

# Funky Dice: Loaded and Relabeled

William Gasarch - University of MD

## If You Roll Two Standard 6-Sided Dice Then

1. 2: (1,1). ONE way. Prob  $\frac{1}{36}$ .
2. 3: (1,2), (2,1). TWO ways. Prob  $\frac{1}{18}$ .
3. 4: (1,3), (2,2), (3,1). THREE ways. Prob  $\frac{1}{12}$ .
4. 5: (1,4), (2,3), (3,2), (4,1). FOUR ways. Prob  $\frac{1}{9}$ .
5. 6: (1,5), (2,4), (3,3), (4,2), (5,1) FIVE ways. Prob  $\frac{5}{36}$ .

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5. 6: (1,5), (2,4), (3,3), (4,2), (5,1) FIVE ways. Prob  $\frac{5}{36}$ .
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7. 8: (2,6), (3,5), (4,4), (5,3), (6,2) FIVE ways. Prob  $\frac{5}{36}$ .
8. 9: (3,6), (4,5), (5,4), (6,3) FOUR ways. Prob  $\frac{1}{9}$ .
9. 10: (4,6), (5,5), (6,4) THREE ways. Prob  $\frac{1}{12}$ .
10. 11: (5,6), (6,5) TWO ways. Prob  $\frac{1}{18}$ .
11. 12: (6,6) ONE way. Prob  $\frac{1}{36}$ .

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**Question 1** Can we load two 6-sided dice so that every number from 2 to 12 has the **same** probability. Called **fair sums**.

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Will we get the same probabilities as the standard labelling?

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Probably not.

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**Question 2** Is there **some** non-standard way to label two 6-sided die so that you get the same probability of sums as the standard labelling?

# Loaded Dice

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## Sums are Unfair!

**How Unfair?:**  $1/6 - 1/36 \sim 0.139$  unfair.

# What Are Loaded Dice?

**Def:** A **Die** is a 6-tuple  $(p_1, p_2, p_3, p_4, p_5, p_6)$  such that  $0 \leq p_i \leq 1$  and  $\sum_{i=1}^6 p_i = 1$ .

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3. Answer on next slide.

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$$p_1q_4 + p_2q_3 + p_3q_2 + p_4q_1 = \text{Prob}(\text{sum} = 5)$$

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**The coefficient of  $x^i$  is  $\text{Prob}(\text{sum} = i)$**

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**Assume** they yield **fair sums**, all sums have prob  $1/11$ . Then

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Continued on Next Slide.

## No Dice (cont)

From last slide: If there are two loaded dice that give fair sums then there exist reals  $(p_1, \dots, p_6), (q_1, \dots, q_6)$  such that

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1.  $r$  root of  $x^{10} + \dots + x + 1 \implies r$  root of  $x^{11} - 1$  &  $r \neq 1$ .
2.  $r$  root of  $x^{11} - 1$  &  $r \neq 1 \implies r$  root of  $x^{10} + \dots + x + 1$ .

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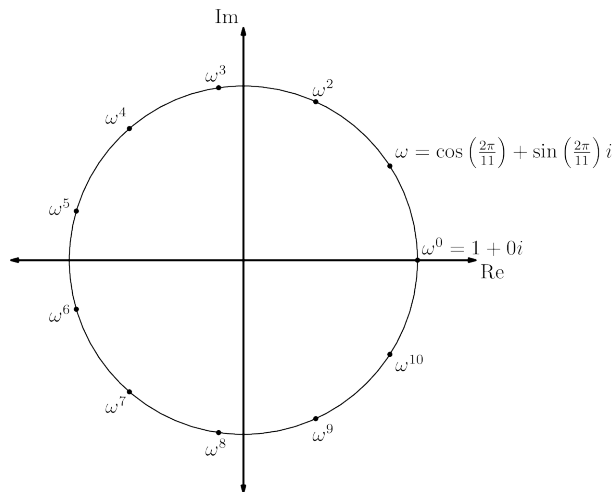
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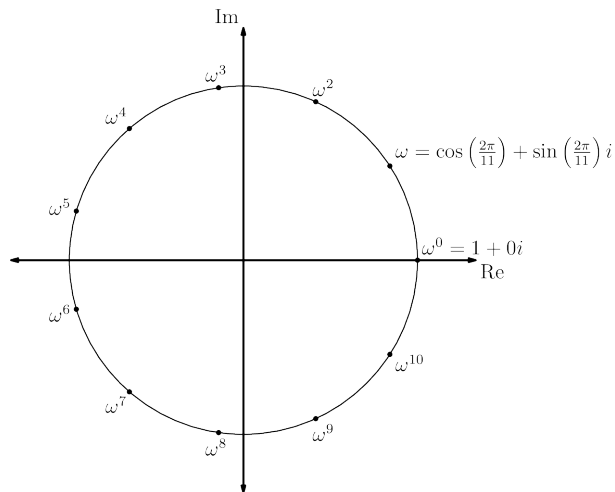
The roots of  $x^{11} - 1$  are on the complex unit circle. See Next Slide.

# The 11th Roots of Unity: Only Real one is 1



1 is only real 11th root of unity.

