Fun Problems!

Casino Game!

Casino Game

You make 2 bets. LOSE only once.

1. You have \$1 and will make two bets x, y, $x + y \leq 1$.
2. You bet x. Casino says WIN (W) or LOSE (L)
3. You bet y. Casino says W or L.
4. Casino can only say L once.

Strategy Tips: If you bet \$$x$ in 1st round:

1. If on the 1st bet Casino says W then bet 0 for 2nd bet.
2. If on the 1st bet Casino says L then bet $1 - x$ for 2nd bet.

Casino Game Example

DO WITH YOUR GROUP: If you bet $\frac{3}{4}$? $\frac{1}{2}$? $\frac{1}{4}$? then what will Casino do to maximize its profit. How much do you end up with?

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You bet $\frac{1}{4}$.

- Casino says W, bet 0 in round 2, end with $\frac{5}{4}$.
- Casino says L, bet $\frac{3}{4}$, end with $\frac{3}{2}$.

Casino will say W and you end up with $\frac{5}{4}$.

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You bet $\frac{1}{2}$.

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- Casino says L, bet $\frac{1}{2}$ in round 2, end with 1.

Casino will say L and you end up with 1

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- Casino says W, bet 0 in round 2, end with $\frac{3}{2}$.
- Casino says L, bet $\frac{1}{2}$ in round 2, end with 1.

Casino will say L and you end up with 1

You bet $\frac{3}{4}$.

- Casino says W, bet 0 in round 2, end with $\frac{7}{4}$.
- Casino says L, bet $\frac{1}{4}$, end with $\frac{1}{2}$.

Casino will say L and you end up with $\frac{1}{2}$.

How Much Should You Bet?

What is the optimal strategy for the good casino game. How much should you bet in the 1st and 2nd rounds to maximize your profit?
Work in your Groups!

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Solution: If you bet x in the 1st round then

- If Casino says W then bet 0 in 2nd round, end with $1 + x$
- If Casino says L then bet $1 - x$ in 2nd, end with $2 - 2x$

Set $1 + x = 2 - 2x$, so $x = \frac{1}{3}$, you end up with $\frac{4}{3}$.

Note: If began with z, bet $\frac{z}{3}$, end up with $\frac{4z}{3}$.

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We solved problem for betting TWICE.

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Solution: If you bet x in the 1st round then

- If Casino says W then bet 0 in the 2nd round, end with $1 + x$
- If Casino says L then you have $1 - x$ and two rounds, use last strategy to end up with $\frac{4(1-x)}{3}$.

Want these to be EQUAL, which means $x = \frac{1}{7}$ and you end up with $\frac{8}{7}$.

Note: If began with z then bet $\frac{z}{7}$ and end up with $\frac{8z}{7}$.

How Much Should You Bet? – 4 Rounds, n Rounds

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We solved the problem for betting THRICE: $\frac{8z}{7}$

Can you spot a pattern?

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Can you spot a pattern?
Yes!
If make n bets then $\frac{2^n z}{2^n-1}$.
Solution: If you bet x in the 1st round then

- If Casino says W then bet 0 in the 2nd round, end with $1+x$
- If Casino says L then you have $n-1$ rounds, end up with $\frac{2^{n-1}(1-x)}{2^{n-1}-1}$. $\frac{4(1-x)}{3}$.

Leave it to you to work out the algebra.

Principles

1. Do not give your opponent a chance to make a decision – make both cases equal.
2. Gather evidence, sport patterns, then prove them.
3. Generalize – leave it to you to look at
    - n rounds
    - Casino says W M times

Change of a Dollar!

Change of a n cents: Pennies and Nickels

How many ways can you make change for 0 cents:

Change of a n cents: Pennies and Nickels

How many ways can you make change for 0 cents:
only 1.

