On the two-colour disjunctive Rado number for the equations $\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c_j$, $j = 1, 2$

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Abstract

Given a system of linear equations $\mathscr{S}$, the disjunctive Rado number for the system $\mathscr{S}$ is the least positive integer $R = \mathscr{R}_d(\mathscr{S})$, if it exists, such that every 2-colouring of the integers in $[1, R]$ admits a monochromatic solution to at least one equation in $\mathscr{S}$. We determine $\mathscr{R}_d(\mathscr{S})$ when $\mathscr{S}$ is the pair of equations $\{ \sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c_1, \sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c_2 \}$ for some range of values of c_1 and c_2.

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1 Introduction

By an r-colouring of $\{1, \ldots, N\}$ we mean a mapping $\chi : \{1, \ldots, N\} \to \{1, \ldots, r\}$. In 1916, Schur showed that for every positive integer r, there exist a least positive integer $s = s(r)$ such that for every r-coloring of the integers in the interval $[1, s]$, there exists $x, y, x + y \in [1, s]$ such that $\chi(x) = \chi(y) = \chi(x + y)$. Schur's Theorem was generalized in a series of results in the 1930's by Rado leading to a complete resolution to the following problem: characterize systems of linear homogeneous equations with integral coefficients $\mathscr{S}$ such that for a given positive integer r, there exists a least positive integer $n = \mathscr{R}(\mathscr{S}; r)$ such that every r-coloring of the integers in the interval $[1, n]$ yields a monochromatic solution to the system $\mathscr{S}$. There has been a growing interest in the determination of the Rado numbers $\mathscr{R}(\mathscr{S}; r)$, particularly when $\mathscr{S}$ is a single equation and $r = 2$; for instance, see [1, 4, 5, 6, 7, 8, 10]. When $r = 2$, we denote this number simply by $\mathscr{R}(\mathscr{S})$.

The problem of disjunctive Rado numbers was introduced by Johnson & Schaal in [9]. The 2-colour disjunctive Rado number for the set of equations $\mathcal{E}_1, \ldots, \mathcal{E}_k$ is the least positive integer N such that any 2-colouring of $\{1, \ldots, N\}$ admits a monochromatic solution to at least one of the equations

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$\mathcal{E}_1, \ldots, \mathcal{E}_k$; we denote this by $\mathscr{R}_{\mathrm{d}}(\mathcal{E}_1, \ldots, \mathcal{E}_k)$. Johnson & Schaal gave necessary and sufficient conditions for the existence of the 2-colour disjunctive Rado number for the additive equations $x_1 - x_2 = a$ and $x_1 - x_2 = b$ for all pairs of distinct positive integers a, b, and also determined exact values when it exists. They also determined exact values for the pair of multiplicative equations $ax_1 = x_2$ and $bx_1 = x_2$ whenever a, b are distinct positive integers; for alternate proofs, see [13]. Dileep, Moondra & Tripathi [14] extended the results of Johnson & Schaal to the set of equations $x_1 - x_2 = a_i$, $1 \leq i \leq k$, giving conditions for the existence of the 2-colour disjunctive Rado numbers, exact values in some cases, and upper and lower bounds in all cases. They also investigated and obtained parallel results for the set of multiplicative equations $y = a_i x$, $1 \leq i \leq k$. Further, they gave a general search-based algorithm with a run time of $O(ka_k \log a_k)$ for the case of additive equations, which is exponentially better than the brute-force algorithm for the problem. Lane-Harvard & Schaal [12] determined exact values of 2-colour disjunctive Rado number for the pair of equations $ax_1 + x_2 = x_3$ and $bx_1 + x_2 = x_3$ for all distinct positive integers a, b. Sabo, Schaal & Tokaz [15] determined exact values of 2-colour disjunctive Rado number for $x_1 + x_2 - x_3 = c_1$ and $x_1 + x_2 - x_3 = c_2$ whenever c_1, c_2 are distinct positive integers. Kosek & Schaal [11] determined the exact value of 2-colour disjunctive Rado number for the equations $x_1 + \cdots + x_{m-1} = x_m$ and $x_1 + \cdots + x_{n-1} = x_n$ for all pairs of distinct positive integers m, n.

