

COMMUNICATIONS OF THE MOSCOW MATHEMATICAL SOCIETY

On the chromatic number of a space

To cite this article: A M Raigorodskii 2000 *Russ. Math. Surv.* 55 351

View the [article online](#) for updates and enhancements.

You may also like

- [Estimating the chromatic numbers of Euclidean space by convex minimization methods](#)

Elena S Gorskaya, Irina M Mitricheva,
Vladimir Yu Protasov et al.

- [The problems of Borsuk and Grunbaum on lattice polytopes](#)

A M Raigorodskii

- [On the colouring of spheres embedded](#)

in \mathbb{R}^n

Andrei B Kupavskii

On the chromatic number of a space

A. M. Raigorodskii

1. Introduction and main result. We define the quantity $\chi(\mathbb{R}^n)$, the *chromatic number of the Euclidean space* \mathbb{R}^n , to be the minimal number of colours needed for a colouring of the points of \mathbb{R}^n in which Euclidean distance equal to one is never realized between points of the same colour. In other words, $\chi(\mathbb{R}^n)$ is the chromatic number of the following graph: the set of vertices of the graph coincides with all of \mathbb{R}^n , and the edges join exactly those vertices at distance one from each other. In fact, the problem of finding $\chi(\mathbb{R}^n)$ was first considered by Hadwiger [1], who showed that if there is an arbitrary cover of $\mathbb{R}^n = \bigcup_{i=1}^{n+1} \Omega_i$ by closed sets Ω_i , then there is an element Ω_i of this cover such that $\mathbb{R}^+ \cup \{0\} = \{|x - y| : x, y \in \Omega_i\}$. It is known (see [2]–[4]), that $4 \leq \chi(\mathbb{R}^2) \leq 7$ and that, moreover, $5 \leq \chi(\mathbb{R}^3) \leq 18$. Raiskii obtained in [5] the estimate $\chi(\mathbb{R}^n) \geq n + 2$. This estimate has been improved many times: thus, in [6] it was shown that $\chi(\mathbb{R}^n) \gg n^2$, then in [7] it was shown that $\chi(\mathbb{R}^n) \gg n^3$, and, finally, in [8] it was established that $\chi(\mathbb{R}^n)$ has exponential growth. In other words, it was proved in [8] that $\chi(\mathbb{R}^n) \geq (1.207 + o(1))^n$ (here one should note that according to [6] one has the inequality $\chi(\mathbb{R}^n) \leq (3 + o(1))^n$). In this paper we use the technique proposed in [9] to prove the following result.

Theorem. Let $A_1 = \frac{1}{2}(x + 2y)$, $A_2 = \sqrt{-3A_1^2 + 6A_1 + 1}$, $A_3 = \frac{1}{4}(1 + \frac{A_1 - 1}{A_2})$, and $A_4 = \frac{1}{8}(1 + 3A_1 - A_2)$ (here x and y are independent real variables). In addition, define x_0, y_0 to be the roots of the system of non-linear equations

$$2A_3 \log A_4 + (1 - 4A_3) \log(A_1 - 2A_4) + (2A_3 - 1) \log(1 - A_1 + A_4) - \log y + \log(x - y) = 0, \tag{1}$$

$$\frac{(1 - x)^2 y}{(x - y)^3} = 1 \tag{2}$$

such that $x_0 = 0.36063\dots$ and $y_0 = 0.063907\dots$. Finally, define γ by

$$\gamma^{-1} = (A_4^0)^{A_4^0} (A_1^0 - 2A_4^0)^{A_1^0 - 2A_4^0} (1 - A_1^0 + A_4^0)^{1 - A_1^0 + A_4^0} (1 - x_0)^{1 - x_0} (x_0 - y_0)^{x_0 - y_0} y_0^{y_0},$$

where $A_i^0 = A_i(x_0, y_0)$. Then $\chi(\mathbb{R}^n) \geq (\gamma + o(1))^n = (1.239\dots + o(1))^n$.

2. Proof of the theorem. First we note that if $\Sigma \subset \mathbb{R}^n$ is an (M, D) -critical configuration in \mathbb{R}^n with critical distance $d > 0$ (that is, $\text{card } \Sigma = M$ and in every set $Q \subset \Sigma$ such that $\text{card } Q = D + 1$, there is a pair of elements $x, y \in Q$, between which the distance is d), then $\chi(\mathbb{R}^n) \geq M/D$ (see [6]), and thus to complete the proof it suffices to construct some (M, D) -critical configuration Σ with $M/D \geq (\gamma + o(1))^n$.

Let $n \in \mathbb{N}$ be sufficiently large, $a = [x_0 n]$, $b = [y_0 n]$, and p the smallest odd prime number greater than $[(a + 2b)/2]$. It is known (see [10], German p. 323), that for sufficiently large x and for some $\alpha < 1$ (for example, for $\alpha = 38/61$) there is a prime number between x and $x + x^\alpha$. Therefore, we may assume that $p < [(a + 2b)/2] + [(a + 2b)/2]^\alpha$. We note that the choice of α affects only the estimate of $o(1)$ in the statement of the theorem. We set $\Sigma = \{\mathbf{x} = (x_1, \dots, x_n) : x_i \in \{0, 1, -1\}; \text{card}\{x_i = \pm 1\} = a; \text{card}\{x_i = -1\} = b\} \subset \mathbb{R}^n$. Clearly $M = \text{card } \Sigma = \binom{n}{a} \binom{a}{b}$. It is also clear that $\text{conv } \Sigma$ is a cross-polytope. At the same time the estimates for $\chi(\mathbb{R}^n)$ in [8] were based on

This work was supported financially by the Russian Foundation for Basic Research (grant nos. 99-01-00357, 00-15-96109).

