



Note

An 18-colouring of 3-space omitting distance one

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Abstract

It is known that the number of colours $\chi(\mathbb{R}^3)$ necessary to colour each point of 3-space so that no two points lying distance 1 apart have the same colour lies between 5 and 21. The upper bound of 21 was established by constructing a colouring with the 1-excluded property. In this paper a 1-excluded, 18-colouring is constructed.

1. Introduction to the colouring problem

In 1944 Hadwiger [3] showed that if $\mathbb{R}^n$ is covered by $n+1$ closed sets ($n \geq 1$) then one of the sets S contains all non-negative reals as (euclidean) distances between its points, that is, S generates $\mathbb{R}^+ \cup \{0\}$ as the set $\{\|x-y\| : x, y \in S\}$. Raiskii in 1970 [4] showed that in dimensions greater than one the condition ‘closed’ could be dropped from Hadwiger’s result, that is, for $n \geq 2$ the result was true for any collection of $n+1$ sets in $\mathbb{R}^n$. Woodall [8] independently showed the same thing in 1973.

The chromatic number of $\mathbb{R}^n$, $\chi(\mathbb{R}^n)$, is defined to be the minimum number of colours (sets S_i) necessary to colour $\mathbb{R}^n$ ($\bigcup_{i=1}^k S_i = \mathbb{R}^n$) so that the unit distance is excluded between points of the same colour ($\|x-y\| \neq 1$ for all $x, y \in S_i$ for all i). Raiskii’s theorem tells us that $\chi(\mathbb{R}^n) \geq n+2$.

It is known that in the Euclidean plane $\mathbb{R}^2$

$$4 \leq \chi(\mathbb{R}^2) \leq 7.$$

The lower bound was established by Raiskii’s theorem or the Moser spindle [2] (see Fig. 1) which needs at least 4 colours to colour it. The upper bound of 7 was established by a colouring based on congruent plane-filling hexagonal cells.

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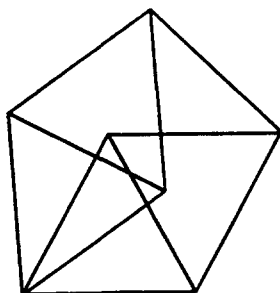


Fig. 1. The Moser spindle.

In 3-space it is known that

$$5 \leq \chi(\mathbb{R}^3) \leq 21.$$

The lower bound for $\chi(\mathbb{R}^3)$ was established by Raiskii's theorem or a 3-space analog to the Moser spindle. The upper bound of 21 was found [6] using a colouring based on the lattice $D_3 \cong A_3$, which is the face-centred cubic lattice.

For a more general and detailed account of the colouring problem see [6,7] and for the problem's history see [5].

In this paper it is demonstrated that there is an 18-colouring (with 1 as an excluded distance) based on the lattice D_3^* (which is the body-centred cubic lattice). This colouring scheme is then perturbed to give an 18-colouring with a larger range of excluded distances.

2. A background on lattices

We define a *lattice* Λ in $\mathbb{R}^n$ to be the span over the integers of a basis $\mathbf{b}_1, \dots, \mathbf{b}_n$ for $\mathbb{R}^n$, which we call also a basis for Λ . Its *gram matrix* is $(a_{ij}) = ((\mathbf{b}_i \cdot \mathbf{b}_j))$, where $(\mathbf{x} \cdot \mathbf{y})$ denotes the usual dot product.

The *determinant* of Λ , denoted $|\Lambda|$, is the determinant of the gram matrix associated with any basis and is the square of the volume of the parallelepiped generated by any basis of Λ . A sublattice Γ of Λ is a subgroup of Λ when both are viewed as additive abelian groups. The index of Γ in Λ is as defined for groups. Note that if Γ has index m in Λ then

$$\det(\Gamma) = m^2 \det(\Lambda).$$

The *Voronoi region* about a lattice vector $\lambda \in \Lambda$ is defined by

$$\mathcal{V}_\Lambda(\lambda) := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \lambda\| \leq \|\mathbf{x} - \lambda'\| \text{ for all } \lambda' \in \Lambda\}.$$

The Voronoi regions are all convex polytopes which are all congruent under translations by lattice vectors, and they tessellate $\mathbb{R}^n$. We write $\mathcal{V}_A := \mathcal{V}_A(\mathbf{0})$. Let $\mathcal{P}$ be the solid parallelepiped generated by a basis of A . As

$$\bigcup_{\lambda \in A} (\lambda + \mathcal{P}) = \mathbb{R}^n = \bigcup_{\lambda \in A} (\lambda + \mathcal{V}_A)$$

and since the index set of the two unions is A , we have

$$\text{volume}(\mathcal{V}_A) = \text{volume}(\mathcal{P}) = |A|^{1/2}.$$

The *covering radius* of A is $\rho_A := \max\{\|x\| : x \in \mathcal{V}_A\}$.

