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A 15-colouring of 3-space omitting distance one

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Abstract

It is known that the number $\chi(\mathbb{R}^3)$ of colours necessary to colour each point of 3-space so that no two points lying distance 1 apart have the same colour lies between 5 and 18. All optimal colourings (which establish the upper bound for $\chi(\mathbb{R}^n)$) have to date been found using lattice–sublattice colouring schemes. This paper shows that in $\mathbb{R}^n$ such colouring schemes must use at least $2^{n+1}-1$ colours to have an excluded distance. In addition this paper constructs a 1-excluded colouring of $\mathbb{R}^3$ using a lattice–sublattice scheme with 15 colours—the least number of colours possible for such schemes.

MSC: primary 05C15; 11H31 secondary 05B40; 52C17

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1. Introduction to the colouring problem

The chromatic number $\chi(\mathbb{R}^n)$ of $\mathbb{R}^n$ is defined to be the minimum number of colours (sets S_i) necessary to colour $\mathbb{R}^n$ (such that $\bigcup_{i=1}^{\chi} S_i = \mathbb{R}^n$) so that the unit distance is excluded between points of the same colour ($||x-y|| \neq 1$ for all $x, y \in S_i$ for all i). Raiskii's work [3] tells us that $\chi(R^n) \geqslant n+2$.

It is known that in the euclidean plane $\mathbb{R}^2$

$$4 \leqslant \chi(\mathbb{R}^2) \leqslant 7.$$

The lower bound is established by the equation of Raiskii or the Moser spindle (see Fig. 1) which needs at least 4 colours to colour it.

The upper bound of 7 is established by a colouring based on congruent plane-filling regular hexagonal cells.

In 3-space

$$5 \leqslant \chi(R^3) \leqslant 18.$$

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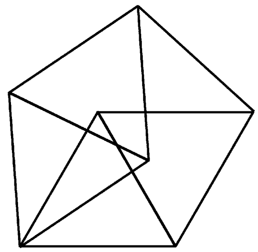


Fig. 1. The Moser spindle.

The lower bound for $\chi(\mathbb{R}^3)$ is established by Raiskii's work or a three-space analog to the Moser spindle. The upper bound of 18 (see [2]), is found using a lattice–sublattice colouring based on a perturbation of the body-centred cubic lattice $A_3^* \cong D_3^*$.

For more background on this and other colouring problems see [4,5].

2. A background on lattices

We define a *lattice* Λ in $\mathbb{R}^n$ to be the span over the integers of a basis $\beta = \mathbf{b}_1, \ldots, \mathbf{b}_n$ for $\mathbb{R}^n$, which we call also a basis for Λ. Its *gram matrix* is $(a_{ij}) = ((\mathbf{b}_i \cdot \mathbf{b}_j))$, where $(x \cdot y)$ denotes the usual dot product.

The *determinant* of Λ, denoted $|\Lambda|$, is the determinant of the gram matrix associated with any basis and is the square of the volume of the parallelepiped generated by any basis of Λ. A sublattice Γ of Λ is a subgroup of Λ when both are viewed as additive abelian groups. The index of Γ in Λ is as defined for groups. Note that if Γ has index m in Λ then

$$\det(\Gamma) = m^2 \det(\Lambda).$$

The *Voronoi region* about a lattice vector $\lambda \in \Lambda$ is defined by

$$\mathscr{V}_\Lambda(\lambda) = \{x \in \mathbb{R}^n : ||x - \lambda|| \leqslant ||x - \lambda'|| \text{ for all } \lambda' \in \Lambda\}.$$

The Voronoi regions are all convex polytopes which are all congruent under translations by lattice vectors, and they tesselate $\mathbb{R}^n$. We abbreviate $\mathscr{V}_\Lambda(0)$ to $\mathscr{V}_\Lambda$. Let $\mathscr{P}$ be the solid parallelepiped generated by a basis of Λ. As

$$\bigcup_{\lambda \in \Lambda} (\lambda + \mathscr{P}) = \mathbb{R}^n = \bigcup_{\lambda \in \Lambda} (\lambda + \mathscr{V}_\Lambda)$$

and since the index set of the two disjoint unions is Λ, we have

$$\text{volume}(\mathscr{V}_\Lambda) = \text{volume}(\mathscr{P}) = |\Lambda|^{1/2}.$$

The *covering radius* of Λ is $\rho_\Lambda = \max\{||x|| : x \in \mathscr{V}_\Lambda\}$. The *covering diameter* of Λ is $\max\{||x - y|| : x, y \in \mathscr{V}_\Lambda\} = 2\rho_\Lambda$.

