NIM with Cash

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Standard NIM Games

A is a finite set of natural numbers.

1. n stones on the table.
2. Players alternate rm $a \in A$ stones.
3. Play until someone can't move. That player loses.

Notation: $W_A(n) =$ whoever wins if start with n stones.
Example: If $A = \{1, 3, 4\}$ then

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Note: NIM is a well known game. NIM and many variants have been studied.

NIM SWITCH THEOREM

Theorem

Let A be a finite set and $n \in N$.

$$W_A(n) = \mathbf{I} \text{ iff } (\exists a \in A, a \leq n)[W_A(n-a) = \mathbf{II}].$$

NIM with Cash (**NWC**)

Let A be a finite subset of N. We assume $1 \in A$ throughout.

1. n stones on the table. Player **I** has \$$d$, Player **II** has \$$e$.
2. Players alternate rm $a \in A$ stones. If rm a then loses \$$a$.
3. Play until someone can't move. That player loses.
   Note: Either there are no stones or that player is broke.

Notation: $W_A^{\mathrm{cash}}(n; d, e)$ is who wins if start with n stones on the table, Player **I** has \$$d$, Player **II** has \$$e$.

NIM WITH CASH SWITCH THEOREM

Theorem

Let A be a finite set and $n, d, e \in N$.

$$W_A^{\text{cash}}(n; d, e) = \mathbf{I} \text{ iff } (\exists a \in A, a \le n)[W_A(n-a; e, d-a) = \mathbf{II}].$$

Main Question

Given A:

Math Person: We want an exact win condition for $W_A^{\rm cash}(n; d, e)$.
(e.g, Player **I** wins iff $n \equiv 0 \pmod 8$ and $e < d^2$)

CS Person: Can solve using dynamic programming in $O(n^3)$ time.

Math Person: OKAY. We want an easily understood $(\log n)^{O(1)}$ algorithm to determine $W^{\rm cash}(n; d, e)$ (assuming constant time arithmetic operations).

Convention: **Win Condition** means $(\log n)^{O(1)}$ **Algorithm**

Rich, Poor, and Middle Class

There will be three cases:

1. At least one of the players is **Rich**! Using the same strategy as you would in standard NIM is **The Normal Strategy**.
2. At least one player is **Poor**. Always rm 1 until the other player is broke is **The Miserly Strategy**.
3. Both are **Middle Class**. This is the hard case.

If At Least One Player Is Rich: Example

$A = \{1, 3, 4\}$. $n = 10$. Player **I** needs ??? to win.

If At Least One Player Is Rich: Example

$A = \{1, 3, 4\}$. $n = 10$. Player **I** needs ??? to win.

1. Player **I** rm 3 leaving 7 stones.
2. Player **II** rm 1,3,4 leaving 6,4,3 stones.
3. Player **I** rm 4,4,3 leaving 2,0,0 stones.
4. If 0 stones left then DONE- Player **I** wins, else there are 2 stones
   4.1 Player **II** rm 1 leaving 1
   4.2 Player **I** rm 1 leaving 0 and he wins!

Player **I** rm at most 3+4+1=8 stones.
Upshot: If Player **I** has \$8 then Player **I** wins **no matter how much Player II has**. $W^{\text{cash}}(10; 8, \infty) = \mathbf{I}$. Player **I** **normally**.

If At Least One Player Is Rich: How Rich?

$f^{\mathbf{II}}(0) = 0$ (Player **II** wins and needs 0 to win.)

If $W_A(n) = \mathbf{I}$ then

$$f^{\mathbf{I}}(n) = \min_{a \in A, a \le n} \{f^{\mathbf{II}}(n-a) + a : W_A(n-a) = \mathbf{II}\}$$

If $W_A(n) = \mathbf{II}$ then

$$f^{\mathbf{II}}(n) = \max_{a \in A, a \le n} \{f^{\mathbf{I}}(n-a)\}$$

Easy to prove:

- If $W_A(n) = \mathbf{I}$ then Player **I** wins $(n; f^{\mathbf{I}}(n), \infty)$.
- If $W_A(n) = \mathbf{II}$ then Player **II** wins $(n; \infty, f^{\mathbf{II}}(n))$.