# The 11th Roots of Unity: Only Real one is 1



1 is only real 11th root of unity.  $x^{10} + \dots + 1 = 0$ : **no** real roots.

# No Dice (cont)

## Recap

If there exists two 6-sided dice that give fair sums then there exists reals  $p_1, \dots, p_6, q_1, \dots, q_6$  such that

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The Right Hand Side has 0 real roots.

## Contradiction

# What About Two $d$ -Sided Dice?

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**VOTE:**

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4. UNKNOWN TO SCIENCE!

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Answer on next slide

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**Answer** No  $d$ .

1. The proof that for **even  $d$**  you **cannot** load two  $d$ -sided dice to get fair sums is similar to what we did for two 6-sided dice.

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**Answer** No  $d$ .

1. The proof that for **even  $d$**  you **cannot** load two  $d$ -sided dice to get fair sums is similar to what we did for two 6-sided dice.
2. The proof that for **odd  $d$**  you **cannot** load two  $d$ -sided dice to get fair sums requires a few tricks.

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**Answer** No  $d$ .

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2. The proof that for **odd  $d$**  you **cannot** load two  $d$ -sided dice to get fair sums requires a few tricks. We leave that to you.

# Can You Ever Load Dice to Get Fair Sums?

Is there a  $d_1, d_2 \geq 2$  such that there are  $d_1$ -sided and  $d_2$ -sided dice that give fair sums?

**VOTE:** YES or NO or UNKNOWN TO SCIENCE.

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**VOTE:** YES or NO or UNKNOWN TO SCIENCE.

Answer on next slide.

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**Fame!** One paper refers to **The Gasarch-Kruskal Thm**.

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How far are normal dice from uniform?

$$2(1/11 - 1/36)^2 + 2(1/11 - 1/18)^2 + 2(1/11 - 1/12)^2 + 2(1/9 - 1/11)^2 +$$

$$2(5/36 - 1/11)^2 + (1/6 - 1/11)^2 \sim 0.0217$$

## How Close To Uniform Can You Get? (cont)

**Thm** The optimal pair of 6-sided dice is  $(\frac{1}{2}, 0, 0, 0, 0, \frac{1}{2})$  and  $(\frac{1}{8}, \frac{3}{16}, \frac{3}{16}, \frac{3}{16}, \frac{3}{16}, \frac{1}{8})$ .

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If you use two  $n$ -sided fair dice then the the distance from uniform sums is  $\frac{2n^2+1}{3n^3} - \frac{1}{2n-1} \sim \frac{1}{6n}$ .

# Different Labels on Dice

William Gasarch - University of MD

# Can You Label Dice To Get Same Probs?

A **labeling** of a 6-sided die has any positive natural numbers as labels. We allow using a number twice. We allow using numbers higher than 6. So  $(1, 2, 2, 3, 5, 8)$  would be allowed.

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YES. There a non-standard labeling of a pair of 6-sided dice so that the dice yield the SAME probabilities as the standard dice.

We prove this on the next slide.

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**Coefficient of  $x^n$  is number of ways to get  $n$ .**

## Example of Non-Standard Labelings

Label dice  $(1, 2_1, 2_2, 2_3, 5_1, 5_2)$  and  $(1, 3_1, 3_2, 3_3, 3_4, 7)$ ?

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## Example of Non-Standard Labelings

Label dice  $(1, 2_1, 2_2, 2_3, 5_1, 5_2)$  and  $(1, 3_1, 3_2, 3_3, 3_4, 7)$ ?

$$(2x^5 + 3x^2 + x)(x^7 + 4x^3 + x) = 2x^{12} + 3x^9 + 9x^8 + 2x^6 + 12x^5 + 4x^4 + 3x^3 + x^2$$

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1. 12: TWO ways. Prob  $\frac{1}{18}$ .
2. 9: THREE ways. Prob  $\frac{1}{12}$ .
3. 8: NINE ways. Prob  $\frac{1}{4}$ .
4. 6: TWO ways. Prob  $\frac{1}{18}$ .

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- 5: TWELVE ways. Prob  $\frac{1}{3}$ .
- 4: FOUR ways. Prob  $\frac{1}{9}$ .
- 3: THREE ways. Prob  $\frac{1}{12}$ .
- 2: ONE ways. Prob  $\frac{1}{36}$ .

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$$(x^6 + x^5 + x^4 + x^3 + x^2 + x)(x^6 + x^5 + x^4 + x^3 + x^2 + x)^2.$$

## Is there a Non-Standard Labeling That... Cont.

$$(x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6})(x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4} + x^{b_5} + x^{b_6}) =$$

$$(x^6 + x^5 + x^4 + x^3 + x^2 + x)^2 = x^2(x^5 + x^4 + x^3 + x^2 + x + 1) =$$

$$x^2(x+1)^2(x^2-x+1)^2(x^2+x+1)^2.$$

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So desired dice are (1, 2, 2, 3, 3, 4) and (1, 3, 4, 5, 6, 8).

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The proof is similar to what we did, though requires some thought.

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Or maybe just **Unknown to Bill**.

# Parting Thoughts

William Gasarch - University of MD

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3. It is remarkable that a problem about dice lead to looking at complex roots of polynomials!