Journal of Combinatorial Theory, Series A **93**, 173–176 (2001)
doi:10.1006/jcta.2000.3065, available online at http://www.idealibrary.com on IDEAL®

NOTE

Monochromatic Translates of Configurations in the Plane

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Communicated by the Managing Editors

Received October 29, 1999

It is shown that every red–blue coloring of the plane, without two blue points distance 1 apart, must have a red translate of every three-point configuration. A seven-point configuration S and a red–blue coloring are exhibited, which avoids both distance one in blue and translates of S in red. © 2001 Academic Press

Key Words: Euclidean Ramsey; translates.

1. INTRODUCTION: RESULTS

Juhasz showed that for any red–blue coloring of the plane with no two blue points distance one apart, there is a red congruent copy of every 4 point configuration. In the same paper, she found a 12-point configuration and a red–blue coloring of the plane that forbids distance 1 in the blue set and congruent copies of that configuration in the red [3]. What happens if "congruent copy" is replaced by "translate"? Certainly it should be easier to find configurations and colorings that forbid translates of those configurations; however, here it is shown that under these conditions it still is impossible to find very small configurations and colorings.

An n-coloring of A is a partition of A into color classes $C_1, ..., C_n$. In this paper, all two colorings are red and blue. Call a red–blue coloring of a Euclidean space *admissible* if no two blue points are distance one apart, and call an n-coloring *proper* if each color class forbids the distance one. The chromatic number of a Euclidean space S, denoted $\chi(S)$, is the smallest n such that there exists a proper n-coloring of that space. An *n-point configuration* is a set of n points $\{a_1, ..., a_n\}$ in m-dimensional Euclidean space $\mathbb{R}^m$. A translate of the configuration A is $A + v$, for some vector v.

0097-3165/01 $35.00

Theorem 1. *Every admissible coloring of the plane has a red translate of every three point configuration. In fact, every admissible coloring of* $\mathbb{R}^m$ *has a red translate of every* n *point configuration, where* $n \leqslant (1+o(1))(1.2)^n$.

Theorem 2. *There exists a seven-point configuration and an admissible red-blue coloring of the plane so that the seven-point configuration is forbidden in the red set.*

2. PROOFS

Proposition 1. *If there exists an* n*-point configuration* A *and an admissible coloring of* $\mathbb{R}^m$ *forbidding red translates of* A, $\chi(\mathbb{R}^m) \leqslant n$.

Proof. Suppose such a coloring and such a configuration A exist. Label the vertices of A $a_1, ..., a_n$. Color each point p of $\mathbb{R}^m$ with the ith color if $p+a_i$ is blue. If $p+a_i$ is blue for more than one value of i, pick the smallest i. Now each point in the plane is colored, because there are no red translates of A. Suppose that two points b and c colored with the ith color were distance 1 apart. Then in the red–blue coloring of the plane given by assumption, $b+a_i$ is blue, and $c+a_i$ is again blue. But if b and c are distance 1 apart, $b+a_i$ and $c+a_i$ are distance one apart, and we have two blue points a distance 1 from each other, a contradiction. ∎

Now Theorem 1 follows from the known bounds for the chromatic number of $\mathbb{R}^m$. If there were an admissible coloring of the plane and a three-point configuration forbidden in the red set by the coloring, we would have a three coloring of the plane forbidding distance 1. But $\chi(\mathbb{R}^m) > 3$ by [2], so such a coloring does not exist. Similarly, if there were an n-point configuration and admissible coloring of $\mathbb{R}^m$ we would have a proper n-coloring, but by [1], $\chi(\mathbb{R}^m) > n$.

Also, we have a partial converse to Proposition 1.

If an n-coloring of $\mathbb{R}^m$ has color classes $C_1, ..., C_n$, and $C_i = C_1 + v_i$, for some fixed vectors $v_1, ..., v_n$, call the coloring *regular*.

