THE DETERMINATION OF 2-COLOR ZERO-SUM GENERALIZED SCHUR NUMBERS

Aaron Robertson
Department of Mathematics, Colgate University, Hamilton, New York
arobertson@colgate.edu

Bidisha Roy
Harish-Chandra Research Institute, HBNI, Jhunsi, Allahabad, India
bidisharoy@hri.res.in

Subha Sarkar
Harish-Chandra Research Institute, HBNI, Jhunsi, Allahabad, India
subhasarkar@hri.res.in

Received: , Revised: , Accepted: , Published:

Abstract

Consider the equation $\mathcal{E} : x_1 + \cdots + x_{k-1} = x_k$ and let k and r be positive integers such that $r \mid k$. The number $S_{\mathfrak{z},2}(k;r)$ is defined to be the least positive integer t such that for any 2-coloring $\chi : [1,t] \to \{0,1\}$ there exists a solution $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k)$ to equation $\mathcal{E}$ satisfying $\displaystyle\sum_{i=1}^{k} \chi(\hat{x}_i) \equiv 0 \pmod{r}$. In a recent paper, the first author posed the question of determining the exact value of $S_{\mathfrak{z},2}(k;4)$. In this article, we solve this problem and show, more generally, that $S_{\mathfrak{z},2}(k,r) = kr - 2r + 1$ for all positive integers k and r with $r \mid k$ and $k \geq 2r$.

1. Introduction

For $r \in \mathbb{Z}^+$, there exists a least positive integer $S(r)$, called a *Schur number*, such that within every r-coloring of $[1, S(r)]$ there is a monochromatic solution to the linear equation $x_1 + x_2 = x_3$.

In 1933, Rado [10] generalized the work of Schur to arbitrary systems of linear equations. For any integer $k \geq 2$ and $r \in \mathbb{Z}^+$, there exists a least positive integer $S(k;r)$, called a *generalized Schur number*, such that every r-coloring of $[1, S(k;r)]$ admits a monochromatic solution to the equation $\mathcal{E} : x_1 + \cdots + x_{k-1} = x_k$. Indeed, Rado's result proves, in particular, that the number $S(k,r)$ exists (is finite). In [2], Beutelspacher and Brestovansky proved the exact value $S(k;2) = k^2 - k - 1$.

Before we analogize the above number, we need the following definition.

Definition 1. Let $r \in \mathbb{Z}^+$. We say that a set of integers $\{a_1, a_2, \ldots, a_n\}$ is *r-zero-sum* if $\sum_{i=1}^{n} a_i \equiv 0 \pmod{r}$.

The Erdős-Ginzburg-Ziv Theorem [5] is one of the cornerstones of zero-sum theory (see, for instance, [1] and [9]). It states that any sequence of $2n-1$ integers must contain an n-zero-sum subsequence of n integers. In recent times, zero-sum theory has made remarkable progress (see, for instance, [3], [4], [6], [7], [8]).

In [11], the first author replaced the "monochromatic property" of the generalized Schur number by the "zero-sum property" and introduced the following new number which is called a *zero-sum generalized Schur number.*

Notation. Throughout the article, we represent the equation $x_1 + \cdots + x_{k-1} = x_k$ by $\mathcal{E}$.

Definition 2. Let k and r be positive integers such that $r \mid k$. We define $S_{\mathfrak{z}}(k;r)$ to be the least positive integer t such that for any r-coloring $\chi : [1,t] \to \{0,\ldots,r-1\}$ there exists a solution $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k)$ to equation $\mathcal{E}$ satisfying $\displaystyle\sum_{i=1}^{k} \chi(\hat{x}_i) \equiv 0 \pmod{r}$.

We only make the above defintion (and Defintion 3, below) for $r \mid k$ since the coloring of $\mathbb{Z}^+$ by coloring every integer with color 1 shows that we cannot guarantee an r-zero-sum solution if $r \nmid k$.

Since $r \mid k$, note that if $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k)$ is a monochromatic solution to equation $\mathcal{E}$, then clearly it is an r-zero-sum solution. Hence, we get, $S_{\mathfrak{z}}(k;r) \leq S(k;r)$ and therefore, $S_{\mathfrak{z}}(k;r)$ is finite.

In [11], the first author calculated lower bounds of this number for some r. In particular, he proved the following result.