Schaal & Zinter [16] studied the 2-colour Rado number for the equation $x_1 + 3x_2 + c = x_3$ for $c \geq 3$, giving a lower bound in all cases and upper bounds in some. Dwivedi & Tripathi [2] generalized this to investigate the 2-colour Rado number for the equation $x_1 + ax_2 - x_3 = c$ for positive integers a, giving conditions for existence, upper and lower bounds in all cases and exact results in a few. The same authors [3] further generalized this to investigate the 2-colour Rado number for the equation $\sum_{i=1}^{m-2} x_i + ax_{m-1} + x_m = c$ when $4 \leq m \leq a$. They give a necessary and sufficient condition for the Rado number to exist, give upper and lower bounds in all cases, and exact values in many cases. This paper investigates the disjunctive Rado problem for the pair of equations $\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c_1$ and $\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c_2$. We reproduce some pertinent results from [3] for ready reference.

Theorem 1. **([3, Theorem 1])** *Let $a, c, m \in \mathbb{Z}$ and $4 \leq m \leq a$. If $a+m$ and c are both odd, then $\mathscr{R}(c)$ does not exist.*

Proposition 2. **([3, Proposition 1])** *For $a \in \mathbb{N}$ and $4 \leq m \leq a$, $\mathscr{R}(a+m-3) = 1$.*

Theorem 3. **([2, Theorem 3], [3, Theorem 5])**
Let a, m be integers of the same parity, with $a \geq 3$ and $m \geq 3$. Let $a' = a + m - 3$. If either of

(i) $m = 3$ *and* $c \leq -\frac{a(a-3)}{2}$;

(ii) $m \geq 4$ *and* $c < -(a'+3)(a-2)$,

then

$$\mathscr{R}(c) = (a'+3)(a'-c) + 1.$$

2 Results for $\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c_i$, $i = 1, 2$

We study the disjunctive Rado numbers for the pair of equations

$$\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c_1, \tag{1a}$$

$$\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c_2, \tag{1b}$$

where $a \geq 3$, $m \geq 3$, and c_1, c_2 are any integers. Throughout this paper, we denote these 2-colour Rado numbers by $\mathtt{Rad}_2\big(\underbrace{1, \ldots, 1}_{m-2 \text{ times}}, a, -1; c_1, c_2\big)$, or more briefly by $\mathscr{R}(c_1, c_2)$.

By assigning the colour of x_i in the solution of eqn. (1a) and eqn. (1b) to $x_i - 1$, we note that this is equivalent to determining the smallest positive integer R for which every 2-colouring of $[0, R-1]$ contains a monochromatic solution to

$$\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c_1', \tag{2a}$$

$$\text{or} \quad \sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c_2', \tag{2b}$$

where $c_j' = c_j - a'$, $j \in \{1, 2\}$, and $a' = a + m - 3$.

Proposition 4. *Let $\langle 1, \ldots, 1, a\rangle$ be a list of positive integers, where there are n occurrences of 1 and where $a \geq 3$. Let $\lambda \in \mathbb{N}$, $\lambda \geq a - 1$. Then for each $N \in \{0, \ldots, \lambda(a+n)\}$ the equation*

$$\sum_{k=1}^{n} x_k + ax_{n+1} = N \tag{3}$$

admits a solution with each $x_i \in \{0, \ldots, \lambda\}$.

Proof. If $N = \lambda(a+n)$, then $x_i = \lambda$ for $i \in \{1, \ldots, n+1\}$ is a solution to eqn. (3).

If $0 \leq N < \lambda(a+n)$, we can write $N = q(a+n) + \epsilon a + r$, where $0 \leq q < \lambda$, $0 \leq r \leq n$ and $\epsilon \in \{0, 1\}$. Then $x_i = q + 1$ for $1 \leq i \leq r$, $x_i = q$ for $r + 1 \leq i \leq n$, and $x_{n+1} = q + \epsilon$ is a solution to eqn. (3). ■

Theorem 5. *Let $4 \leq m \leq a$ and $c_i = k_i(a + m - 3)$, $1 < k_i \leq a + m - 2$, $i \in \{1, 2\}$. Then*

$$\mathscr{R}(c) = \min\{k_1, k_2\}.$$

Proof. Let $k = \min\{k_1, k_2\}$. The colouring $\Delta : [1, k-1] \to \{0, 1\}$ be defined by $\Delta(x) = 0$ is a valid colouring, since $\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m \leq (a+m-2)(k-1) - 1 = k(a+m-3) + (k-2) - (a+m-3) < k(a+m-3)$. Hence $\mathscr{R}(c) \geq k$.