For a detailed account of lattices see [1].

3. A lattice-based colouring scheme

The method we use to set up a colouring is as follows.

We choose a lattice A and use it to tessellate the underlying space (using the Voronoi regions of A) and choose a sublattice Γ of A such that the minimal distance between $\mathcal{V}_A$ and $\gamma + \mathcal{V}_A$ where $\gamma \in \Gamma$ is at least twice the covering radius ρ_A of A .

By colouring the interiors of the Voronoi regions of A according to the coset of A (mod Γ and the boundaries by any colour of the two or more adjacent interiors we get a colouring with chromatic number $|A/\Gamma|$, the index of Γ in A . Furthermore by rescaling we can make 1 the excluded distance.

4. The 21-colouring

We illustrate this lattice-based colouring scheme by explaining the 21-colouring of $\mathbb{R}^3$.

Let D_3 be the set of points in $\mathbb{R}^3$ with integer coordinates which sum to an even number. This is the face-centred cubic lattice, and is the lattice the centres of oranges will assume when stacked greengrocer fashion. The Voronoi region $\mathcal{V}_{D_3}$ has vertices $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$, $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$; thus its covering radius is $\rho_{D_3} = 1$.

Let us tessellate $\mathbb{R}^3$ by the Voronoi regions of the lattice D_3 . A basis for D_3 is $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$. Let Γ be the sublattice of D_3 spanned by $(1, 2, 3)$, $(-3, 1, -2)$ and $(-1, -3, 2)$. These bases for D_3 and Γ have gram matrices

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 14 & -7 & -1 \\ -7 & 14 & -4 \\ -1 & -4 & 14 \end{pmatrix},$$

with determinants 4 and 4×21^2 , and so Γ is a sublattice of index 21 in D_3 . We shall prove that the 21-colouring obtained from Γ omits all distances d in the interval $(2, \sqrt{14/3})$.

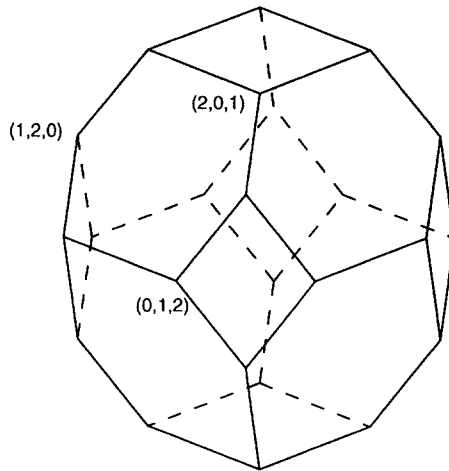


Fig. 2. The truncated octahedron Voronoi region $\mathcal{V}_{D_3^*}$ of the lattice D_3^* .

The shortest non-zero vectors in Γ have length $\sqrt{14}$, with the next shortest having length $\sqrt{18} > 2 + \sqrt{14/3}$. Thus it suffices to prove that $\mathcal{V}_{D_3}$ and $\gamma + \mathcal{V}_{D_3}$ are distance $\sqrt{14/3}$ apart for each vector γ of length $\sqrt{14}$ in Γ . But these vectors γ are all equivalent under permutations and negations of coordinates, which leave $\mathcal{V}_{D_3}$ unchanged, and so it suffices to consider $\gamma = (1, 2, 3)$. Now $(\frac{1}{3}, \frac{1}{3}, \frac{2}{3})$ and $(1 - \frac{1}{3}, 2 - \frac{1}{3}, 3 - \frac{2}{3})$ are two points contained in $\mathcal{V}_{D_3}$ and $\gamma + \mathcal{V}_{D_3}$ that are exactly $\sqrt{14/3}$ apart. To see that there are no closer points in these polytopes, let $y := (1, 4, 5)$; then the planes $x \cdot y = 5$ and $x \cdot y = 19$ are exactly distance $\sqrt{14/3}$ apart, and it is easy to check that $v \cdot y \leq 5$ for all $v \in \mathcal{V}_{D_3}$, and $w \cdot y \geq 19$ for all $w \in \gamma + \mathcal{V}_{D_3}$.