Theorem 1. *[11] Let k and r be positive integers such that $r \mid k$. Then,*

$$S_{\mathfrak{z}}(k;r) \geq \begin{cases} 3k-3 & \text{when } r = 3; \\ 4k-5 & \text{when } r = 4; \\ 2(k^2-k-1) & \text{when } r = k \text{ is odd.} \end{cases}$$

In the same article, he introduced another number meant only for 2-colorings, but keeping the r-zero-sum notion.

Definition 3. Let k and r be positive integers such that $r \mid k$. We denote by $S_{\mathfrak{z},2}(k;r)$ the least positive integer such that every 2-coloring of $\chi : [1, S_{\mathfrak{z},2}(k;r)] \to \{0,1\}$ admits a solution $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k)$ to equation $\mathcal{E}$ satisfying $\displaystyle\sum_{i=1}^{k} \chi(\hat{x}_i) \equiv 0 \pmod{r}$.

Since any 2-coloring of $[1, S_{\mathfrak{z}}(k;r)]$ is also an r-coloring (for $r \geq 2$), we see that $S_{\mathfrak{z},2}(k;r) \leq S_{\mathfrak{z}}(k;r)$ and hence $S_{\mathfrak{z},2}(k;r)$ is finite. Furthermore, in the case when $k = r$ we recover the generalized Schur number $S(k;2)$.

In [11], the first author proved the following theorem related to these 2-color zero-sum generalized Schur numbers.

Theorem 2. *[11] Let k and r be two positive integers such that $r \mid k$. Then,*

$$S_{\mathfrak{z},2}(k;r) = \begin{cases} 2k-3; & \text{if } r=2 \\ 3k-5; & \text{if } r=3 \text{ and } k\neq 3 \\ k^2-k-1; & \text{if } r=k \end{cases}$$

One notes that the exact values of $S_{\mathfrak{z},2}(k;r)$ for $r = 2, 3$ and $S_{\mathfrak{z},2}(r,r)$ do not show any obvious generalization to $S_{\mathfrak{z},2}(k;r)$ for any k which is a multiple of r. However, the computations given in [11] when $r = 4$ and $k = 4, 8, 12$, and when $r = 5$ and $k = 5, 10, 15$, were enough for us to conjecture a general formula, which turns out to hold. To this end, by Theorem 3 below, we answer a question posed by the first author in [11] and, more generally, determine the exact values of $S_{\mathfrak{z},2}(k;r)$.

Theorem 3. *Let k and r be positive integers such that $r \mid k$ and $k \geq 2r$. Then, $S_{\mathfrak{z},2}(k;r) = rk - 2r + 1$.*

2. Preliminaries

We start by presenting a pair of lemmas useful for proving our upper bounds.

Lemma 1. *Let k and r be positive integers such that $r \mid k$ and $k \geq 2r$. Let $\chi : [1, rk - 2r + 1] \to \{0, 1\}$ be a 2-coloring such that $\chi(1) = \chi(r - 1) = 0$. Then there exists an r-zero-sum solution to equation $\mathcal{E}$ under χ.*

Proof. Consider the solution $(1, 1, \ldots, 1, k-1)$. If $\chi(k-1) = 0$, then, since $\chi(1) = 0$, we are done. Hence, we shall assume that $\chi(k - 1) = 1$.

Next, we look at the solution

$$(\underbrace{1, \ldots, 1}_{k-r}, \underbrace{k-1, k-1, k-1}_{r-1}, rk - 2r + 1).$$

Since $\chi(1) = 0$ and $\chi(k-1) = 1$, we can assume that $\chi(rk-2r+1) = 0$; otherwise, we have exactly r integers of color 1 and so the solution is r-zero-sum.

Since $(1, r, r, \ldots, r, rk - 2r + 1)$ is a solution to $\mathcal{E}$, we can assume that $\chi(r) = 1$. Finally, consider

$$(\underbrace{r-1, \ldots, r-1}_{r-1}, \underbrace{r, \ldots, r}_{k-r}, rk - 2r + 1).$$

Since
$\chi(r - 1) = 0, \chi(r) = 1$, $\chi(rk - 2r + 1) = 0$, and $r \mid k$, this solution is r-zero-sum, thereby proving the lemma. □

Lemma 2. *Let k and r be positive integers such that $r \mid k$ and $k \geq 2r$. Let $\chi : [1, rk-2r+1] \to \{0,1\}$ be a coloring such that $\chi(1) = 0$ and $\chi(r-1) = 1$. If one of the following holds, then there exists an r-zero-sum solution to equation $\mathcal{E}$:*

(a) $\chi(r) = 0$;

(b) $\chi(k-2) = 1$;

(c) $\chi(k-1) = 0$;

(d) $\chi(k) = 0$

(e) $\chi(rk-2r-1) = 0$;

(f) $\chi(rk-2r+1) = 1$.

