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Recent trends in Euclidean Ramsey theory

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Abstract

We give a brief summary of several new results in Euclidean Ramsey theory, a subject which typically investigates properties of configurations in Euclidean space which are preserved under finite partitions of the space.

1. Introduction

Ramsey theory typically deals with problems of the following type. We are given a set S, a family $\mathscr{F}$ of subsets of S, and a positive integer r. We would like to decide whether or not for every partition of $S = C_1 \cup \cdots \cup C_r$ into r subsets, some C_i contains some $F \in \mathscr{F}$. If so, we write $S \xrightarrow{r} \mathscr{F}$ (for a more complete treatment of Ramsey theory, see [13]).

In Euclidean Ramsey theory, S is usually the set of points of some Euclidean space $\mathbb{E}^N$, and the sets on $\mathscr{F}$ are determined by various geometric considerations. For example, suppose X is some finite subset of $\mathbb{E}^k$, and let $\mathscr{F} = \mathscr{F}_N(X)$ denote the set of *congruent copies* of X in $\mathbb{E}^N$. We say that X is *Ramsey* if for all r, there exists $N = N(X, r)$ such that $\mathbb{E}^N \xrightarrow{r} \mathscr{F}_N(X)$. In this case we will use the abbreviation $\mathbb{E}^N \xrightarrow{r} X$ (cf. [2]).

Instead of letting $\mathscr{F} = \mathscr{F}_N(X)$ be determined by letting the special orthogonal group $SO(k)$ act on X, we could let $\mathscr{F} = \mathscr{F}'_N(X)$ be the family of all *homothetic copies* $tX + \bar{a}$ of X (where t is a positive real and $\bar{a} \in \mathbb{E}^N$). Thus, $\mathscr{F}'_N(X)$ consists of all dilated (by t) and translated (by $\bar{a}$) copies of X. In this case, the assertion $\mathbb{E}^N \xrightarrow{r} \mathscr{F}'_N(X)$, $N = \dim(X)$, is a standard result in classical Ramsey theory due independently to Gallai and Witt (see [13]). However, for this situation the much stronger density theorem holds (due to Furstenberg [8]). What we mean by this is illustrated by the following example. For $X = \{1, 2, \ldots, k\}$, the assertion $\mathbb{E} \xrightarrow{r} \mathscr{F}'(X)$ is just van der Waerden's theorem [21, 13], which asserts that if $\mathbb{N} = \{0, 1, 2, \ldots\}$ is partitioned into finitely many classes C_i, then some C_i contains k-term arithmetic progressions (= homothetic copies of $\{1, 2, \ldots, k\}$) for every k. However, this is an immediate consequence of Szemerédi's result [20] that

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if $S \subset \mathbb{N}$ has positive upper density, i.e.,

$$\limsup_{N \to \infty} \frac{|S \cap \{1, 2, \ldots, N\}|}{N} > 0,$$

then S contains k-term arithmetic progressions for every k. The theorem of van der Waerden is a partition theorem; the (more difficult) theorem of Szemerédi is a density version of it.

One way to formulate density theorems for sets X which are arbitrary finite subsets of $\mathbb{E}^n$ (rather than subsets of the integer lattice points of $\mathbb{E}^n$) is to identify the lattice generated by integer linear combinations of the $x \in X$ with the corresponding integer lattice points in the Euclidean space $\mathbb{E}^{|X|}$ (we omit details).

2. Ramsey sets

The fundamental question, which remains unanswered at the time of this writing, is to characterize Ramsey sets. Let us say that X is *spherical* if X is contained on the surface of some sphere (with finite radius). A basic result in Euclidean Ramsey theory is the following.

Theorem (Erdős et al. [2]). *If X is Ramsey then X is spherical.*

Thus, the simplest sets which are not Ramsey are sets X_3 of three collinear points. It is known [19] that $\mathbb{E}^N$ can be always partitioned in 16 sets, none of which contains a congruent copy of X_3.

On the other hand, Frankl and Rödl [5] have recently shown that any simplex X^* (i.e., $n+1$ points spanning $\mathbb{E}^n$) is Ramsey. Also, it is known [2] that if X and X' are Ramsey then so is their Cartesian product $X \times X'$. Quite recently, Křiž settled an old question in Euclidean Ramsey theory by showing that the set of 5 vertices of a regular pentagon is Ramsey. More generally, he showed [14] that if X has a transitive automorphism group which is solvable then X is Ramsey.