For a detailed account of lattices see Conway and Sloane [1].

3. Lattice–sublattice colouring schemes

We choose an n-dimensional lattice Λ and use it to tesselate the underlying space $\mathbb{R}^n$ (using the Voronoi regions $\lambda + \mathscr{V}_\Lambda$ of $\lambda \in \Lambda$) and choose a sublattice Γ of Λ such that, for each $\gamma \in \Gamma$, the minimum distance $d(\mathscr{V}_\Lambda, \Gamma)$, between $\mathscr{V}_\Lambda$ and $\gamma + \mathscr{V}_\Lambda$ where $\gamma \in \Gamma$, is greater than the covering diameter $(2\rho_\Lambda)$ of Λ.

By colouring the interiors of the Voronoi regions of Λ according to the coset of $\Lambda \,(\mathrm{mod}\, \Gamma)$ and each boundary point by an appropriately chosen colour from the adjacent interiors we get a colouring with chromatic number $|\Lambda/\Gamma|$, the index of Γ in Λ and excluding the distance $2\rho_\Lambda$. If the minimum distance between $\mathscr{V}_\Lambda$ and $(\gamma + \mathscr{V}_\Lambda)$ is always greater than the covering diameter, then no particular care is needed with the boundary points, and the colouring excludes all distances in some interval $(2\rho_\Lambda, 2\rho_\Lambda + \varepsilon)$. In either case, by rescaling, we can make 1 an excluded distance.

It is possible to relax the requirement $d(\mathscr{V}_\Lambda, \Gamma) > 2\rho_\Lambda$ to $d(\mathscr{V}_\Lambda, \Gamma) \geqslant 2\rho_\Lambda$ by including in $\mathscr{V}_\Lambda$ only one vertex of any diametrically opposite pair of Voronoi vertices. We will refer to this changed Voronoi region as an *improvised Voronoi region.* (We may for example include a vertex v in $\mathscr{V}_\Lambda$ if and only if $v \cdot w > 0$ where w is any vector in $\mathbb{R}^n$ so that $v \cdot w \neq 0$ for all vertices v in $\mathscr{V}_\Lambda$.) This small change to $\mathscr{V}_\Lambda$ ensures that all points in $\mathscr{V}_\Lambda$ are strictly less than distance $2\rho_\Lambda$ apart.

4. The lower bound

This section establishes that if a Λ–Γ lattice–sublattice scheme is used to colour $\mathbb{R}^n$ with any excluded distance then $|\Lambda/\Gamma| \geqslant 2^{n+1} - 1$ for all $n > 1$.

Thus in $\mathbb{R}^3$ at least 15 colours are needed for a lattice–sublattice colouring scheme to exclude any distance.

4.1. Preliminaries

For the sake of brevity, in the remainder of this section colouring will mean a lattice–sublattice colouring with lattice Λ and sublattice Γ.

In order to simplify arguments we will in this section assume that all Voronoi regions are improvised and so only include one of any pair of diametrically opposite Voronoi vertices. Note that any colouring using ordinary Voronoi regions certainly remains a colouring if we improvise the Voronoi regions, so we do not lose any generality with this simplification.

We say that a vector $\lambda \in \Lambda$, or the corresponding Voronoi region $\lambda + \mathscr{V}_\Lambda$, *makes contact* with $\mathscr{V}_\Lambda$ if the set F_λ of points of $\mathscr{V}_\Lambda$ that are equidistant from from 0 and λ is nonempty.