On the other hand, since $x_1 = \cdots = x_m = k$ satisfies eqns. (1a), (1b) for $c = k(a + m - 3)$, every colouring $\chi : [1, k] \to \{0, 1\}$ admits a monochromatic soloution to eqns. (1a), (1b). Hence $\mathscr{R}(c) \leq k$. ■

Theorem 6. *Let a, m be integers of the same parity, with $a \geq 3$ and $m \geq 4$. Let $a' = a + m - 3$ and $c'_j = c_j - a'$, $j \in \{1, 2\}$. Then for $c_1 < -(a' + 3)(a - 2)$,*

$$\mathscr{R}(c_1, c_2) = \begin{cases} (a'+3)(a'-c_1)+1 & \text{if } c_1 - a' \leq c_2 \leq c_1; \\ (a'+2)(a'-c_1)+1 & \text{if } (a'+2)c_1 - a'(a'+1) \leq c_2 < c_1 - a'; \\ (a'-c_2)+1 & \text{if } (a'+3)c_1 - a'(a'+2) < c_2 < (a'+2)c_1 - a'(a'+1) \\ (a'+3)(a'-c_1)+1 & \text{if } c_2 \leq (a'+3)c_1 - a'(a'+2). \end{cases}$$

Proof. We note that $a' = a + m - 3$, and that

$$\mathscr{R}(c_1, c_2) \leq \min\{\mathscr{R}(c_1), \mathscr{R}(c_2)\} = (a'+3)(a'-c_1)+1 = -(a'+3)c'_1 + 1$$

by Theorem 3.

Lower Bound

To prove the result in the first and last case, it suffices to exhibit a valid colouring of $[1, (a+m)(a+m-c_1-3)]$ with respect to eqn. (1a), (1b).

Let $\Delta : [1, (a+m)(a+m-c_1-3)] \to \{0, 1\}$ be defined by

$$\Delta(x) = \begin{cases} 0 & \text{if } x \in [1, a+m-c_1-3] \cup [(a+m-1)(a+m-c_1-3)+1, (a+m)(a+m-c_1-3)]; \\ 1 & \text{if } x \in [a+m-c_1-2, (a+m-1)(a+m-c_1-3)]. \end{cases}$$

Let $A = [1, a+m-c_1-3]$, $B = [a+m-c_1-2, (a+m-1)(a+m-c_1-3)]$, and $C = [(a+m-1)(a+m-c_1-3)+1, (a+m-1)(a+m-c_1-3)]$.

Suppose $x_1, \dots, x_m$ is a solution to eqn. (1a), with $\Delta(x_1) = \cdots = \Delta(x_m)$.

Suppose $\Delta(x_i) = 0$ for $i \in \{1, \dots, m\}$. If $x_1, \dots, x_{m-1}$ all belong to A, then

$$a+m-c_1-2 \leq x_m = \sum_{i=1}^{m-2} x_i + a x_{m-1} - c_1 \leq (a+m-2)(a+m-c_1-3) - c_1 \leq (a+m-1)(a+m-c_1-3).$$

Hence $x_m \in B$, and so $\chi(x_m) = 1$.

If at least one of $x_1, \dots, x_{m-1}$ belongs to C, then

$$x_m = \sum_{i=1}^{m-2} x_i + a x_{m-1} - c_1 \geq (a+m-c_1-3) + \min C = (a+m)(a+m-c_1-3) + 1.$$

Hence x_m is outside the domain of Δ. Therefore $\Delta(x_i) = 1$ for $i \in \{1, \dots, m\}$, and so

$$x_m = \sum_{i=1}^{m-2} x_i + a x_{m-1} - c_1 \geq (a+m-2) \cdot \min B - c_1 \geq (a+m-1)(a+m-c_1-3) + 1.$$

Hence $x_m \in$ C.

This proves that Δ is a valid colouring of $[1, (a+m)(a+m-c_1-3)]$ with respect to eqn. (1a).

In the first case, the same argument applies with respect to eqn. (1b). In the fourth case,

$$x_m = \sum_{i=1}^{m-2} x_i + a x_{m-1} - c_2 > (a+m)(a+m-c_1-3).$$

Hence x_m is outside the domain of Δ. Thus Δ is a valid colouring of $[1,(a+m)(a+m-c_1-3)]$ with respect to eqn. (1b). This concludes the proof of the first and fourth cases.