Using only pennies:
How many ways can you make change for 1,2,3, or 4 cents:

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Using only pennies and nickels:
How many ways can you make change for 5,6,7,8, or 9 cents:

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Using only pennies:
How many ways can you make change for 1,2,3, or 4 cents:
only 1.

Using only pennies and nickels:
How many ways can you make change for 5,6,7,8, or 9 cents:
only 2. Use a nickel or don't.

Change of a n cents: Pennies and Nickels

How many ways can you make change of n using pennies and nickels? Give a formula.

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How many ways can you make change of n using pennies and nickels? Give a formula.

Formula: $f(n)$ is number of ways to make change of n cents using pennies and nickels.

$f(0) = f(1) = f(2) = f(3) = f(4)$.

If $0 \le i \le 4$: $f(5k + i) = f(5k)$. So we only look at $f(5k)$

$f(5k)$: You either use 0 nickels or ≥ 1 nickels.

- If you use 0 nickels then $f(5k) = 1$, all pennies.
- If you ≥ 1 nickels then $f(5k - 5)$.

$$f(5k) = 1+f(5(k-1)) = 2+f(5(k-2)) = \cdots k+f(5(k-k)) = k+1$$

Change of n cents: Pennies and Nickels

How many ways can you make change of n using pennies and nickels? Write a program:

Program: Use Recursion (Top Down)

PROGRAM f

Input n. If $n = 0, 1, 2, 3, 4$ then return 1.

Return $1 + f(n-5)$ (either use 0 nickels or ≥ 1 nickel)

Program: Dynamic Programming (Bottom Up)

Let F be an Array.

$F(0) = 1$, $F(1) = 1$, $F(2) = 1$, $F(3) = 1$, $F(4) = 1$.

For $i = 5$ to n Return $F(i) = F(i-5) + 1$.

Key: When computing $F(i)$ you already have $F(i-5)$.

Adv: End up with $F(0), \ldots, F(n)$, not just $F(n)$.

Change of a n cents: Pennies and Nickels and Dimes

Let $g(n)$ be the number of ways to make change of a n cents using pennies, nickels, and dimes. Write a program to computer $g(n)$ using both Recursion and Dynamic Programming.

When done run it on $n = 1$ to 100 and compare results to a neighboring table to check.

Advice: First THINK about the program on paper and do some examples.

Change of n cents: Pennies and Nickels and Dimes

1. If $0 \le n \le 9$ then $g(n) = f(n)$.
2. If use ≥ 1 dime then $g(n-10)$ ways to make change.

Program: Use Recursion (Top Down)
PROGRAM f

Input n. If $0 \le n \le 9$ then return $f(n)$.

Return $1 + g(n-10)$.

Program: Dynamic Programming (Bottom Up)
Let F be an Array.

For $0 \le n \le 9$ $G(n) = F(n)$.

For $i = 10$ to n Return $G(i) = G(i-10) + 1$.

Key: When computing $G(i)$ you already have $G(i-10)$.
Adv: End up with $G(0), \ldots, G(n)$, not just $F(n)$.
Question: Which program is better: Recursion of Dyn Prog?

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PROGRAM f

Input n. If $0 \le n \le 9$ then return $f(n)$.

Return $1 + g(n-10)$.

Program: Dynamic Programming (Bottom Up)
Let F be an Array.

For $0 \le n \le 9$ $G(n) = F(n)$.

For $i = 10$ to n Return $G(i) = G(i-10) + 1$.

Key: When computing $G(i)$ you already have $G(i-10)$.
Adv: End up with $G(0), \ldots, G(n)$, not just $F(n)$.
Question: Which program is better: Recursion of Dyn Prog?
Dyn Prog!: Recursion may recompute many values. Slow.

Change of a n cents: Pennies and Nickels and Dimes

Is there a **Formula** for how many ways to make change of a n.

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Yes and No.