Hence this sublattice gives a colouring of chromatic number 21 of $\mathbb{R}^3$ which (on rescaling) gives an excluded distance interval of $(1, \sqrt{7/6})$.

5. The 18-colouring

The 18-colouring of $\mathbb{R}^3$ is based on the lattice D_3^* , the body-centred cubic lattice. For convenience of coordinates we dilate the usual lattice by a factor of 4 to be the span of the basis $(4, 0, 0)$, $(2, 2, 2)$, $(0, 0, 4)$. We take the sublattice Γ to be the span of the basis $(8, 0, 4)$, $(4, 8, 0)$, $(0, 4, 8)$ ($4 \times$ the Voronoi vertices labelled in Fig. 2). The gram matrices of these bases are

$$\begin{pmatrix} 16 & 8 & 0 \\ 8 & 16 & 8 \\ 0 & 8 & 16 \end{pmatrix} \quad \text{and} \quad 16 \begin{pmatrix} 5 & 2 & 2 \\ 2 & 5 & 2 \\ 2 & 2 & 5 \end{pmatrix}$$

with determinants $4^3 \times 16$ and $16^3 \times 81$. Thus Γ is a sublattice of index 18 in D_3^* .

We modify slightly the Voronoi region of D_3^* (whose vertices are generated by permuting and negating co-ordinates in the vector $(1, 2, 0)$) (see Fig. 2) to say that the

vertices v of $\mathcal{V}_{D_3^*}$ are in $\mathcal{V}_{D_3^*}$ if and only if the sum of coordinates of v is positive. Note that the covering radius $\rho_{D_3^*} = \sqrt{5}$.

Note also that the translations of $\mathcal{V}_{D_3^*}$ still cover 3-space $\bigcup_{x \in \mathcal{A}_3^*} x + \mathcal{V}_{D_3^*} = \mathbb{R}^3$ and that exactly one of each pair of diametrically opposite Voronoi vertices is included in $\mathcal{V}_{D_3^*}$. Thus all points in $\mathcal{V}_{D_3^*}$ are $< 2\sqrt{5}$ distance apart.

The shortest non-zero vectors in Γ have length $4\sqrt{5}$, with the next shortest having length $2\sqrt{6} > 4 \times \rho_{D_3^*}$. Thus we need only check for these shortest vectors γ (which are equivalent under even permutations and vector negation which all leave $\mathcal{V}_{D_3^*}$ fixed) that $\mathcal{V}_{D_3^*}$ and $\gamma + \mathcal{V}_{D_3^*}$ are a distance greater than $2\sqrt{5}$ apart. Hence it suffices to show that $w + \mathcal{V}_{D_3^*}$ and $\mathcal{V}_{D_3^*}$ are a distance greater than $2\sqrt{5}$ apart for $w = (8, 0, 4)$.

The planes $x \cdot w = 20$ and $x \cdot w = 60$ are distance $2\sqrt{5}$ apart. Furthermore it is easily seen that $x \cdot w \leq 20$ for all $x \in \mathcal{V}_{D_3^*}$ and $y \cdot w > 60$ for all $y \in w + \mathcal{V}_{D_3^*}$ (as $(-2, 0, -1) \notin \mathcal{V}_{D_3^*}$).

Note that as $(2, 0, 1) \in \mathcal{V}_{D_3^*}$ and the excluded vertex $(6, 0, 3)$ of $w + \mathcal{V}_{D_3^*}$ are $2\sqrt{5}$ apart, $2\sqrt{5}$ is the only excluded distance.

Thus on rescaling 1 is the only excluded distance.

6. An improved 18-colouring

Consider the lattice Λ that is the span of the basis $(6, 24, -12)$, $(-12, 6, 24)$, $(24, -12, 6)$ and its sublattice Γ' with basis $(54, 54, 0)$, $(0, 54, 54)$, $(54, 0, 54)$.

It can be checked that Γ' is a sublattice of index 18 in Λ . Moreover with D_3^* and Γ as given in the previous section, $\Lambda/\Gamma' = D_3^*/\Gamma$ when both factor groups are viewed as factor groups of $\mathbb{Z}^3$.

