*

$$\mathcal{G} = \{G = (V, E) : (\forall v \in V)[|\deg(v)| \in A]\}.$$

2. *The following graph property is regular:*

$$\mathcal{G} = \{G = (V, E) : |E| \in A\}.$$

Proof:
1) Let $W = \{w \in \{0,1\}^* : \#_1(w) \in A\}$. W is regular by the definition of $A \subseteq \mathsf{N}$ being regular. Let α be the regular expression such that $L(W) = \alpha$.

A graph G is in $\mathcal{G}$ iff every row of its adjacency matrix is in L. Consider the regular expression

$$\beta = \$(\alpha\$)^*$$

It is easy to see that
$G \in \mathcal{G}$ implies $G \in L(\beta)$.
$G \notin \mathcal{G}$ implies $G \notin L(\beta)$.

2) We leave this to the reader.
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Corollary 2.3 *Let $d, m \in 2$. The following graph properties are regular.*

1. *The set of graphs where $(\forall v)[\deg(v) \geq d]$.*
2. *The set of graphs where $(\forall v)[\deg(v) = d]$.*
3. *The set of graphs where $(\forall v)[\deg(v) \leq d]$.*
4. *Let $A \subseteq \{0, 1, \ldots, d-1\}$. The set of graphs where*

$$\{\deg(v) \pmod{m} : v \in V\} \subseteq A\}.$$

5. *The set of Eulerian graphs. (This is the $m = 2$, $A = \{0\}$ case.)*
6. *The set of graphs $G = (V, E)$ such that $|E| \equiv d \pmod{m}$.*

Open Problem 2.4 Aside from Eulerian graphs are there any other *interesting* graph properties that are regular.

3 Partial Graphs

We will need to deal with graphs that are not completely specified. Here is an example of the adjacency matrix of such a (what we will call) partial graph.

$$\begin{pmatrix}
 & | & 1 & 2 & 3 & 4 & | & 5 & 6 & 7 & 8 & 9 & 10 \\
-- & | & - & - & - & - & - & - & - & - & - & - & - \\
1 & | & 0 & 1 & 0 & 0 & | & 1 & 0 & 1 & 0 & 1 & 0 \\
2 & | & X & 0 & 0 & 0 & | & 1 & 0 & 0 & 1 & 0 & 1 \\
3 & | & X & X & 0 & 4 & | & 0 & 1 & 0 & 0 & 1 & 0 \\
4 & | & X & X & X & 0 & | & 0 & 0 & 0 & 0 & 0 & 1 \\
-- & | & - & - & - & - & - & - & - & - & - & - & -- \\
5 & | & X & X & X & X & | & 0 & ? & ? & ? & ? & ? \\
6 & | & X & X & X & X & | & X & 0 & ? & ? & ? & ? \\
7 & | & X & X & X & X & | & X & X & 0 & ? & ? & ? \\
8 & | & X & X & X & X & | & X & X & X & 0 & ? & ? \\
9 & | & X & X & X & X & | & X & X & X & X & 0 & ? \\
10 & | & X & X & X & X & | & X & X & X & X & X & 0
\end{pmatrix}$$

We note the following:

- $V = \{1, \ldots, 10\}$
- $E = \{(1,2), (1,5), (1,7), (1,9), (2,5), (2,10), (3,4), (3,6), (3,9), (4,10)\}$.
- For all vertices v, (v, v) is not an edge.
- Aside from the prohbition on self-loops, the status of the potential edges in $\{5, 6, 7, 8, 9\} \times \{5, 6, 7, 8, 9\}$ is not specified.
- The numbers $\{1, \ldots, 10\}$ on the top and on the sides are not part of the matrix. They are there so the reader can see which rows and column correpond to which vertices.
- We have divided the vertices into two sets $\{1, 2, 3, 4\}$ and $\{5, \ldots, 10\}$. This is not part of the defintion of a partial graph; however, we will do this in our proofs.

We now define partial graphs formally.

Def 3.1 A *partial graph* is an $n \times n$ matrix (entry (i, j) is denoted x_{ij}) such that

1. For all $1 \le i \le n$, $x_{ii} = 0$.
2. For all $1 \le i < j \le n$, $x_{ij} = x_{ji} \in \{0, 1, ?\}$
3. For all $1 \le i < j, x_{ji} = X$.