It is known from Voronoi's work [6,7] that such a nonempty F_λ will be a face of $\mathscr{V}_\Lambda$ (which is the whole of $\mathscr{V}_\Lambda$ iff $\lambda = 0$) and we say that λ contacts $\mathscr{V}_\Lambda$ on the face F_λ.

Let us call a lattice vector λ *minimal* if it has minimum length among all vectors in the same coset of $\Lambda/2\Lambda$.

It is clear from the definition of $\mathscr{V}_\Lambda$ that if $v \in \mathbb{R}^n$ then

$$v \in \mathscr{V}_\Lambda \iff 2(v \cdot \gamma) \leqslant (\gamma \cdot \gamma) \quad \text{for every } \gamma \in \Lambda, \tag{1}$$

and that v is equidistant from 0 and λ if and only if

$$2(v \cdot \lambda) = (\lambda \cdot \lambda). \tag{2}$$

The following result is known already, but the proof is included as no other reference seems to include it.

Lemma 4.1. *A vector λ makes contact with $\mathscr{V}_\Lambda$ if and only if λ is minimal.*

Proof. Suppose λ is not minimal. Then there is a $\gamma \in \Lambda$ such that $((\lambda + 2\gamma) \cdot (\lambda + 2\gamma)) < (\lambda \cdot \lambda)$, that is,

$$(\lambda \cdot \gamma) + (\gamma \cdot \gamma) < 0. \tag{3}$$

If λ makes contact with $\mathscr{V}_\Lambda$, let $v \in F_\lambda$ so that (2) holds. Also, by (1),

$$2(v \cdot (\lambda + \gamma)) \leqslant ((\lambda + \gamma) \cdot (\lambda + \gamma)) \quad \text{and} \quad 2(v \cdot (-\gamma)) \leqslant (\gamma \cdot \gamma).$$

Adding these two inequalities and using (3) gives

$$2(v \cdot \lambda) \leqslant (\lambda \cdot \lambda) + 2(\lambda \cdot \gamma) + 2(\gamma \cdot \gamma) < (\lambda \cdot \lambda),$$

contradicting (2).

Conversely suppose that λ does not make contact with $\mathscr{V}_\Lambda$. Then $1/2\lambda \notin \mathscr{V}_\Lambda$ and so, by (1), there is a $\gamma \in \Lambda$ such that $2(1/2\lambda \cdot \gamma) > (\gamma \cdot \gamma)$; that is, $0 > (-\lambda \cdot \gamma) + (\gamma \cdot \gamma)$, which gives $(\lambda \cdot \lambda) > ((\lambda - 2\gamma) \cdot (\lambda - 2\gamma))$. This implies that λ is not minimal. $\square$

Voronoi (p. 277 [7]) showed that $\lambda \in \Lambda$ contacts $\mathscr{V}_\Lambda$ on an $(n-1)$ dimensional face if and only if there is exactly one other minimal vector equivalent to $\lambda \,(\mathrm{mod}\, 2\Lambda)$, namely $-\lambda$.

The Voronoi region $\mathscr{V}_\Lambda$ of an n dimensional lattice Λ usually has $2(2^n - 1)$ faces as typically all of the 2^n cosets of $\Lambda/2\Lambda$ (except $0 + \Lambda$) have a minimal pair of coset representatives. We will refer to such lattices as *generic*.

4.2. Colourings with the covering diameter as an excluded distance

Theorem 4.2. *If n-space ($n > 1$) is coloured using a generic lattice Λ, then the colouring requires at least $2^{n+1} - 1$ colours to exclude the distance $2\rho_\Lambda$.*

Proof. Clearly any two Voronoi regions contacting $\mathscr{V}_\Lambda$ must be less than a covering diameter apart unless both of the Voronoi regions contact $\mathscr{V}_\Lambda$ at diametrically opposed Voronoi vertices v and $-v$ (and these Voronoi vertices are ρ_Λ from the origin).

