There Is a 2-Coloring Of the Plane Without a mono Red 2-Stick or a mono Blue Big-Stick

Exposition by William Gasarch-U of MD

Credit Where Credit is Due

The main result in these slides is due to Szlam (2001) and Conlon and Fox (2017).

Recall the Notation $\mathbb{R}^2 \to (\ell_{\mathbf{a}}, \ell_{\mathbf{b}})$

Notation Let $a, b \geq 2$. $\mathbb{R}^2 \to (\ell_n, \ell_m)$ means

Recall the Notation $\mathbb{R}^2 \to (\ell_a, \ell_b)$

Notation Let $a, b \ge 2$. $\mathbb{R}^2 \to (\ell_n, \ell_m)$ means

For all $\mathrm{COL}\colon \mathbb{R}^2 \to [2]$ there exists **Red** ℓ_n or **Blue** ℓ_m.

Recall the Notation $\mathbb{R}^2 \to (\ell_a, \ell_b)$

Notation Let $a, b \ge 2$. $\mathbb{R}^2 \to (\ell_n, \ell_m)$ means

For all $\mathrm{COL}\colon \mathbb{R}^2 \to [2]$ there exists **Red** ℓ_n or **Blue** ℓ_m.

What about $\mathbb{R}^2 \to (\ell_2, \ell_b)$ with $b \ge 3$.

Recall the Notation $\mathbb{R}^2 \rightarrow (\ell_{\mathbf{a}}, \ell_{\mathbf{b}})$

Notation Let $a, b \geq 2$. $\mathbb{R}^2 \rightarrow (\ell_n, \ell_m)$ means

For all $\mathrm{COL}\colon \mathbb{R}^2 \rightarrow [2]$ there exists **Red** ℓ_n or **Blue** ℓ_m.

What about $\mathbb{R}^2 \rightarrow (\ell_2, \ell_b)$ with $b \geq 3$.

We show the following (mod some lemmas):

Recall the Notation $\mathbb{R}^2 \to (\ell_a, \ell_b)$

Notation Let $a, b \ge 2$. $\mathbb{R}^2 \to (\ell_n, \ell_m)$ means

For all $\mathrm{COL}\colon \mathbb{R}^2 \to [2]$ there exists **Red** ℓ_n or **Blue** ℓ_m.

What about $\mathbb{R}^2 \to (\ell_2, \ell_b)$ with $b \ge 3$.

We show the following (mod some lemmas):

There exist d, for all n, $\mathbb{R}^n \to (\ell_2, \ell_{2^{dn}}))$ (Szlam 2001).

Recall the Notation $\mathbb{R}^2 \rightarrow (\ell_a, \ell_b)$

Notation Let $a, b \geq 2$. $\mathbb{R}^2 \rightarrow (\ell_n, \ell_m)$ means

For all $\mathrm{COL}\colon \mathbb{R}^2 \rightarrow [2]$ there exists **Red** ℓ_n or **Blue** ℓ_m.

What about $\mathbb{R}^2 \rightarrow (\ell_2, \ell_b)$ with $b \geq 3$.

We show the following (mod some lemmas):

There exist d, for all n, $\mathbb{R}^n \rightarrow (\ell_2, \ell_{2^{dn}}))$ (Szlam 2001).

There exists d, for all n, $\mathbb{R}^2 \not\rightarrow (\ell_2, \ell_{2^{dn}})$ (Conlon-Wu, 2017).

Main Result I

Over the next few (many?) slides we will show

Main Result I

Over the next few (many?) slides we will show

There exist d, for all n, $\mathbb{R}^n \to (\ell_2, \ell_{2^{dn}}))$ (Szlam 2001).

Chromatic Number of $\mathbb{R}^n$

Definition $c(n)$ is the least number c such that there exists

$$\mathrm{COL}\colon \mathbb{R}^n \to [c]$$

with no mono unit stick. .

Chromatic Number of $\mathbb{R}^n$

Definition $c(n)$ is the least number c such that there exists

$$\mathrm{COL}\colon \mathbb{R}^n \to [c]$$

with no mono unit stick. .

We will need a lower bound on $c(n)$.