It is natural to make the following conjecture.

Conjecture (\$1000). If X is spherical then X is Ramsey.

3. Sphere-Ramsey sets

Let $S^n(\rho)$ denote the sphere of radius ρ centered at the origin in $\mathbb{E}^{n+1}$, i.e.,

$$S^n(\rho) := \left\{ \bar{x} = (x_1, \ldots, x_{n+1}) : \sum_{i=1}^{n+1} x_i^2 = \rho^2 \right\}.$$

We say that X is *sphere-Ramsey* if for all r there exists $N=N(X,r)$ and $\rho=\rho(X,r)$ such that for any partition $S^N(\rho)=C_1\cup\cdots\cup C_r$, some C_i contains a congruent copy of X (which we abbreviate by $S^N(\rho)\xrightarrow{r} X$).

Clearly if X is sphere-Ramsey then X is Ramsey (and therefore spherical). Also, it can be shown (cf. [16]) that if X and Y are sphere-Ramsey then so is the Cartesian product $X\times Y$.

The following recent result of Matoušek and Rödl (see also [5]) shows that simplexes are sphere-Ramsey.

Theorem (Matoušek and Rödl [16]). *Suppose $X\subseteq S^k(1)$ is a simplex. Then for all r and all $\varepsilon>0$, there exists $N=N(X,r,\varepsilon)$ such that $S^N(1+\varepsilon)\xrightarrow{r} X$.*

The ε occurring in the preceding statement is not a defect of the proof but rather an essential ingredient as the following result of the author shows.

Theorem (Graham [11]). *If $X=\{\bar{x}_1,\ldots,\bar{x}_t\}\subseteq S^k(1)$ is* unit-sphere-Ramsey, *i.e.*, $S^{N(X,r)}(1)\xrightarrow{r} X$, *then for any linear dependence*

$$\sum_{i\in I} c_i\bar{x}_i=\bar{0}$$

there must exist a nonempty $J\subseteq I$ so that

$$\sum_{j\in J} c_j=0.$$

As a corollary, if the convex hull of $X\subseteq S^k(1)$ contains the origin $\bar{0}$ then X is *not* unit-sphere-Ramsey (since in this case $\bar{0}=\sum_{i\in I} c_i x_i$ with all $c_i>0$).

There is currently no plausible conjecture characterizing the sphere-Ramsey sets.

4. A question of Furstenberg

Not long ago Bourgain [1] (using tools from harmonic analysis) established the following interesting result, a type of density theorem in which the group $SO(n)$ is enlarged to allow expansions as well. For a set $W\subseteq\mathbb{E}^k$, define the upper density $\bar{\delta}(W)$ of W by

$$\bar{\delta}(W):=\limsup_{R\to\infty}\frac{m(B(0,R)\cap W)}{m(B(0,R))},$$

where $B(0,R)$ denotes the k-ball $\{\bar{x}=(x_1,\ldots,x_k)\colon \sum_{i=1}^k x_i^2\leqslant R^2\}$ centered at the origin, and m denotes Lebesgue measure.

Theorem (Bourgain [1]). *Let $X=\{x_1,\dots,x_k\}\subseteq\mathbb{E}^k$ be a simplex (i.e., X spans a $(k-1)$-space). If $W\subseteq\mathbb{E}^k$ with $\bar{\delta}(W)>0$ then there exists t_0 so that for all $t>t_0$, W contains a congruent copy of tX.*

Furstenberg et al. [9] had earlier results for $k=1$ and 2.

Bourgain also showed that some restriction on X is necessary by exhibiting a set W_0 with $\bar{\delta}(W_0)>0$ for which there are $t_1<t_2<\cdots$ tending to infinity, so that W_0 contains no congruent copy of any t_iX_3, where X_3 is the set of 3 collinear points forming a degenerate $(1,1,2)$-triangle. (In fact, essentially the same construction had already occurred in [2]). Furstenberg [7] asked whether the same phenomenon occurs for *any* nonspherical set X. The following result shows that this is indeed the case.