The colouring Δ, with suitable modifications, also provide a valid colouring in the second and third cases. In the second case, we consider the function Δ, restricted to $[1,(a+m-1)(a+m-c_1-3)] = A \cup B$. A sub-argument used in the first case shows that this is a valid colouring for eqn. (1a), (1b). In the third case, we consider the function Δ, restricted to $[1, a+m-c_2-3] = A \cup B \cup C'$, where $C' = [(a+m-1)(a+m-c_1-3)+1, a+m-c_2-3]$. An argument similar to the one in the first case shows that this is a valid colouring for eqn. (1a), (1b).

In view of Theorem 3, the proof of the first and fourth cases are complete. Since we have provided valid colourings in the second and third cases, it only remains to prove the upper bounds in these two cases.

Upper Bound

By assigning the colour of x_i in the solution of eqn. (1a), (1b) to $x_i - 1$, we equivalently consider monochromatic solutions to eqn. (2a), (2b) under colourings that start with $x = 0$.

Let $\chi : [0, -(a'+3)c_1'] \to \{0,1\}$ be any 2-colouring of $[0, -(a'+3)c_1']$. Without loss of generality, let $\chi(0) = 0$.

Each step in the following sequence forces a colour on some number in the given range in order to avoid a mononchromatic solution to eqn. (2a), (2b).

- $x_i = 0$, $1 \le i \le m-1 \Rightarrow \chi(-c_1') = 1$ and $\chi(-c_2') = 1$.
- $x_i = -c_1'$, $1 \le i \le m-1 \Rightarrow \chi\big(-(a'+2)c_1'\big) = 0$.
- $x_i = 0$, $2 \le i \le m-1$, $x_m = -(a'+2)c_1' \Rightarrow \chi\big(-(a'+1)c_1'\big) = 1$.

We capture this information as Table 2.

0	**1**
0	$-c_j'$
$-(a'+2)c_j'$	$-(a'+1)c_j'$

Table 2. Some initial colourings with $j = 1, 2$

There remain the second and third cases of the main theorem. We must show:

- that for every 2-colouring of $\chi : [0, -(a'+2)c_1'] \to \{0,1\}$ must yield a monochromatic solution to one of eqn. (2a), (2b) for $-(c_1' - a') < -c_2' \le -(a'+2)c_1'$, and

- that for every 2-colouring of $\chi : [0, -c_2'] \to \{0,1\}$ must yield a monochromatic solution to one of eqn. (2a), (2b) for $-(a'+2)c_1' < -c_2' < -(a'+3)c_1'$.

We have assumed, without loss of generality, that $\chi(0) = 0$. There are two possibilities for $\chi(1)$, of which the case $\chi(1) = 1$ is common to both the second and third case. Before we consider the two cases separately, we assume $\chi(1) = 1$. The proof of the respective upper bounds is common in the two cases when $\chi(1) = 1$, but not when $\chi(1) = 0$.

We claim that

$$\chi\big(-tc_1' - a'\big) = \begin{cases} 0 & \text{if } t \text{ is odd;} \\ 1 & \text{if } t \text{ is even} \end{cases}$$

for $t \in \{1, \dots, a'\}$.

With each $x_i = 1$, $2 \le i \le m-1$ and $x_m = -(a'+1)c_1'$ in eqn. (2a), we have $x_1 = -a'(c_1'+1)$, forcing $\chi\big(-a'c_1' - a'\big) = 0$ in order to avoid a monochromatic colouring. This proves the claim for $t = a'$.

Suppose $t \in \{3, \dots, a'\}$, t is odd, and that $\chi\big(-tc_1' - a'\big) = 0$. We begin the inductive step at $t = a'$. To complete the claim, we show that if $\chi\big(-tc_1' - a'\big) = 0$, then $\chi\big(-(t-1)c_1' - a'\big) = 1$ and $\chi\big(-(t-2)c_1' - a'\big) = 0$ for $t \in \{3, \dots, a'\}$.

Each step in the following sequence forces a colour on some number in the given range in order to avoid a mononchromatic solution to eqn. (2a).

- $x_i = 0$, $2 \le i \le m-1$, $x_m = -tc_1' - a' \Rightarrow \chi\big(-(t-1)c_1' - a'\big) = 1$.
- $x_1 = -(t-1)c_1' - a'$, $x_i = 1$, $2 \le i \le m-1 \Rightarrow \chi\big(-tc_1'\big) = 0$.
- $x_i = 0$, $2 \le i \le m-1$, $x_m = -tc_1' \Rightarrow \chi\big(-(t-1)c_1'\big) = 1$.
- $x_i = 1$, $2 \le i \le m-1$, $x_m = -(t-1)c_1' \Rightarrow \chi\big(-(t-2)c_1' - a'\big) = 0$.