The author thanks Professor László A. Székely for kindly giving him a copy of the unpublished survey paper ‘‘Erdős on unit distances and the Szemerédi–Trotter theorems’’.

AMS 2000 *Mathematics Subject Classification*. Primary 52C10; Secondary 05A16, 11B75.

the construction of critical configurations whose convex hull is a $(0, 1)$ -polyhedron. We shall show here that the methods given in [11] and [9] enable us to work with cross-polytopes and by using them to obtain better lower bounds for the chromatic number. In fact, as was already noted, $M = \binom{n}{a} \binom{a}{b}$. Moreover, it is not difficult to see that the following *arithmetic property* holds: for any two vectors $\mathbf{x}, \mathbf{y} \in \Sigma$, $(\mathbf{x}, \mathbf{y}) \equiv a \pmod{p}$ if and only if either $\mathbf{x} = \mathbf{y}$ or $(\mathbf{x}, \mathbf{y}) = a - p$.

With each vector $\mathbf{x} \in \Sigma$ we associate a polynomial $F_{\mathbf{x}} \in \mathbb{Z}/p\mathbb{Z}[x_1, \dots, x_n]$ by setting

$$F_{\mathbf{x}}(\mathbf{y}) = \prod_{i \not\equiv a \pmod{p}} (i - (\mathbf{x}, \mathbf{y})),$$

and then, successively applying the relations $x_i^3 = x_i$, $i = 1, \dots, n$, we pass from the polynomial $F_{\mathbf{x}}$ to the polynomial $\tilde{F}_{\mathbf{x}} \in \mathbb{Z}/p\mathbb{Z}[x_1, \dots, x_n]$. We now consider an arbitrary set $Q = \{\mathbf{x}_1, \dots, \mathbf{x}_s\} \subset \Sigma$ such that $(\mathbf{x}_i, \mathbf{x}_j) \not\equiv a \pmod{p} \quad \forall i \neq j$. In this case one can show that the polynomials $\tilde{F}_{\mathbf{x}_1}, \dots, \tilde{F}_{\mathbf{x}_s}$ are linearly independent over $\mathbb{Z}/p\mathbb{Z}$ (see [9]), and hence

$$s \leq \sum_{i=0}^{p-1} \sum_{j=0}^{\lfloor (p-1-i)/2 \rfloor} \binom{n}{j} \binom{n-j}{p-1-i-2j} = D. \quad (3)$$

It follows immediately from inequality (3) and the arithmetic property that the set Σ is an (M, D) -critical configuration with critical distance $d = \sqrt{2p}$, and therefore $\chi(\mathbb{R}^n) \geq M/D$. Finally, making some fairly simple computations, it is not difficult to see that, in turn, $M/D \geq (\gamma + o(1))^n$. This proves the theorem.

3. Remarks. 1. In the proof of the theorem it is not necessary to assume that p is a prime number. It suffices to assume that $p = q^\alpha$, where q is a prime and $\alpha \geq 1$. Here it is only necessary to make some changes in the definition of the polynomials $F_{\mathbf{x}}$ (see [12]).

2. Although the method proposed in this paper for obtaining lower bounds of the chromatic number of a space also gives best current asymptotic result, nevertheless in small dimensions (at least for $n \leq 24$) we have not succeeded in improving the estimates from [5], [6] and [8] by using it. It appears that further improvements using our method will require an essentially sharper inequality (3).

Bibliography

- [1] H. Hadwiger, *Portugal. Math.* **4** (1944), 140–144.
- [2] H. Hadwiger, *Elem. Math.* **16** (1961), 103–104.
- [3] L. Moser and W. Moser, *Canad. Math. Bull.* **4** (1961), 187–189.
- [4] D. Coulson, *Discrete Math.* **170** (1997), 241–247.
- [5] D. E. Raïskii, *Mat. Zametki* **7** (1970), 319–323; English transl., *Math. Notes* **7** (1970), 194–196.
- [6] D. G. Larman and C. A. Rogers, *Mathematika (London)* **19** (1972), 1–24.
- [7] D. G. Larman, *Comment. Math. Helv.* **53** (1978), 529–535.
- [8] P. Frankl and R. Wilson, *Combinatorica* **1** (1981), 357–368.
- [9] A. M. Raigorodskii, *Uspekhi Mat. Nauk* **54:2** (1999), 185–186; English transl., *Russian Math. Surveys* **54** (1999), 453–454.
- [10] K. Prachar, *Primzahlverteilung*, Springer-Verlag, Berlin-Heidelberg 1965; Russian transl., Mir, Moscow 1967.
- [11] A. Nilli, *Contemp. Math.* **178** (1994), 209–210.
- [12] A. M. Raigorodskii, *Uspekhi Mat. Nauk* **52:6** (1997), 181–182; English transl., *Russian Math. Surveys* **52** (1997), 1324–1325.