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Yes and No.

$$g_0(n) = \frac{1}{100}x^2 + \frac{1}{5}x + 1$$

$$g_1(n) = \frac{1}{100}x^2 + \frac{9}{50}x + \frac{81}{100}$$

$$g_2(n) = \frac{1}{100}x^2 + \frac{4}{25}x + \frac{16}{25}$$

$$g_3(n) = \frac{1}{100}x^2 + \frac{7}{50}x + \frac{49}{100}$$

$$g_4(n) = \frac{1}{100}x^2 + \frac{3}{25}x + \frac{9}{25}$$

$$g_5(n) = \frac{1}{100}x^2 + \frac{1}{5}x + \frac{3}{4}$$

$$g_6(n) = \frac{1}{100}x^2 + \frac{9}{50}x + \frac{14}{25}$$

$$g_7(n) = \frac{1}{100}x^2 + \frac{4}{25}x + \frac{39}{100}$$

$$g_8(n) = \frac{1}{100}x^2 + \frac{7}{50}x + \frac{6}{25}$$

$$g_9(n) = \frac{1}{100}x^2 + \frac{3}{25}x + \frac{11}{100}$$

$$g(n) = g_{n \bmod 10}(n)$$

Would you call that a formula? **Discuss**

Change of a n cents: Pennies and Nickels and Dimes and Quarters

Do on your own.

Find the Missing Numbers

Ground Rules

Alice says **ALL BUT** 1 of the numbs in $\{1, \ldots, 1000\}$ in randomly.

Bob listens to her and wants to determine **The Missing Number**.

Could Bob keep track of ALL of the numbs to det **TMN**?

No- Bob's memory isn't that good. Is there some other way?

- How can Bob do this without writing down too much?
- How go generalize and formalize this problem?

Try To Solve It

Help Bob Determine TMN! **In your groups!**

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Help Bob Determine TMN! **In your groups!**
Solution:
Note that

$$\sum_{i=1}^{1000} = \frac{1000 \times 1001}{2} = 500500$$

Bob keeps a **running sum** of all the numbs he hears.
At the end the sum is S. TMN is $500500 - S$

How to Generalize

Alice says **ALL BUT** 1 of the numbs in $\{1, \ldots, n\}$ in randomly.

- How can Bob do this without writing down too much?
- What is **too much**? How for formalize?

You Solve it!

Alice says **ALL BUT** 1 of the numbs in $\{1, \ldots, n\}$ in randomly. How can Bob find TMN without too much memory.

Point: In school you are presented with clean problems. This is more real world– we will first solve the problem and then figure out how to ask it properly.

Work on it!

You Solve it!

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How can Bob find TMN without too much memory.

Point: In school you are presented with clean problems. This is more real world– we will first solve the problem and then figure out how to ask it properly.

Work on it!
Solution:
Note that $\sum_{i=1}^{n} = \frac{n(n+1)}{2}$.
Bob keeps a **running sum** of all the numbs he hears.
At the end the sum is S. TMN is $\frac{n(n+1)}{2} - S$.

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Bob has to remember a number between 1 and $\frac{n(n+1)}{2}$.
How many bits is such a number?

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How many bits is such a number? $\log_2(\frac{n(n+1)}{2}) \sim 2 \log_2 n$

Clean Questions For You For Later

Alice says **ALL BUT** 1 of the numbs in $\{1, \ldots, n\}$ in randomly.

1. Can Bob det TMN with only $2\log_2 n$ bits of memory? YES!
2. Can Bob det TMN with only $\log_2 n$ bits of memory?
3. Can Bob det TMN with only $(0.5)\log_2 n$ bits of memory?

Alice says **ALL BUT** 2 of the numbs in $\{1, \ldots, n\}$ in randomly.

1. Can Bob det TMNs with only $c\log_2 n$ bits of memory.
2. Find the smallest possible c

Alice says **ALL BUT** k of the numbs in $\{1, \ldots, n\}$ in randomly.

1. Can Bob det TMNs with only $c\log_2 n$ bits of memory.
2. Find the smallest possible c