Proof. We will prove each possibility separately; however, the order in which we do so matters so we will not be proving them in the order listed.

(c) Consider $(1, 1, \ldots, 1, k-1)$. If $\chi(k-1) = 0$ then this solution is r-zero sum.

(d) By considering the solution $(r-1, \ldots, r-1, (r-1)(k-1))$, we can assume that $\chi((r-1)(k-1)) = 0$. Using this in

$$(\underbrace{1, \ldots, 1}_{k-r+1}, \underbrace{k, \ldots, k}_{r-2}, (r-1)(k-1))$$

along with the assumption that $\chi(k) = 0$, we have an r-zero-sum solution.

(f) From part (c), we may assume that $\chi(k-1) = 1$. Looking at $(r-1, \ldots, r-1, k-1, rk-2r+1)$, since $\chi(k-1) = \chi(r-1) = 1$, and we assume that $\chi(rk-2r+1) = 1$, we have an r-zero sum solution.

(a) From part (f), we may assume that $\chi(rk-2r+1) = 0$. With this assumption, we see that $(1, r, \ldots, r, rk-2r+1)$ is r-zero-sum when $\chi(r) = 0$.

(b) From parts (a) and (f), we may assume $\chi(r) = 1$ and $\chi(rk-2r+1) = 0$. Under these assumptions, we find that

$$(\underbrace{1, \ldots, 1}_{k-r-1}, r, \underbrace{k-2, \ldots, k-2}_{r-1}, rk-2r+1)$$

is an r-zero-sum solution with $\chi(k-2) = 1$.

(e) By considering

$$(\underbrace{1, \ldots, 1}_{r-1}, \underbrace{r, \ldots, r}_{k-2r}, \underbrace{2r-3, \ldots, 2r-3}_{r}, rk-2r-1)$$

and using $r \mid k$, we have an r-zero-sum solution when $\chi(rk-2r-1) = 0$. □

3. Proof of the Main Result

In this section we prove that $S_{\mathfrak{z},2}(k;r) = rk - 2r + 1$.

Proof. We start with the lower bound. To prove that $S_{\mathfrak{z},2}(k;r) > rk-2r$, we consider the 2-coloring χ of $[1, rk-2r]$ defined by $\chi(i) = 0$ for $1 \le i \le k-2$ and $\chi(i) = 1$ for $k-1 \le i \le rk-2r$. Assume, for a contradiction, that χ admits an r-zero-sum solution $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k)$ to equation $\mathcal{E}$. Then $\chi(\hat{x}_i) = 1$ for some $i \in \{1, 2, \ldots, k\}$; otherwise the solution is monochromatic of color 0, but $\sum_{i=1}^{k-1} \hat{x}_i \ge k-1$, meaning that $\hat{x}_k$ cannot be of color 0.

Assuming that $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k)$ is r-zero-sum and not monochromatic of color 0, we must have $\chi(x_j) = 1$ for at least r of the x_j's. Since the minimum integer under χ that is of color 1 is $k-1$, this gives us

$$\sum_{i=1}^{k-1} x_i \ge (r-1)(k-1) + 1(k-r) = rk - 2r + 1 > rk - 2r,$$

which is out of bounds, a contradiction. Hence, χ does not admit an r-zero-sum solution to $\mathcal{E}$ and we conclude that $S_{\mathfrak{z},2}(k;r) \ge rk - 2r + 1$.

We now move on to the upper bound. We let $\chi : [1, rk-2r+1] \to \{0,1\}$ be an arbitrary 2-coloring. We may assume that $\chi(1) = 0$ since χ admits an r-zero-sum solution if and only if the induced coloring $\overline{\chi}$ defined by $\overline{\chi}(i) = 1 - \chi(i)$ also does so.

The cases $r = 2, 3$ have been done by Theorem 2. Hence, we may assume that $r \ge 4$. We must handle the case $r = 4$ separately; we start with this case.