4 CLIQ_k, COL_k, PLANAR, Are Not Regular

Def 4.1 Let $k \ge 1$.

1. CLIQ_k is the set of graphs that have a k-clique.
2. COL_k is the set of graphs that are k-colorable.
3. PLANAR is the set of graphs that are planar.

There are trivial cases of these problems that are regular. We leave the proof of the following to the reader.

Theorem 4.2

1. CLIQ_1 *is regular. This is the set of all graphs except the empty graph.*
2. CLIQ_2 *is regular. This is the set of all graphs that have an edge.*
3. COL_1 *is regular. This is the set of all graphs that have no edges.*

We will show the following:

1. For $k \ge 3$, CLIQ_k is not regular.
2. For $k \ge 2$, COL_k is not regular.
3. PLANAR is not regular.

The proofs are essentially the same.

4.1 A Thought Experiment for $\mathbf{CLIQ_5}$ and $\mathbf{COL_4}$

Consider two graphs on 10 vertices.

G_1: $E_1 = \{1\} \times \{5, 6, 7, 8\}$.

G_2: $E_2 = \{2\} \times \{9, 10, 11, 12\}$.

We represent the matrices for these graphs but also put in for clarity the vertices each row and column represent.

Here is the matrix for G_1:

$$
\begin{pmatrix}
 & | & 1 & 2 & 3 & 4 & | & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
-- & | & - & - & - & -- & | & - & - & - & - & - & - & - & - \\
1 & | & 0 & 0 & 0 & 0 & | & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
2 & | & \$ & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & | & \$ & \$ & 0 & 0 & | & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & | & \$ & \$ & \$ & 0 & | & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-- & | & - & - & - & - & | & - & - & - & - & - & - & - & - \\
5 & | & \$ & \$ & \$ & \$ & | & 0 & ? & ? & ? & ? & ? & ? & ? \\
6 & | & \$ & \$ & \$ & \$ & | & \$ & 0 & ? & ? & ? & ? & ? & ? \\
7 & | & \$ & \$ & \$ & \$ & | & \$ & \$ & 0 & ? & ? & ? & ? & ? \\
8 & | & \$ & \$ & \$ & \$ & | & \$ & \$ & \$ & 0 & ? & ? & ? & ? \\
9 & | & \$ & \$ & \$ & \$ & | & \$ & \$ & \$ & \$ & 0 & ? & ? & ? \\
10 & | & \$ & \$ & \$ & \$ & | & \$ & \$ & \$ & \$ & \$ & 0 & ? & ? \\
11 & | & \$ & \$ & \$ & \$ & | & \$ & \$ & \$ & \$ & \$ & \$ & 0 & ? \\
12 & | & \$ & \$ & \$ & \$ & | & \$ & \$ & \$ & \$ & \$ & \$ & \$ & 0
\end{pmatrix}
$$

Here is the matrix for G_2:

	1	2	3	4	5	6	7	8	9	10	11	12
−−	−	−	−	−−	−	−	−	−	−	−	−	−
1	0	0	0	0	0	0	0	0	0	0	0	0
2	$	0	0	0	1	0	0	0	1	1	1	1
3	$	$	0	0	0	0	0	0	0	0	0	0
4	$	$	$	0	0	0	0	0	0	0	0	0
−−	−	−	−	−	−	−	−	−	−	−	−	−
5	$	$	$	$	0	?	?	?	?	?	?	?
6	$	$	$	$	$	0	?	?	?	?	?	?
7	$	$	$	$	$	$	0	?	?	?	?	?
8	$	$	$	$	$	$	$	0	?	?	?	?
9	$	$	$	$	$	$	$	$	0	?	?	?
10	$	$	$	$	$	$	$	$	$	0	?	?
11	$	$	$	$	$	$	$	$	$	$	0	?
12	$	$	$	$	$	$	$	$	$	$	$	0

Imagine that CLIQ_5 or COL_4 or PLANAR are regular via DFA M. Imagine that you input the first 4 rows of G into M and get state q. Imagine that you input the first 4 rows of H into M and you also get state q. This can be used to get a contradiction.