Theorem. *Let $X=\{\bar{x}_1,\dots,\bar{x}_n\}\subseteq\mathbb{E}^k$ be nonspherical. Then for any N there exists a set $W\subseteq\mathbb{E}^N$ with $\bar{\delta}(W)>0$ and a set $T\subset\mathbb{R}$ with $\underline{\delta}(T)>0$ so that W contains no congruent copy of tX for any $t\in T$.*

Proof. We first claim that there must exist constants $c_2,c_3,\dots,c_n$ such that

(i) $\sum_{i=2}^n c_i(\bar{x}_i-\bar{x}_1)=0$,

(ii) $\sum_{i=2}^n c_i(\bar{x}_i\cdot\bar{x}_i-\bar{x}_1\cdot\bar{x}_1)=1$

(so the c_i are not all zero).

To see this, assume without loss of generality that X is *minimally* nonspherical (consequently, $\{\bar{x}_1,\dots,\bar{x}_{n-1}\}$ is spherical). Now, since X is nonspherical, X cannot be a simplex, and consequently the vectors $\bar{x}_i-\bar{x}_1$, $i=2,3,\dots,n$, must be dependent. That is, there exist c_i (not all zero) such that (i) holds. By the minimality assumption, we can assume $c_n\neq 0$, and that $\bar{x}_1,\dots,\bar{x}_{n-1}$ lie on some sphere, say with center $\bar{w}$ and radius r. Since

$$\bar{x}_i\cdot\bar{x}_i-\bar{x}_1\cdot\bar{x}_1=(\bar{x}_i-\bar{w})\cdot(\bar{x}_i-\bar{w})-(\bar{x}_1-\bar{w})\cdot(\bar{x}_1-\bar{w})+2(\bar{x}_i-\bar{x}_1)\cdot\bar{w}$$

then

$$\begin{aligned}\sum_{i=2}^n c_i(\bar{x}_i\cdot\bar{x}_i-\bar{x}_1\cdot\bar{x}_1)&=\sum_{i=2}^n c_i((\bar{x}_i-\bar{w})\cdot(\bar{x}_i-\bar{w})-(\bar{x}_1-\bar{w})\cdot(\bar{x}_1-\bar{w}))\\&\quad+2\sum_{i=2}^n c_i(\bar{x}_i-\bar{x}_1)\cdot\bar{w}\\&=c_n((\bar{x}_n-\bar{w})\cdot(\bar{x}_n-\bar{w})-r^2)=b\neq 0,\end{aligned}$$

since by assumption $\bar{x}_n$ is *not* on the sphere with center $\bar{w}$ and radius r. We can now rescale the c_i to make b equal to 1, and so (ii) also holds, and the claim is proved.

Now, set

$$c'_1 = -\sum_{i=2}^{n} c_i,$$

$$c'_i = c_i, \quad 2 \leqslant i \leqslant n.$$

Then by (i) and (ii) we have

(i′) $\sum_{i=1}^{n} c'_i \bar{x}_i = \bar{0}$

(ii′) $\sum_{i=1}^{n} c'_i \bar{x}_i \cdot \bar{x}_i = 1$

(iii′) $\sum_{i=1}^{n} c'_i = 0$.

Next, we define the set W. For $1 \leqslant i \leqslant n$, define

$$W_i := \{\bar{x} \in \mathbb{E}^N \colon \| c'_i \bar{x} \cdot \bar{x} \| < 1/10n\},$$

where $\|y\|$ denotes the distance from y to the nearest integer, and set

$$W := \bigcap_{i=1}^{n} W_i.$$

By standard results in diophantine approximation, $\bar{\delta} W > 0$. Note that W consists of spherical shells centered at the origin.

Consider now the expanded copy tX of X and suppose a congruent copy of it occurs in W. By the spherical symmetry of W, there must exist a point $\bar{a} \in \mathbb{E}^N$ such that the *translate* $tX + \bar{a}$ also is a subset of W. However,

$$\sum_{i=1}^{n} c'_i(t\bar{x}_i + \bar{a}) \cdot (t\bar{x}_i + \bar{a}) = t^2 \sum_{i=1}^{n} c_i \bar{x}_i \cdot \bar{x}_i + 2t\bar{a} \cdot \left(\sum_{i=1}^{n} c'_i \bar{x}_i \right) + \bar{a} \cdot \bar{a} \sum_{i=1}^{n} c'_i \qquad (1)$$