We will show that $4k - 7$ serves as an upper bound for $S_{\mathfrak{z}}(4; r)$. Consider the following solution to $\mathcal{E}$:

$$(1, 1, 1, \underbrace{2, \ldots, 2}_{k-8}, 3, 3, k, k, 4k-7).$$

Noting that $r - 1 = 3$ and $rk - 2r + 1 = 4k - 7$, by Lemmas 1 and 2, we may assume $\chi(3) = 1$, $\chi(k) = 1$, and $\chi(4k-7) = 0$. Since k is a multiple of 4 and $k \ge 8$, we see that $k - 8$ is also a multiple of 4. Hence, the color of 2 does not affect whether or not this solution is 4-zero-sum. Of the integers not equal to 2, we have exactly four of them of color 1. Hence, this solution is 4-zero-sum. This, along with the lower bound above, proves that $S_{\mathfrak{z},2}(k;4) = 4k - 7$.

We now move on to the cases where $r \ge 5$. We proceed by assuming that no r-zero-sum solution occurs under an arbitrary 2-coloring $\chi : [1, rk-2r+1] \to \{0,1\}$. From Lemmas 1 and 2, we may assume the following table of colors holds.

color 0	color 1
1	$r-1$
$k-2$	r
$rk-2r+1$	$k-1$
	k
	$rk-2r-1$.

In order for the solution

$$(\underbrace{1,\dots,1}_{k-r},\underbrace{k-2,\dots,k-2}_{r-2},k+r-3,rk-2r+1)$$

not to be r-zero-sum, we deduce that $\chi(k+r-3)=1$. Using this in the solution

$$(\underbrace{1,\dots,1}_{k-r},\underbrace{k,\dots,k}_{r-2},k+r-3,rk-3)$$

we may assume that $\chi(rk-3)=0$. In turn, we use this in

$$(\underbrace{2,\dots,2}_{k-r-1},r,r-1,\underbrace{k,\dots,k}_{r-2},rk-3)$$

to deduce that $\chi(2)=1$. Modifying this last solution slightly, we consider

$$(\underbrace{3,\dots,3}_{k-r-1},r,r,r,\underbrace{k,\dots,k}_{r-3},rk-3)$$

to deduce that $\chi(3)=1$. Finally, since $r\geq 5$, we can consider

$$(\underbrace{2,\dots,2}_{k-2r+6},\underbrace{3,\dots,3}_{r-5},\underbrace{k-1,\dots,k-1}_{r-2},rk-2r-1).$$

We see that this solution is monochromatic (of color 1), and, hence, is r-zero-sum. This proves that $S_{\mathfrak{z},2}(k;r)\leq rk-2r+1$ for $r\geq 5$, which, together with the lower bound at the beginning of the proof, gives us $S_{\mathfrak{z},2}(k;r)=rk-2r+1$, thereby completing the proof. □

Acknowledgement. We would like to sincerely thank Prof. R Thangadurai for his insightful remarks. This work was done while the second and third authors were visiting the Department of Mathematics, Ramakrishna Mission Vivekananda Educational and Research Institute and they wish to thank this institute for the excellent environment and for their hospitality.

References

[1] S. D. Adhikari, *Aspects of Combinatorics and Combinatorial Number Theory*, Narosa, New Delhi, 2002.

[2] A. Beutelspacher and W. Brestovansky, *Generalized Schur Numbers*, Lecture Notes in Mathematics **969** (1982), 30-38.

[3] A. Bialostocki and P. Dierker, *On the Erdős-Ginzburg-Ziv theorem and the Ramsey numbers for stars and matchings*, Discrete Math. **110** (1992), 1-8.

[4] Y. Caro, *Zero-sum problems, a survey*, Discrete Math. **152** (1996), 93-113.

[5] P. Erdős, A. Ginzburg, and A. Ziv, *Theorem in additive number theory*, Bulletin Research Council Israel **10F** (1961), 41-43.

[6] W. Gao and A. Geroldinger, *Zero-sum problems in finite abelian groups: a survey*, Expo. Math **24** (2006), 337-369.

[7] D. Grynkiewicz, *Structural Additive Theory*, Springer, 2013.

[8] D. Grynkiewicz, *A weighted Erdős-Ginzburg-Ziv theorem*, Combinatorica **26** (2006), 445-453.

[9] Melvyn B. Nathanson, *Additive Number Theory: Inverse Problems and the Geometry of Sumsets*, Springer, 1996.

[10] R. Rado, *Studien zur Kombinatorik (German)*, Math. Z. **36** (1933), no. 1, 424-470.

[11] A. Robertson, *Zero-sum generalized Schur numbers.*, arXiv:1802.03382v1.