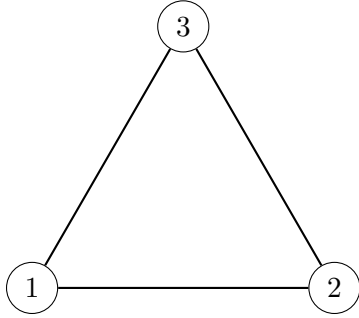




1 Our Starting Point

We start with a theorem that is easy; however, it is the starting point for our line of research.

Theorem 1.1 . *For all $\text{COL}: \mathbb{R}^2 \rightarrow [2]$ there exists 2 points, same color, 1 inch apart.*

Proof:

Look at an equilateral triangle in the plane with side length 1.

Since there are 3 vertices and 2 colors, 2 of the vertices are the same color. these two points are the same color and 1 inch apart. ■

There are two directions (actually many more) to investigate as variants of Theorem 1.1.

1. What if we use more colors?
2. What if we want three monochromatic colinear points p, q, r such that $d(p, q) = d(q, r) = 1$?

The second question, with more parameters, is the theme of this book. We will consider the first question as well since it will help us with the second question.

2 The Chromatic Number of the Plane

Consider the graph $G = (V, E)$ where

$$\begin{aligned} V &= \mathbb{R}^2 \\ E &= \{(x, y) : d(x, y) = 1\} \end{aligned}$$

Notation 2.1 χ is the chromatic number of this graph.

Theorem 1.1 can be rephrased as

Theorem 2.2 $\chi \geq 3$.

We investigate what χ can be.

Theorem 2.3 $\chi \geq 4$.

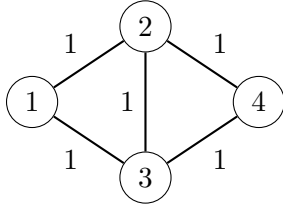


Figure 1: A Graph Gadget For 3-Coloring the Plane

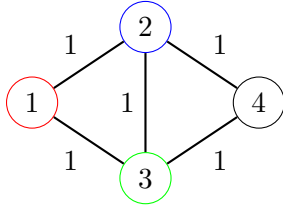


Figure 2: Coloring of Graph Gadget for 3-coloring

Proof:

Assume, by way of contradiction, that COL is a proper 3-coloring of the plane. Glue together two unit equilateral triangles to form the configuration in Figure 1

We can assume that $\text{COL}(1) = \text{RED}$. Since COL is a proper 3-coloring

$$\text{COL}(2) \neq \text{RED} \text{ and } \text{COL}(3) \neq \text{RED}.$$

Since $d(2, 3) = 1$, $\text{COL}(2) \neq \text{COL}(3)$. Hence we can assume $\text{COL}(2) = \text{BLUE}$ and $\text{COL}(3) = \text{GREEN}$. Figure 2 shows all that we know at this point.

Since $\text{COL}(2) = \text{BLUE}$, $\text{COL}(3) = \text{GREEN}$, $d(2, 4) = d(3, 4) = 1$, we have $\text{COL}(4) = \text{RED}$. Figure 3 shows all that we know at this point.

Note that $d(1, 4) = \sqrt{3}$. Hence

- If p, q are $\sqrt{3}$ apart then $\text{COL}(p) = \text{COL}(q)$.
- If $\text{COL}(p) = \text{RED}$ then circle of radius $\sqrt{3}$ around p is RED.

■

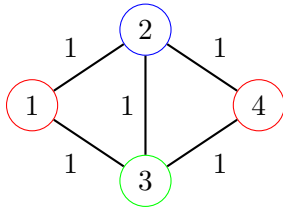


Figure 3: Coloring of Graph Gadget for 3-coloring: All Four Points