Note: Only defined $f^{\mathbf{I}}(n)$ when $W_A(n) = \mathbf{I}$. Sim $f^{\mathbf{II}}(n)$.

Wrong Guy Problem

What if $W_A(n) = \mathbf{I}$ but Player **I** has $f^{\mathbf{I}}(n) - 1$ dollars?
How much does Player **II** need to win?

If $W_A(n) = \mathbf{II}$ then

$$f^{\mathbf{I}}(n) = \min_{a \in A, a \le n} \{ f^{\mathbf{II}}(n-a) + a : f^{\mathbf{I}}(n-a) = f^{\mathbf{II}}(n) \}.$$

It $W_A(n) = \mathbf{I}$ then

$$f^{\mathbf{II}}(n) = \max_{a \in A, a \le n} \{ f^{\mathbf{I}}(n-a) \}.$$

Rich Man Theorem

Theorem

Let A, n, d, e be given. Let $f^{\mathbf{I}}, f^{\mathbf{II}}$ be as defined above.

1. *If $d \geq f^{\mathbf{I}}(n)$ and $e < f^{\mathbf{II}}(n)$ then $W_A^{\text{cash}}(n; d, e) = \mathbf{I}$.*
2. *If $d < f^{\mathbf{I}}(n)$ and $e \geq f^{\mathbf{II}}(n)$ then $W_A^{\text{cash}}(n; d, e) = \mathbf{II}$*
3. *If $d \geq f^{\mathbf{I}}(n)$ and $e \geq f^{\mathbf{II}}(n)$ then $W_A^{\text{cash}}(n; d, e) = W_A(n)$.*

Upshot: Have covered **ALL** cases where at least one player is rich.

Example $A = \{1, 3, 4\}$

1. $f^{\mathbf{I}}(7k) = f^{\mathbf{II}}(7k) = 5k$.
2. $f^{\mathbf{I}}(7k+1) = 5k+1$ and $f^{\mathbf{II}}(7k+1) = 5k$.
3. $f^{\mathbf{I}}(7k+2) = f^{\mathbf{II}}(7k+2) = 5k+1$.
4. $f^{\mathbf{I}}(7k+3) = 5k+2$ and $f^{\mathbf{II}}(7k+3) = 5k+1$.
5. $f^{\mathbf{I}}(7k+4) = 5k+4$ and $f^{\mathbf{II}}(7k+4) = 5k+2$.
6. $f^{\mathbf{I}}(7k+5) = f^{\mathbf{II}}(7k+5) = 5k+4$.
7. $f^{\mathbf{I}}(7k+6) = 5k+5$ and $f^{\mathbf{II}}(7k+6) = 5k+4$.

Note: For **all** sets A we have looked at the functions $f^{\mathbf{I}}$ and $f^{\mathbf{II}}$ are roughly of the form above.

If At Least One Player is Poor: Example

$A = \{1, 3, 4\}$. $n = 10$, Player **I** has \$4 , Player **II** has \$4.

If At Least One Player is Poor: Example

$A = \{1, 3, 4\}$. $n = 10$, Player **I** has \$4 , Player **II** has \$4.

1. Player **II** always rm just one stone.
2. Player **I** runs out of money and loses.

If At Least One Players is Poor: How Poor?

$g^{\mathbf{I}}(n) = \lfloor \frac{n}{2} \rfloor + 1$

$g^{\mathbf{II}}(n) = \lfloor \frac{n}{2} \rfloor + (n \bmod 2)$

Theorem Let $n, d, e \in \mathbb{N}$.

1. If $d \geq g^{\mathbf{I}}(n)$ and $e < g^{\mathbf{II}}(n)$ then $W_A^{\text{cash}}(n; d, e) = \mathbf{I}$.
   (**II** is poor, **I** is not)
2. If $d < g^{\mathbf{I}}(n)$ and $e \geq g^{\mathbf{II}}(n)$ then $W_A^{\text{cash}}(n; d, e) = \mathbf{II}$.
   (**I** is poor, **II** is not)
3. If $d < g^{\mathbf{I}}(n)$ and $e < g^{\mathbf{II}}(n)$ (both poor) then
   - 3.1 If $d > e$ then $W_A^{\text{cash}}(n; d, e) = \mathbf{I}$
   - 3.2 If $d = e$ then $W_A^{\text{cash}}(n; d, e) = \mathbf{II}$
   - 3.3 If $d < e$ then $W_A^{\text{cash}}(n; d, e) = \mathbf{II}$

Upshot: Have covered all cases where at least one player is poor.