Lemma Needed For Lower Bound on $c(n)$

Lemma Needed For Lower Bound on $c(n)$

Lemma

Lemma Needed For Lower Bound on $c(n)$

Lemma

Let q be a prime power. Let $k \in \mathbb{N}$.

Lemma Needed For Lower Bound on $c(n)$

Lemma

Let q be a prime power. Let $k \in \mathbb{N}$.
Let $F_1, \ldots, F_{\binom{[n]}{q-1}+1} \in \binom{[n]}{2q-1}$.

Lemma Needed For Lower Bound on $c(n)$

Lemma

Let q be a prime power. Let $k \in \mathbb{N}$.
Let $F_1, \ldots, F_{\binom{[n]}{q-1}+1} \in \binom{[n]}{2q-1}$.
Then there exists $1 \le i < j \le \binom{[n]}{q-1} + 1$ with

$$|F_i \cap F_j| = q - 1.$$

Lower Bound on Chromatic Number of $\mathbb{R}^n$

theorem For all n,

$$c(n) > \left\lceil \max_{q \text{ prime power}} \frac{\binom{n}{2q-1}}{\binom{n}{q-1} + 1} \right\rceil$$

Lower Bound on Chromatic Number of $\mathbb{R}^n$

theorem For all n,

$$c(n) > \left\lceil \max_{q \text{ prime power}} \frac{\binom{n}{2q-1}}{\binom{n}{q-1}+1} \right\rceil$$

Let $\mathrm{COL}\colon \mathbb{R}^n \to [c(n)]$.

Lower Bound on Chromatic Number of $\mathbb{R}^n$

theorem For all n,

$$c(n) > \left\lceil \max_{q \text{ prime power}} \frac{\binom{n}{2q-1}}{\binom{n}{q-1}+1} \right\rceil$$

Let $\mathrm{COL}\colon \mathbb{R}^n \to [c(n)]$.

We will define a finite set of points $S \subseteq \mathbb{R}^n$ and then restrict COL to S.

The set S and a Set System

The set S and a Set System

Let $S \subseteq \mathbb{R}^n$ be all of the vectors such that

- $n - 2q - 1$ of the components are 0.
- $2q - 1$ of the components are $\frac{1}{\sqrt{2q}}$.

Let $F\colon S \to \binom{[n]}{2q-1}$ by viewing each vector in S as a bit vector though with $\frac{1}{\sqrt{2q}}$ instead of 1.

Claim about S and F and Proof

Claim about S and F and Proof

Claim Let $u, v \in S$. If $|F(u) \cap F(v)| = f$ then $d(u, v) = 2 - \frac{f+1}{q}$.
Hence $d(u, v) = 1$ iff $|F(u) \cap F(v)| = q - 1$.

Claim about S and F and Proof

Claim Let $u, v \in S$. If $|F(u) \cap F(v)| = f$ then $d(u, v) = 2 - \frac{f+1}{q}$.
Hence $d(u, v) = 1$ iff $|F(u) \cap F(v)| = q - 1$.
Assume $|F(u) \cap F(v)| = f$ then:

Claim about S and F and Proof

Claim Let $u, v \in S$. If $|F(u) \cap F(v)| = f$ then $d(u,v) = 2 - \frac{f+1}{q}$.
Hence $d(u,v) = 1$ iff $|F(u) \cap F(v)| = q - 1$.
Assume $|F(u) \cap F(v)| = f$ then:
There are f coordinates where u and v both have $\frac{1}{\sqrt{2q}}$.

Claim about S and F and Proof

Claim Let $u, v \in S$. If $|F(u) \cap F(v)| = f$ then $d(u, v) = 2 - \frac{f+1}{q}$.
Hence $d(u, v) = 1$ iff $|F(u) \cap F(v)| = q - 1$.
Assume $|F(u) \cap F(v)| = f$ then:
There are f coordinates where u and v both have $\frac{1}{\sqrt{2q}}$.
There are $2q - 1 - f$ coordinates where u has $\frac{1}{2q}$ and v has 0.