Thus for a generic lattice Λ (where $\mathscr{V}_\Lambda$ has $2(2^n-1)$ faces of dimension $n-1$), a colouring will need 1 colour for $\mathscr{V}_\Lambda$ and 1 for each of the Voronoi regions contacting other faces of $\mathscr{V}_\Lambda$. Thus a generic lattice colouring requires at least $2^{n+1}-1$ different colours to exclude distance $2\rho_\Lambda$. □

Theorem 4.3. *If n-space* $(n>1)$ *is coloured using any lattice* Λ, *then the colouring requires at least* $2^{n+1}-1$ *colours to exclude the distance* $2\rho_\Lambda$.

Proof. The only situation where vectors contact $\mathscr{V}_\Lambda$ at faces $\geqslant 2\rho_\Lambda$ apart is when two vectors contact $\mathscr{V}_\Lambda$ at Voronoi vertices a covering diameter apart. Suppose now λ and $-\lambda$ are two such vectors. Then (as $n>1$) these Voronoi vertices are not $n-1$ dimensional faces and so by Voronoi ([7, p. 277]) λ and $-\lambda$ must be in a minimal set of size greater than 2 (hence at least 4). Thus we may colour (using the improvisation) Voronoi regions $\lambda+\mathscr{V}_\Lambda$ and $-\lambda+\mathscr{V}_\Lambda$ with the same colour but the minimal set containing these two vectors (and at least two more) will still contribute at least 2 different colours.

Hence to exclude the distance $2\rho_\Lambda$ we need 1 colour for the coset $0+\Lambda$ and at least 2 colours for the minimal vectors of each of the other 2^n-1 cosets of $\Lambda/2\Lambda$. □

4.3. Colourings with any excluded distance

Let us call a line segment *locally maximal in* $\mathscr{V}_\Lambda$, if the line segment is contained in $\mathscr{V}_\Lambda$ and no small perturbations of the line segment (that respect the property of inclusion in $\mathscr{V}_\Lambda$) increase the length of the line segment.

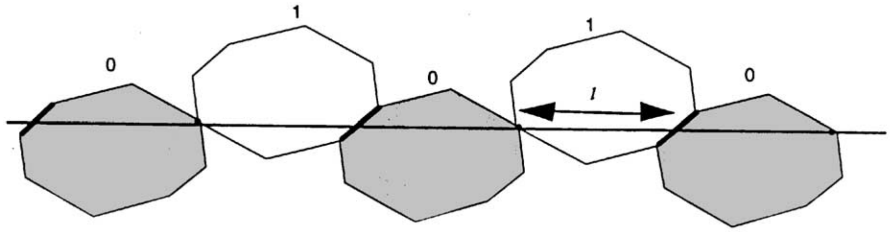
Lemma 4.4. *A line segment is locally maximal in* $\mathscr{V}_\Lambda$ *if and only if the endpoints of the segment lie on diametrically opposed Voronoi vertices of* $\mathscr{V}_\Lambda$.

Proof. The set of differences between endpoints $\{u=v-w : v,w\in\mathscr{V}_\Lambda\}$ is equal to $2\mathscr{V}_\Lambda$, with the vertices $2v$ of $2\mathscr{V}_\Lambda$ obtained uniquely as the difference of diametrically opposed Voronoi vertices v and $-v$. □

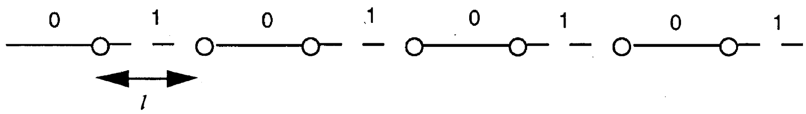
Theorem 4.5. *If n-space* $(n>1)$ *is coloured using any lattice* Λ, *then the colouring requires at least* $2^{n+1}-1$ *colours to exclude any distance.*

Proof. Suppose we colour $\mathbb{R}^n$ with $|\Lambda/\Gamma|<2^{n+1}-1$. Then there are two Voronoi regions $\lambda_1+\mathscr{V}_\Lambda$ and $\lambda_2+\mathscr{V}_\Lambda$ $(\lambda_1-\lambda_2\in\Gamma)$ that are necessarily coloured the same (colour 1) and make contact with $\mathscr{V}_\Lambda$ (coloured with colour 0 say). Furthermore we can choose λ_1 and λ_2 so that contact is not at diametrically opposite vertices.