Informally, Γ has been perturbed (see Fig. 3 below) to give $\Gamma' \cong D_3$, a lattice with optimal packing properties (see [1]) and the underlying lattice D_3^* perturbs correspondingly to Λ .

The Voronoi region of Λ (see Fig. 4) is the convex hull, with volume 17496, of the 24 vertices consisting of even permutations and vertex negations of the Voronoi vertices given in Table 1. The covering radius of Λ , $\rho_\Lambda = \sqrt{339}$.

Consider $\lambda = (54, 54, 0)$. We will show that $\mathcal{V}_\Lambda$ and $\lambda + \mathcal{V}_\Lambda$ are distance $2\sqrt{393}$ apart. The two vectors $(13, 13, 1) \in \mathcal{V}_\Lambda$ and $(54 - 13, 54 - 13, -1) \in \lambda + \mathcal{V}_\Lambda$ are distance $2\sqrt{393}$

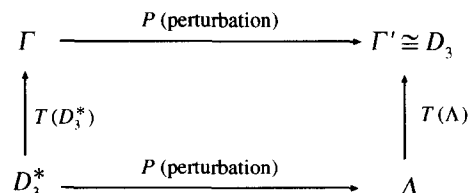


Fig. 3. A commutative diagram illustrating the linear perturbation P improving the 18-colouring. (The map T is a bijective linear map from the underlying lattice to the sublattice.)

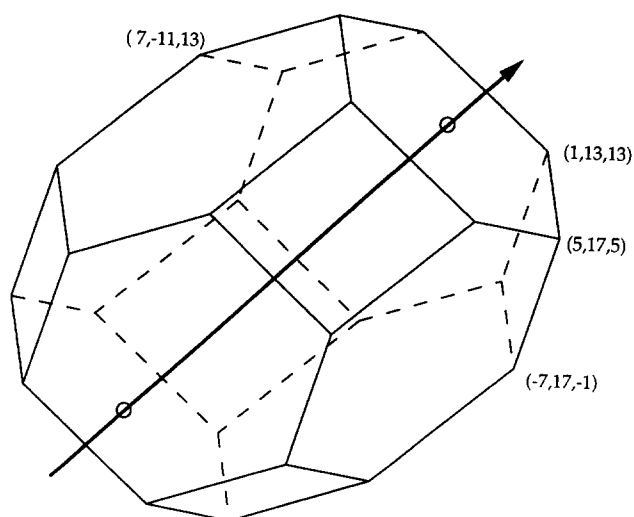


Fig. 4. The Voronoi region $\mathcal{V}_A$ of the lattice A . A cyclic permutation of coordinates to the right corresponds to an anticlockwise rotation by $2\pi/3$ about the central axis.

Table 1

The Voronoi vertices of the lattice A modulo even permutations and vertex negations

Voronoi vertex	Closest lattice vectors in A
(13, 13, 1)	(0, 0, 0), (18, 18, 18), (30, 12, -6), (6, 24, -12)
(5, 5, 17)	(0, 0, 0), (18, 18, 18), (12, -6, 30), (-12, 6, 24)
(17, -1, -7)	(0, 0, 0), (12, -6, 24), (30, 12, -6), (24, -12, 6)
(13, 7, -11)	(0, 0, 0), (12, -6, -24), (30, 12, -6), (6, 24, -12)

apart. Conversely let $w = (14, 14, -1)$. The planes $x \cdot w = 363$ and $x \cdot w = 1149$ are distance $2\sqrt{393}$ apart and it can be checked that $x \cdot w \leq 363$ for all $x \in \mathcal{V}_A$ and $y \cdot w \geq 1149$ for all $y \in \lambda + \mathcal{V}_A$.

Now since the shortest nonzero vectors in Γ' are all equivalent under even permutations and vertex negations which leave $\mathcal{V}_A$ fixed it follows that $\mathcal{V}_A$ and $\gamma + \mathcal{V}_A$ are distance $2\sqrt{393}$ apart for all shortest nonzero vectors γ of Γ' .

All nonzero vectors x of Γ' that are not of shortest length have length $\|x\| \geq 108 > 2\sqrt{339} + 2\sqrt{393}$. Thus Voronoi regions $\gamma + \mathcal{V}_A$ about distinct vectors γ of Γ' are at least distance $2\sqrt{393}$ apart. Hence $(2\sqrt{339}, 2\sqrt{393})$ is an excluded distance interval for this improved 18-cloung which on lattice rescaling is $(1, \sqrt{131/113})$.

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