Claim about S and F and Proof

Claim Let $u, v \in S$. If $|F(u) \cap F(v)| = f$ then $d(u, v) = 2 - \frac{f+1}{q}$.
Hence $d(u, v) = 1$ iff $|F(u) \cap F(v)| = q - 1$.
Assume $|F(u) \cap F(v)| = f$ then:
There are f coordinates where u and v both have $\frac{1}{\sqrt{2q}}$.
There are $2q - 1 - f$ coordinates where u has $\frac{1}{2q}$ and v has 0.
There are $2q - 1 - f$ coordinates where v has $\frac{1}{2q}$ and u has 0.

Claim about S and F and Proof

Claim Let $u, v \in S$. If $|F(u) \cap F(v)| = f$ then $d(u, v) = 2 - \frac{f+1}{q}$.
Hence $d(u, v) = 1$ iff $|F(u) \cap F(v)| = q - 1$.
Assume $|F(u) \cap F(v)| = f$ then:
There are f coordinates where u and v both have $\frac{1}{\sqrt{2q}}$.
There are $2q - 1 - f$ coordinates where u has $\frac{1}{2q}$ and v has 0.
There are $2q - 1 - f$ coordinates where v has $\frac{1}{2q}$ and u has 0.
There are $n - 4q + f + 2$ coordinates where u and v are both 0.

Claim about S and F and Proof

Claim Let $u, v \in S$. If $|F(u) \cap F(v)| = f$ then $d(u,v) = 2 - \frac{f+1}{q}$.
Hence $d(u,v) = 1$ iff $|F(u) \cap F(v)| = q - 1$.
Assume $|F(u) \cap F(v)| = f$ then:
There are f coordinates where u and v both have $\frac{1}{\sqrt{2q}}$.
There are $2q - 1 - f$ coordinates where u has $\frac{1}{2q}$ and v has 0.
There are $2q - 1 - f$ coordinates where v has $\frac{1}{2q}$ and u has 0.
There are $n - 4q + f + 2$ coordinates where u and v are both 0.
Hence $d(u,v) = 2 \times (2q - 1 - f) \times \frac{1}{2q} = \frac{2q-1-f}{q} = 2 - \frac{f+1}{q}$.

Finish the Proof

Finish the Proof

Restrict COL to S.

Finish the Proof

Restrict COL to S.
Since $|S| = \binom{n}{2q-1}$ and there are c colors, RED must occur

$$\geq \binom{n}{2q-1}/c = \binom{n}{q-1} + 1 \text{ times}$$

Finish the Proof

Restrict COL to S.
Since $|S| = \binom{n}{2q-1}$ and there are c colors, RED must occur

$$\geq \binom{n}{2q-1}/c = \binom{n}{q-1} + 1 \text{ times}$$

Let S' be the subset of S that has that color.

Finish the Proof

Restrict COL to S.
Since $|S| = \binom{n}{2q-1}$ and there are c colors, RED must occur

$$\geq \binom{n}{2q-1}/c = \binom{n}{q-1} + 1 \text{ times}$$

Let S' be the subset of S that has that color.
Since $S' \subseteq \binom{[n]}{2q-1}$ and $|S'| \geq \binom{n}{q-1} + 1$.

Finish the Proof

Restrict COL to S.
Since $|S| = \binom{n}{2q-1}$ and there are c colors, RED must occur

$$\geq \binom{n}{2q-1}/c = \binom{n}{q-1} + 1 \text{ times}$$

Let S' be the subset of S that has that color.
Since $S' \subseteq \binom{[n]}{2q-1}$ and $|S'| \geq \binom{n}{q-1} + 1$.
By Lemma there exists two elements of S with intersection of size $q-1$, say $F(u)$ and $F(v)$.

Finish the Proof

Restrict COL to S.
Since $|S| = \binom{n}{2q-1}$ and there are c colors, RED must occur

$$\geq \binom{n}{2q-1}/c = \binom{n}{q-1} + 1 \text{ times}$$

Let S' be the subset of S that has that color.
Since $S' \subseteq \binom{[n]}{2q-1}$ and $|S'| \geq \binom{n}{q-1} + 1$.
By Lemma there exists two elements of S with intersection of size $q-1$, say $F(u)$ and $F(v)$.
Since $|F(u) \cap F(v)| = q-1$, by the Claim, $d(u,v) = 1$.