We hence have the situation depicted using octogons in the diagram below. (Note that these octogons cannot be Voronoi regions but this shape illustrates the construction better than actual Voronoi regions.)



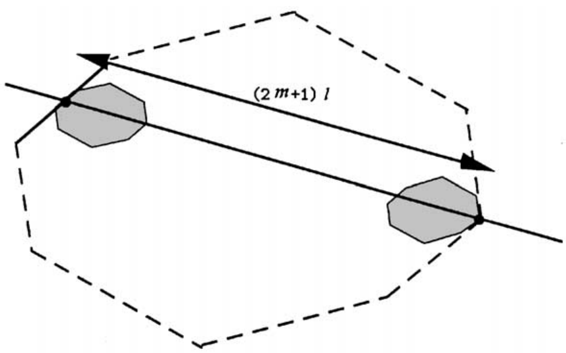
Consider the straight line that passes through the centroids of the contact faces between $\mathscr{V}_\Lambda$, $\lambda_1+\mathscr{V}_\Lambda$ and $\lambda_2+\mathscr{V}_\Lambda$. This line is only interior to Voronoi regions coloured 0 and 1. Considering solely this line we have the situation as shown in the diagram below.



Let ℓ be the length of the line in any one Voronoi region. Note that the points corresponding to the circles may have colours different from 0 and 1 (if the contact faces are of dimension less than $n-1$).

Clearly the only candidates for excluded distance are odd multiples of l, $(2m+1)l$, $m\in\{0,1,2,3,\dots\}$.

Let us contract $\mathbb{R}^n$ by a factor of $(2m+1)$, that is map x to $x/(2m+1)$. The situation for a segment of the line of length $(2m+1)l$ is shown in the diagram below.



Note that the endpoints will locally look as they do on the boundary of $\mathscr{V}_\Lambda$. Clearly the line segment can be shortened infinitesimally (with both endpoints the same colour) and as the endpoints of the segment are not at diametrically opposite vertices of $\mathscr{V}_\Lambda$ (that is the line segment is not locally maximal in $\mathscr{V}_\Lambda$) the line segment can be lengthened infinitesimally (with both endpoints the same colour) and hence by continuity the length of the line segment can remain the same (with both endpoints the same colour).

Thus $(2m+1)l$ cannot be an excluded distance if $|\Lambda/\Gamma| < 2^{n+1}-1$. From this it follows that if we colour $\mathbb{R}^n$ with less than $2^{n+1}-1$ colours there can be no excluded distance. $\square$

5. The 15-colouring

We use the lattice-based colouring scheme to construct a 15-colouring of $\mathbb{R}^3$.

Consider the lattice Λ with basis $\{(2,3,0),(-2,0,3),\ (2,-3,0)\}$ which is of index 36 in the lattice $\mathbb{Z}^3$. Let us tesselate $\mathbb{R}^3$ by the Voronoi regions of the lattice Λ. The Voronoi region (surrounding the origin) is the convex hull generated by the images of the 3 vertices $(0,5/6,13/6)$, $(1,3/2,3/2)$, $(2,5/6,5/6)$ under the group G consisting of changing the sign in any coordinate and swapping the second and third coordinate. This is a set of 24 vertices in all. The covering diameter of this Voronoi region is $\sqrt{22}$.

Let Γ be the sublattice of Λ spanned by $(6,6,3)$, $(-6,-3,6)$ and $(6,-6,-3)$. Now $(6,6,3)=3(2,3,0)+1(-2,0,3)+1(2,-3,0)$, $(-6,-3,6)=-1(2,3,0)+2(-2,0,3)$ and $(6,-6,-3)=-1(-2,0,3)+2(2,-3,0)$. The determinant of

$$\begin{pmatrix} 3 & 1 & 1 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix}$$

is 15, so Γ is an index 15 sublattice of Λ.