We define a graph H:

H has vertices $\{5,6,7,8,9,10,11,12\}$ and edges between all vertices of

$$\{5,6,7,8\}.$$

- Feed $G_1 \cup H$ into M. This computation will first read G_1, and get to state q, then read H, and get to state r. $G_1 \cup H$ is isomorphic to the union of some isolated vertices and K_5. Hence $G_1 \cup H \in \text{CLIQ}_5$. Therefore r is an accepting state. Note that $G_1 \cup H \notin \text{COL}_4$ and $G_1 \cup H \notin$ PLANAR .

- Feed $G_2 \cup H$ into M. This computation will first read G_2, and get to state q, then read H, and get to state r. $G_2 \cup H$ is isomorphic to the union of some isolated vertices and $K_{1,4} \cup K_4$. Hence $G_2 \cup H \notin \text{CLIQ}_5$. Therefore r is a rejecting state. Note that $G_2 \cup H \in \text{COL}_4$ and $G_2 \cup H \in$ PLANAR.

Since r cannot be both accepting and rejecting, we have our contradiction. We will generalize this example in the next section.

5 In Non-Trivial Cases CLIQ_k and COL_k Are Not Regular

Theorem 5.1 *Let $k \ge 3$.*

1. CLIQ_k *is not Regular.*
2. COL_{k-1} *is not regular.*
3. PLANAR *is not regular.*

Proof:

We will prove CLIQ_k is not regular and (1) during the proof make comments relevant to COL_{k-1} and PLANAR, and (2) after the proof that CLIQ_k is not regular we will make observations that show COL_{k-1} and PLANAR are not regular.

Assume, by way of contradiction, that CLIQ_k is regular. Let M be the DFA that accepts CLIQ_k. Assume M has n states.

Consider the following $n+1$ partial graphs on $n+(k-1)(n+1)+1$ vertices

G_1: $E_1 = \{1\} \times \{n+2, \ldots, n+k\}$.
G_2: $E_2 = \{2\} \times \{n+k+1, \ldots, n+2k-1\}$.
G_3: $E_3 = \{3\} \times \{n+2k, \ldots, n+3k-2\}$.
$\vdots$
G_i: $E_i = \{i\} \times \{n+(k-1)i-k+3, \ldots, n+(k-1)i+1\}$.
$\vdots$
G_n: $E_n = \{n\} \times \{n+(k-1)n-k+3, \ldots, n+(k-1)n+1\}$.
G_{n+1}: $E_{n+1} = \{n+1\} \times \{n+(k-1)(n+1)-k+3, \ldots, n+(k-1)(n+1)+1\}$.

Note that we do not specify the status of the edges between the vertices of

$$\{n+2, \ldots, n+(k-1)(n+1)+1\}.$$

Each of these partial graphs can be represented by the first $n+1$ rows of a matrix. Feed each of these $n+1$ graphs into M. Since there are $n+1$ of them, there must be G_i and G_j that end up in the same state q of M.

We define a graph H:

H is the complete graph on the vertices

$$\{n+(k-1)i-k+3,\ldots,n+(k-1)i+1\}.$$

- Feed $G_i \cup H$ into M. This computation will first read G_i, and get to state q, then read H, and get to state r. $G_i \cup H$ is isomorphic to the union of a set of isolated points and K_k. Hence $G_i \cup H \in \mathrm{CLIQ}_k$. Therefore r is an accepting state. Note that, since $k \geq 3$, $G_i \cup H \notin \mathrm{COL}_{k-1}$. Note that if $k=5$ then $G_i \cup H \notin$ PLANAR.

- Feed $G_j \cup H$ into M. This computation will first read G_j, and get to state q, then read H, and get to state r. $G_j \cup H$ is isomorphic to a set the union of a set of isolated points and $K_{1,k-1} \cup K_{k-1}$. Hence, since $k \geq 3$, $G_j \cup H \notin \mathrm{CLIQ}_k$. Therefore r is a rejecting state. Note that, since $k \geq 3$, $G_j \cup H \in \mathrm{COL}_{k-1}$. Note that if $k=5$ then $G_j \cup H \in$ PLANAR.

Since r cannot be both accepting and rejecting, we have our contradiction.

Since $G_i \cup H \notin \mathrm{COL}_{k-1}$ and $\mathcal{G}_j \cup H \in \mathrm{COL}_{k-1}$, the proof shows that COL_{k-1} is not regular.

Since in the $k=5$ case $G_i \cup H \notin$ PLANAR and $\mathcal{G}_j \cup H \in$ PLANAR, the proof shows that PLANAR is not regular.

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