So What is $c(n)$ Really?

So What is $c(n)$ Really?

From

$$c(n) > \left\lceil \max_{q \text{ prime power}} \frac{\binom{n}{2q-1}}{\binom{n}{q-1} + 1} \right\rceil$$

So What is $c(n)$ Really?

From

$$c(n) > \left\lceil \max_{q \text{ prime power}} \frac{\binom{n}{2q-1}}{\binom{n}{q-1}+1} \right\rceil$$

- One can show there exists d, for all n, $c(n) \ge 2^{dn}$. (KELIN-HAVE WE DONE THAT YET?)
- Kelin plotted the graph and it seems that $c(n) \le 2.54 \times 2^{0.266n}$.

$\mathbb{R}^n \to (\ell_2, \ell_{2^{dn}})$

Theorem There exists d such that $\mathbb{R}^n \to (\ell_2, \ell_{2^{dn}})$.

$\mathbb{R}^n \to (\ell_2, \ell_{2^{dn}})$

Theorem There exists d such that $\mathbb{R}^n \to (\ell_2, \ell_{2^{dn}})$.
Let $\mathrm{COL}\colon \mathbb{R}^n \to [2]$.

$\mathbb{R}^n \to (\ell_2, \ell_{2^{dn}})$

Theorem There exists d such that $\mathbb{R}^n \to (\ell_2, \ell_{2^{dn}})$.
Let $\mathrm{COL}\colon \mathbb{R}^n \to [2]$.
There exists d, $c(n) > 2^{dn}$. Let $m = 2^{dn}$.

$\mathbb{R}^n \to (\ell_2, \ell_{2^{dn}})$

Theorem There exists d such that $\mathbb{R}^n \to (\ell_2, \ell_{2^{dn}})$.
Let $\mathrm{COL}\colon \mathbb{R}^n \to [2]$.
There exists d, $c(n) > 2^{dn}$. Let $m = 2^{dn}$.
$\vec{1}$ is the vector $(1, 0, \ldots, 0)$ in $\mathbb{R}^n$.

$\mathbb{R}^n \to (\ell_2, \ell_{2^{dn}})$

Theorem There exists d such that $\mathbb{R}^n \to (\ell_2, \ell_{2^{dn}})$.
Let $\mathrm{COL}\colon \mathbb{R}^n \to [2]$.
There exists d, $c(n) > 2^{dn}$. Let $m = 2^{dn}$.
$\vec{1}$ is the vector $(1, 0, \ldots, 0)$ in $\mathbb{R}^n$.
Case 1 There is a BLUE ℓ_m. Done

$\mathbb{R}^n \to (\ell_2, \ell_{2^{dn}})$

Theorem There exists d such that $\mathbb{R}^n \to (\ell_2, \ell_{2^{dn}})$.
Let $\mathrm{COL}\colon \mathbb{R}^n \to [2]$.
There exists d, $c(n) > 2^{dn}$. Let $m = 2^{dn}$.
$\vec{1}$ is the vector $(1, 0, \ldots, 0)$ in $\mathbb{R}^n$.
Case 1 There is a BLUE ℓ_m. Done
Case 2 There is no BLUE ℓ_m. See next slide.

$\mathbb{R}^n \to (\ell_2, \ell_{2^{dn}})$ Case 2

$\mathbb{R}^n \to (\ell_2, \ell_{2^{dn}})$ Case 2

Case 2 There is no BLUE ℓ_m. We form a coloring $\mathrm{COL}\colon \mathbb{R}^n \to [m]$ as follows:

$\mathbb{R}^n \rightarrow (\ell_2, \ell_{2^{dn}})$ **Case 2**

Case 2 There is no BLUE ℓ_m. We form a coloring $\mathrm{COL}\colon \mathbb{R}^n \rightarrow [m]$ as follows:
Given point $p \in \mathbb{R}^n$ look at

$$p + \vec{1}, p + 2\vec{1}, \ldots, p + m\vec{1}.$$

$\mathbb{R}^n \to (\ell_2, \ell_{2^{dn}})$ Case 2

Case 2 There is no BLUE ℓ_m. We form a coloring $\mathrm{COL}\colon \mathbb{R}^n \to [m]$ as follows:
Given point $p \in \mathbb{R}^n$ look at

$$p + \vec{1}, p + 2\vec{1}, \ldots, p + m\vec{1}.$$

Since there is no BLUE ℓ_m, there exists i such that $\mathrm{COL}(p + i\vec{1})$ is RED.