$$= t^2$$

by (i′)–(iii′). Since each $tx_i + \bar{a} \in W \subseteq W_i$, $1 \leqslant i \leqslant n$, then

$$\| c'_i(t\bar{x}_i + \bar{a}) \cdot (t\bar{x}_i + \bar{a}) \| < \frac{1}{10n},$$

i.e.,

$$\| c'_i(t\bar{x}_i + \bar{a}) \cdot (t\bar{x}_i + \bar{a}) = M_i + \varepsilon_i,$$

where M_i is an integer and $|\varepsilon_i| < 1/10n$. Then, by (1),

$$t^2 = \sum_{i=1}^{n} c'_i(t\bar{x}_i + \bar{a}) \cdot (t\bar{x}_i + \bar{a})$$

$$= \sum_{i=1}^{n} (M_i + \varepsilon_i) = M + \sum_{i=1}^{n} \varepsilon_i = M + \varepsilon, \qquad (2)$$

where M is an integer and $|\varepsilon| < 1/10$. This is clearly impossible if $\|t^2\| > 1/10$ (and certainly the lower density of such t is positive). This completes the proof of the theorem. □

5. Partition variants

The example of Bourgain (mentioned in the previous section) of a set W with $\bar{\delta}(W)>0$ and not containing congruent copies of $t_i X_3$ with $t_1<t_2<\cdots$ going to infinity, and X_3 consisting of 3 collinear points (with distances 1) can be strengthened by the following example.

Example. Define a partition of $\mathbb{E}^N$ into four sets C_i, $1\leqslant i\leqslant 4$, defined by

$$C_i := \{\bar{x}: \lfloor \bar{x}\cdot\bar{x}\rfloor \equiv i (\text{mod } 4)\}$$

where $\lfloor \cdot \rfloor$ denotes the floor (=greatest integer) function. Then no C_i contains a congruent copy of $(2t+1)X_3$ when t is an integer. To see this, suppose for some integer t, $(2t+1)X=\{x,y,z\}\subset C_i$ for some $i=0,1,2$, or 3 (see Fig. 1). By the law of cosines,

$$\bar{x}\cdot\bar{x}=\bar{y}\cdot\bar{y}+(2t+1)^2-2(2t+1)(\bar{y}\cdot\bar{y})^{1/2}\cos\theta,$$

$$\bar{z}\cdot\bar{z}=\bar{y}\cdot\bar{y}+(2t+1)^2+2(2t+1)(\bar{y}\cdot\bar{y})^{1/2}\cos\theta,$$

which implies

$$\bar{x}\cdot\bar{x}+\bar{z}\cdot\bar{z}-2\bar{y}\cdot\bar{y}=2(2t+1)^2. \tag{3}$$

Since $\{x,y,z\}\subset C_i$ then

$$\bar{x}\cdot\bar{x}=4M_x+i+\varepsilon_x,$$

$$\bar{y}\cdot\bar{y}=4M_y+i+\varepsilon_y,$$

$$\bar{z}\cdot\bar{z}=4M_z+i+\varepsilon_z.$$

Substituting these values into (3) yields

$$4(M_x-2M_y+M_z)+\varepsilon_z-2\varepsilon_y+\varepsilon_z=2(2t+1)^2,$$

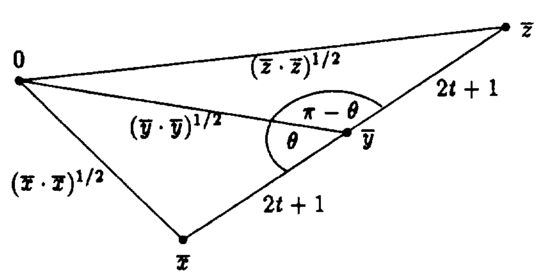

Fig. 1.

which implies

$$4M+\varepsilon_x-2\varepsilon_y+\varepsilon_z=2$$

for some integer M (since $(2t+1)^2\equiv 1 \pmod 8$). However, since $0\leqslant \varepsilon_x, \varepsilon_y, \varepsilon_z<1$, this is impossible.

One suspects that the same result should hold for any nonspherical set X but this is not currently known. The same argument can be applied if the corresponding c_i' expressing the linear dependence of the $\bar{x}_i \cdot \bar{x}_i$ in (ii′) are all rational.