If Both Players are Middle Class: Example

$n = 12$. Player **I** needs \$9 to win normally. If Player **I** has \$8 dollars than Player **II** can wrong-guy-win with \$9. What if both players have \$8? We are in scenario $(12; 8, 8)$.

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1. Player **I** rm 1.
2. If Player **II** rm 3 or 4 then he is poor and lose. So he rm 1.
3. Game is now $(10; 7, 7)$. Player **I** is rich and can win normally.

Note: Typical: Play **miserly** until you are **Rich**.

If Both Players are Middle Class

Let $n, d, e \in \mathbb{N}$. If any of the following happens then the previous slides determine who wins:

- $d \geq f^{\mathbf{I}}(n)$
- $e \geq f^{\mathbf{II}}(n)$
- $d \leq g^{\mathbf{I}}(n)$
- $e \leq g^{\mathbf{II}}(n)$

Def: Both players are **Middle Class** if none of the above happens.

Different Viewpoint

A is a finite set, $1 \in A$. Normal Nim has periodicity p. $(n; d, e)$ is such that both players are middle class.

We map $(n; d, e)$ to $(n \bmod p; b, b^\dagger)$ where

- $b = f^{\mathbf{I}}(n) - d - 1$. How much Player **I** is short of $f^{\mathbf{I}}(n)$.
- $b^\dagger = f^{\mathbf{II}}(n) - e - 1$. How much Player **II** is short of $f^{\mathbf{II}}(n)$.

A set A is **nice** if from $(n \bmod p, b, b^\dagger)$ and $a \in A$ you can determine what $(n - a \bmod p, {b^\dagger}', b')$ you are in.
We assume A is nice.

The Magic Set X

$X \subseteq [p] \times \mathrm{N} \times \mathrm{N}$ is **WINNING** if:

I: For all $(i, b, b^{\dagger}) \in X$ if rm 1 get $(i', b^{\dagger\prime}, b')$ where EITHER

- $(i, b^{\dagger\prime}, b')$ is NOT in X.
- $b' < 0$ and $b^{\dagger\prime} > 0$.
- $b' < 0$ and $b^{\dagger\prime} < 0$ and $W_A(i') =$ **II**.

II: For all $(i, b, b^{\dagger}) \notin X$, if rm $a \in A$ get $(i', b^{\dagger\prime}, b')$ then EITHER

- $(i', b^{\dagger\prime}, b') \in X$.
- $b^{\dagger\prime} < 0$ and $b' \geq 0$.
- $b^{\dagger\prime} < 0$ and $b' < 0$ and $W_A(i') =$ **I**.

Middle Class Theorem

Theorem

Let A be a nice finite set. Assume there exists an p, X as above. Let $n, d, e \geq 0$ Assume that with $(n; d, e)$ both players are middle class. Let $b = f^{\mathbf{I}}(n) - d - 1$ and $b^{\dagger} = f^{\mathbf{II}}(n) - e - 1$.

$$W_A^{\mathrm{cash}}(n; d, e) = \mathbf{I} \textit{ iff } (n \bmod p; b, b^{\dagger}) \in X.$$

Example of a set X

If $A = \{1, 3, 4\}$ then the following set X works.
Take the union of the following sets of $(i; b, b^\dagger)$.

1. $i \in \{0, 2, 5\}$ and $b \leq \lfloor b^\dagger/2 \rfloor \times 2$.
2. $i \in \{1, 3, 6\}$ and $b^\dagger > \lfloor b/2 \rfloor \times 2$.
3. $i \in \{4\}$ and $b^\dagger \geq \lfloor b/2 \rfloor \times 2$.

Complete Theorem

Let A be a nice finite set. Let $f^{\mathbf{I}}, f^{\mathbf{II}}, g^{\mathbf{I}}, g^{\mathbf{II}}$ be defined as above. Assume there exists an p, X as above. Let $n, d, e \geq 0$.