$\mathbb{R}^n \to (\ell_2, \ell_{2^{dn}})$ Case 2

Case 2 There is no BLUE ℓ_m. We form a coloring $\mathrm{COL}\colon \mathbb{R}^n \to [m]$ as follows:
Given point $p \in \mathbb{R}^n$ look at

$$p + \vec{1}, p + 2\vec{1}, \ldots, p + m\vec{1}.$$

Since there is no BLUE ℓ_m, there exists i such that $\mathrm{COL}(p + i\vec{1})$ is RED.
Color p with the least such i.

$\mathbb{R}^n \rightarrow (\ell_2, \ell_{2^{dn}})$ Case 2

Case 2 There is no BLUE ℓ_m. We form a coloring $\mathrm{COL}\colon \mathbb{R}^n \rightarrow [m]$ as follows:
Given point $p \in \mathbb{R}^n$ look at

$$p + \vec{1}, p + 2\vec{1}, \ldots, p + m\vec{1}.$$

Since there is no BLUE ℓ_m, there exists i such that $\mathrm{COL}(p + i\vec{1})$ is RED.
Color p with the least such i.
Since $c(n) > 2^{dn}$, $\exists$ $u, v \in \mathbb{R}^n$ and $1 \leq i \leq m$ such that

$\mathbb{R}^n \to (\ell_2, \ell_{2^{dn}})$ Case 2

Case 2 There is no BLUE ℓ_m. We form a coloring $\mathrm{COL}\colon \mathbb{R}^n \to [m]$ as follows:
Given point $p \in \mathbb{R}^n$ look at

$$p + \vec{1}, p + 2\vec{1}, \ldots, p + m\vec{1}.$$

Since there is no BLUE ℓ_m, there exists i such that $\mathrm{COL}(p + i\vec{1})$ is RED.
Color p with the least such i.
Since $c(n) > 2^{dn}$, $\exists$ $u, v \in \mathbb{R}^n$ and $1 \le i \le m$ such that $d(u, v) = 1$, and

$\mathbb{R}^n \rightarrow (\ell_2, \ell_{2^{dn}})$ **Case 2**

Case 2 There is no BLUE ℓ_m. We form a coloring $\mathrm{COL}\colon \mathbb{R}^n \rightarrow [m]$ as follows:
Given point $p \in \mathbb{R}^n$ look at

$$p + \vec{1}, p + 2\vec{1}, \ldots, p + m\vec{1}.$$

Since there is no BLUE ℓ_m, there exists i such that $\mathrm{COL}(p + i\vec{1})$ is RED.
Color p with the least such i.
Since $c(n) > 2^{dn}$, $\exists$ $u, v \in \mathbb{R}^n$ and $1 \le i \le m$ such that
$d(u, v) = 1$, and
u, v are the same color.

$\mathbb{R}^n \to (\ell_2, \ell_{2^{dn}})$ Case 2

Case 2 There is no BLUE ℓ_m. We form a coloring $\mathrm{COL}\colon \mathbb{R}^n \to [m]$ as follows:
Given point $p \in \mathbb{R}^n$ look at

$$p + \vec{1}, p + 2\vec{1}, \ldots, p + m\vec{1}.$$

Since there is no BLUE ℓ_m, there exists i such that $\mathrm{COL}(p + i\vec{1})$ is RED.
Color p with the least such i.
Since $c(n) > 2^{dn}$, $\exists$ $u, v \in \mathbb{R}^n$ and $1 \le i \le m$ such that $d(u, v) = 1$, and
u, v are the same color.
Hence $u + i\vec{1}$ and $v + i\vec{1}$ are both RED. They form a RED ℓ_2.