6. The chromatic number of $\mathbb{E}^n$

An old question in Euclidean Ramsey theory asks for the minimum number $\chi(n)$ with the property that there is a partition of $\mathbb{E}^n = C_1 \cup \cdots \cup C_{\chi(n)}$ such that no C_i contains two points at mutual distance 1. This first seems to have been raised for the case of $\mathbb{E}^2$ by Nelson in 1950 (see [18] for an historical discussion) who pointed out (still) the best bounds available:

$$4\leqslant \chi(n)\leqslant 7. \tag{4}$$

The lower bound follows by considering the 7 points shown in Fig. 2, where edges between points indicate unit distance. The upper bound follows from an appropriate 7-coloring of a hexagonal tiling of the plane by regular hexagons of diameter $1-\varepsilon$. In spite of continued efforts, the bounds in (4) have not moved in 40 years.

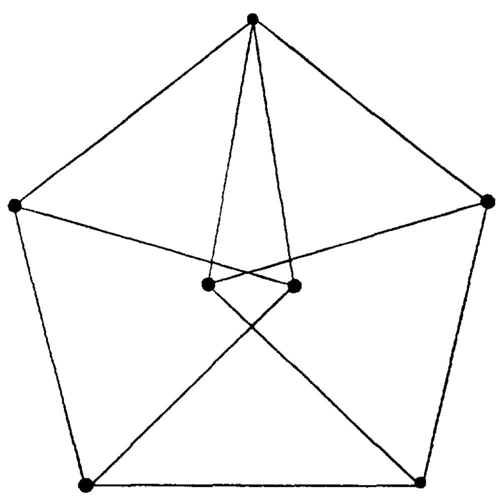

Fig. 2. The Moser graph.

For a general n, we have (see [6])

$$(1+o(1))(1.2)^n < \chi(n) < (3+o(1))^n.$$

The relatively recent lower bound, due to Frankl and Wilson, relies on one of their powerful set intersection theorems (see [6]).

7. Partition theorems in fixed dimension

Since even two points at unit distance can be prevented in partitions of $\mathbb{E}^2$ into 7 sets, one might ask what Euclidean Ramsey theorems could hold when the number of sets in the partition is arbitrary (but finite) and the space, e.g., $\mathbb{E}^2$, is fixed. Of course, when we allow a sufficiently large group in defining $\mathscr{F}$, such as the affine group for van der Waerden's theorem, then we have the classical results. However, there are other possibilities, as the following result shows.

Theorem (Graham [10]). *For any partition of* $\mathbb{E}^n$ *into finitely many classes, some class contains, for all* $\alpha > 0$ *and all sets of lines* $L_1, \ldots, L_n$ *which span* $\mathbb{E}^n$, *a simplex having volume* α *and edges through one vertex parallel to the* L_i.

This result follows from the following result which has a more discrete flavor.

Theorem (Graham [10]). *For any* r *there exists a positive integer* $T(r)$ *so that in any partition of the integer lattice points of* $\mathbb{E}^2$ *into* r *classes, some class contains the vertices of a right triangle with area* $T(r)$.

We remark that Kunen has shown [15] that under the continuum hypothesis, it is possible to partition $\mathbb{E}^2$ into $\aleph$ classes so that no class contains the vertices of any triangle with a *rational* area.

We close with one of our favorite problems in this topic, namely, the growth rate of the van der Waerden function $W(n)$, which is defined to be the least W such that in partition of $\{1, 2, \ldots, W\}$ into two classes, some class must always contain an n-term arithmetic progression. A recent breakthrough of Shelah [17] (finally) showed that $W(n)$ was upper bounded by a primitive recursive function, and in fact

$$W(n) \le \left.\begin{array}{c} 2^{2} \\ 2^{2^{2}} \\ \vdots \\ 2^{2^{\cdot^{\cdot^{2}}}} \\ 2^{2^{\cdot^{\cdot^{2}}}} \\ 2^{2^{2^{2^{\cdot^{\cdot^{2}}}}}} \end{array}\right\} n \text{ layers}$$

The best-known lower bound grows roughly like $n \cdot 2^n$.

Conjecture ($1000).

For all n,

$$W(n) \leq 2^{2^{2^{\cdot^{\cdot^{\cdot^{2^{2}}}}}}} \Big\} n$$

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