1. If $d \geq f^{\mathbf{I}}(n)$ and $e < f^{\mathbf{II}}(n)$ then $W_A^{\text{cash}}(n; d, e) = \mathbf{I}$.
2. If $d < f^{\mathbf{I}}(n)$ and $e \geq f^{\mathbf{II}}(n)$ then $W_A^{\text{cash}}(n; d, e) = \mathbf{II}$
3. If $d \geq f^{\mathbf{I}}(n)$ and $e \geq f^{\mathbf{II}}(n)$ then $W_A^{\text{cash}}(n; d, e) = W_A(n)$.
4. If $d \geq g^{\mathbf{I}}(n)$ and $e < g^{\mathbf{II}}(n)$ then $W_A^{\text{cash}}(n; d, e) = \mathbf{I}$.
5. If $d < g^{\mathbf{I}}(n)$ and $e \geq g^{\mathbf{II}}(n)$ then $W_A^{\text{cash}}(n; d, e) = \mathbf{II}$.
6. If $d < g^{\mathbf{I}}(n)$ and $e < g^{\mathbf{II}}(n)$ (both poor) then
   6.1 If $d > e$ then $W_A^{\text{cash}}(n; d, e) = \mathbf{I}$
   6.2 If $d = e$ then $W_A^{\text{cash}}(n; d, e) = \mathbf{II}$
   6.3 If $d < e$ then $W_A^{\text{cash}}(n; d, e) = \mathbf{II}$
7. If none of the above hold then let $b = f^{\mathbf{I}}(n) - d - 1$ and $b^{\dagger} = f^{\mathbf{II}}(n) - e - 1$. Player **I** wins iff $(n \bmod p; b, b^{\dagger}) \in X$.

Upshot: Have covered **ALL** cases.

Conjecture about $f^{\mathbf{I}}, f^{\mathbf{II}}$ for $A = \{L, \ldots, M\}$

Conjecture 1: There is an offset Θ, depending on L, M such that

$$(\forall 0 \le n \le \Theta - 1)[f^{\mathbf{I}}(n), f^{\mathbf{II}}(n) \text{ some stuff }]$$

$$(\forall n \ge \Theta)[f^{\mathbf{I}}(n + L + M) = f^{\mathbf{I}}(n) + M]$$

$$(\forall n \ge \Theta)[f^{\mathbf{II}}(n + L + M) = f^{\mathbf{II}}(n) + M]$$

Conjecture 2: $\Theta \le 5(M - L)^2 + 2$

Conjecture 3: If $M \ge 2L$ then $\Theta = 2(L + 1)$.

Conjecture about Magic X for $A = \{L, \ldots, M\}$

Conjecture: The Magic set X for $A = \{L, \ldots, M\}$ is $(i, b, b^\dagger)$ such that:

- $0 \le i < L + M$ and $b, b^\dagger \ge 0$
- if $i < L$ then $\left\lfloor \frac{b}{L} \right\rfloor \le \left\lfloor \frac{b^\dagger}{L} \right\rfloor$
- If $L \le i < 2L$ then $\left\lfloor \frac{b}{L} \right\rfloor \le \left\lfloor \frac{b^\dagger - L}{L} \right\rfloor$
- If $2L \le i < 3L$ then $\left\lfloor \frac{b}{L} \right\rfloor \le \left\lfloor \frac{b^\dagger - 3L + i + 1}{L} \right\rfloor$
- If $i \ge 3L$ then $\left\lfloor \frac{b}{L} \right\rfloor \le \left\lfloor \frac{b^\dagger}{L} \right\rfloor$

What Have We Done/What Can We Do

1. Have exact win conditions for
   1.1 $\{1, L\}$
   1.2 $\{1, L, L+1\}$.
2. We have a program that on input A:
   - Outputs $f^{\mathbf{I}}, f^{\mathbf{II}}, g^{\mathbf{I}}, g^{\mathbf{II}}$ (easy).
   - Outputs a conjecture for X (it has never been wrong).
3. For $A = \{L, \ldots, M\}$ we have a conjecture that is surely true.

Future Directions

Conjectures:

1. Exists fast alg to find win cond for W_A^{cash}.
2. Exists alg to find win cond for W_A^{cash}.
3. Exists a win cond for W_A^{cash}.
4. The functions $f^{\mathbf{I}}$ and $f^{\mathbf{II}}$ are always some sort of mod pattern.