These bases for Λ and Γ have gram matrices

$$\begin{pmatrix} 13 & -4 & -5 \\ -4 & 13 & -4 \\ -5 & -4 & 13 \end{pmatrix} \text{ and } \begin{pmatrix} 81 & -36 & -9 \\ -36 & 81 & -36 \\ -9 & -36 & 81 \end{pmatrix}.$$

The sublattice Γ can be characterised quite nicely as the rectangular lattice R with basis $(12,0,0)$, $(0,3,9)$, $(0,-9,3)$ together with the translate $(6,-3,6)+R$. Notice that the translate $(6,-3,6)+R$ includes all the voronoi vertices of R (Fig. 2).

The shortest non-zero vectors in Γ are $\pm(6,6,3)$, $\pm(-6,-3,6)$, $\pm(6,-6,-3)$, $\pm(-6,3,-6)$ with length 9, the next shortest vectors are $\pm(0,3,9)$, $\pm(0,-9,3)$ with length $3\sqrt{10}$.

Consider $\lambda=(6,6,3)$. We will show that $\mathscr{V}_\Lambda$ and $\lambda+\mathscr{V}_\Lambda$ are distance $389/\sqrt{6613}$ apart. The two vectors $13/34(4,3,3)\in\mathscr{V}_\Lambda$ and $(6,6,3)-13/34(4,3,3)\in\lambda+\mathscr{V}_\Lambda$ are distance $\sqrt{389/17}\approx 4.78355$ apart. Conversely let $\mathbf{w}=(50,63,12)$. The planes $(\mathbf{x}\cdot\mathbf{w})=325/2$ and $(\mathbf{x}\cdot\mathbf{w})=1103/2$ are distance $\sqrt{389/17}$ apart and it can be checked that $(\mathbf{x}\cdot\mathbf{w})\leqslant 325/2$ for all $\mathbf{x}\in\mathscr{V}_\Lambda$ and $(\mathbf{y}\cdot\mathbf{w})\geqslant 1103/2$ for all $\mathbf{y}\in\lambda+\mathscr{V}_\Lambda$.

Now since the shortest nonzero vectors in Γ are all equivalent under the symmetries in the group G it follows that $\mathscr{V}_\Lambda$ and $\gamma+\mathscr{V}_\Lambda$ are distance $\sqrt{389/17}$ apart for all shortest nonzero vectors γ of Γ.

On looking at the next layer of vectors in Γ of distance $\sqrt{90}$ away from the origin we see that if λ is distance $\sqrt{90}$ away from the origin then $\mathscr{V}_\Lambda$ and $\lambda+\mathscr{V}_\Lambda$ are at least $\sqrt{90}-\sqrt{22}>\sqrt{389/17}$ apart.

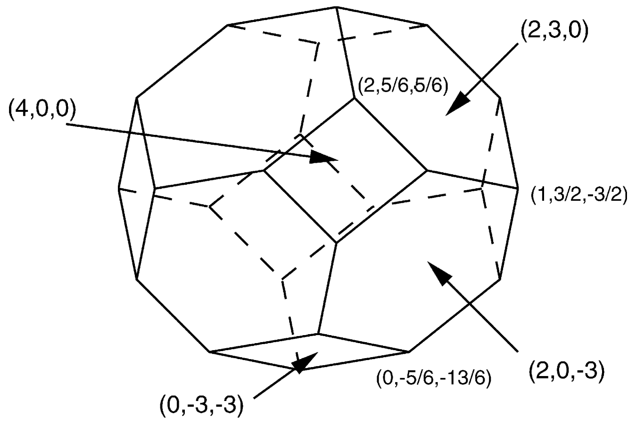


Fig. 2. The Voronoi region of the lattice Λ.

Hence $(\sqrt{22}, \sqrt{389/17})$ is an excluded distance interval for this improved 15-colouring which on lattice rescaling is $(1, \sqrt{389/(22 \times 17)}) \approx (1, 1.01986)$.

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