Proposition 2. *If* $\mathbb{R}^m$ *can be properly* n*-colored by a regular coloring, then there exists an admissible two-coloring of* $\mathbb{R}^m$ *and an* n*-point configuration* A *so that translates of* A *are forbidden in the red set.*

Proof. Assume $v_1 = 0$. Let A be the set of points $\{v_1, ..., v_n\}$, and two-color the plane by letting C_1 be blue, and every other point be red. We wish to show that each point of any translate of A in the original coloring lies in a different color class, so that since there are n classes, some point of A will always be in C_1. Suppose not. Then for some point p, and some i and j, $1 \leqslant i < j$, $p + v_i$ and $p + v_j$ are both in the same color class, say C_a. By assumption, $C_a = C_1 + v_a$, so $(p + v_i - v_a)$ and $(p + v_j - v_a)$ are both in C_1. But $(p + v_i - v_a) + v_j = (p + v_j - v_a) + v_i$, a contradiction, because the left-hand side is colored C_j, but the right is colored C_i. ∎

Now Theorem 2 follows easily by applying Proposition 2 to the famous hexagonal coloring of Isbell [2].

A connection can be made with these problems and the problem considered by Juhasz.

THEOREM 3. *Either every admissible coloring of the plane has a red translate of every four-point configuration, or there exists an admissible coloring of the plane and a seven-point configuration so that congruent copies of the seven point configuration are forbidden in the red.*

Proof. If there existed an admissible red–blue coloring and a four-point configuration $A = \{a_1, a_2, a_3, a_4\}$ so that translates of the configuration were forbidden in the red, we would have a proper four-coloring of the plane. If we allow $a_1 = 0$, the blue from the two-coloring will be the color 1 of the four-coloring, and the other three color classes will be a partition of the red. Now consider the seven-point configuration shown in Fig. 1, a Moser spindle [2].

Clearly it takes more than three colors to properly color the spindle, so since the red set is partitioned into only three different color classes, it cannot contain a Moser spindle, so we are done. ∎

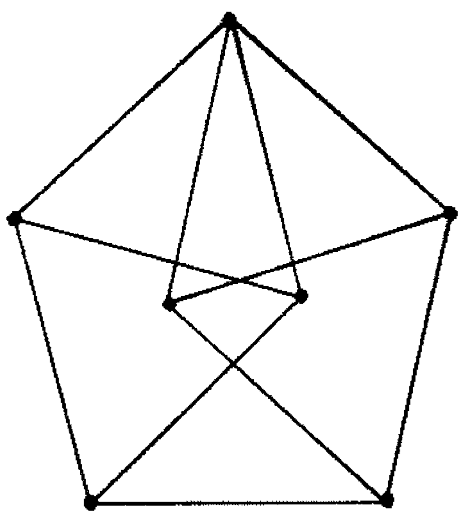

FIG. 1. A Moser spindle, with vertices distance one apart adjacent.

ACKNOWLEDGMENTS

Thanks to Peter Johnson and the Auburn University summer REU, 1999, and the NSF, for making this research possible, and to Vojtech Rödl and Dwight Duffus, for helpful suggestions.

REFERENCES

1. P. Frankl and R. M. Wilson, Intersection theorems with geometric consequences, *Combinatorica* **4** (1981), 357–368.
2. H. Hadwiger, H. DeBrunner, and V. Klee, "Combinatorial Geometry in the Plane," Holt, Rinehart & Winston, New York, 1964.
3. R. Juhasz, Ramsey type theorems in the plane, *J. Combin. Theory Ser. A* **27** (1979), 152–160.
4. I. Kříž, Permutation groups in Euclidean Ramsey theory, *Proc. Amer. Math. Soc.* **112** (1991), 899–907.
5. R. L. Graham, Recent trends in Euclidean Ramsey theory, *Discrete Math* **136** (1994), 119–127.
6. M. Jiří and V. Rödl, On Ramsey sets in spheres, *J. Combin. Theory Ser. A* **70** (1995), 30–35.