In particular, from the above claim, $\chi\big(-c_1' - a'\big) = 0$. We note that $\chi\big((a+m-1)c_1'\big) = 0$ from Table 2.

- $x_1 = -c_1' - a'$, $x_i = -c_1' + 1$, $2 \le i \le m-1$, $x_m = -(a+m-1)c_1' \Rightarrow \chi\big(-c_1' + 1\big) = 1$.
- $x_1 = -c_1' + 1$, $x_i = 1$, $2 \le i \le m-1 \Rightarrow \chi\big(-2c_1' + (a'+1)\big) = 0$.
- $x_i = 0$, $2 \le i \le m-1$, $x_m = -2c_1' + (a'+1) \Rightarrow \chi\big(-c_1' + (a'+1)\big) = 1$.

Now $x_i = 1$, $1 \le i \le m-1$, $x_m = -c_1' + (a'+1)$ forms a monochromatic solution to eqn. (2a). This completes the proof of the second and third cases when $\chi(1) = 1$.

For the remainder of the proof, we consider the second and the third cases when $\chi(1) = 0$. We claim that

$$\chi(n) = 0 \text{ for } 0 \le n \le \left\lfloor \frac{-2c_1'}{a'} \right\rfloor = K. \tag{4}$$

By way of contradiction, assume $\chi(n) = 1$ for some $n \le K$. We claim this implies

$$\chi\big(-tc_1'\big) = \begin{cases} 0 & \text{if } t \text{ is even;} \\ 1 & \text{if } t \text{ is odd} \end{cases}$$

for $t \in \{1, \dots, a'\}$.

From Table 2, we have $\chi(-c_1') = 1$. Let $t \in \{1, \dots, a'-2\}$. Assuming $\chi(-tc_1') = 1$ when t is odd, we show that $\chi\big(-(t+1)c_1'\big) = 0$ and $\chi\big(-(t+2)c_1'\big) = 1$.

Each step in the following sequence forces a colour on some number in the given range in order to avoid a mononchromatic solution to eqn. (2a).

- $x_1 = -tc_1'$, $x_i = n$, $2 \le i \le m-1 \Rightarrow \chi\big(-(t+1)c_1' + na'\big) = 0$.
- $x_1 = -(t+1)c_1' + na'$, $x_i = 0$, $2 \le i \le m-1 \Rightarrow \chi\big(-(t+2)c_1' + na'\big) = 1$.
- $x_i = n$, $2 \le i \le m-1$, $x_m = -(t+2)c_1' + na' \Rightarrow \chi\big(-(t+1)c_1'\big) = 0$.
- $x_1 = -(t+1)c_1'$, $x_i = 0$, $2 \le i \le m-1 \Rightarrow \chi\big(-(t+2)c_1'\big) = 1$.

In particular, we have $\chi\big(-a'c_1'\big) = 1$. We note that the maximum allowable value of numbers used in the previous steps is $-a'c_1' + Ka'$ from the second step, and this lies in the domain of χ. In order that the numbers lie in the domain of χ, we must have $-a'c_1' + na' \leq -(a'+2)c_1'$, in particular. This implies $n \leq -\frac{2c_1'}{a'}$.

To complete the claim that $\chi(n) = 0$ for $0 \leq n \leq K$, we show that $\chi\big(-a'c_1'\big) = 0$ by using Table 2.

- $x_i = n$, $2 \leq i \leq m-1$, $x_m = -(a'+1)c_1' \Rightarrow \chi\big(-a'c_1' - na'\big) = 0$.
- $x_i = 0$, $2 \leq i \leq m-1$, $x_m = -a'c_1' - na' \Rightarrow \chi\big(-(a'-1)c_1' - na'\big) = 1$.
- $x_1 = -(a'-1)c_1' - na'$, $x_i = n$, $2 \leq i \leq m-1 \Rightarrow \chi\big(-a'c_1'\big) = 0$.

This contradiction completes the proof of the claim that $\chi(n) = 0$ for $0 \leq n \leq K$. It can be shown that $a \leq K$, and so we have $\chi(n) = 0$ for $0 \leq n \leq a$ in particular.

For the rest of this proof, we consider the second and third cases separately.

Case (ii) $\big((a'+2)c_1 - a'(a'+1) \leq c_2 < c_1 - a'\big)$

By assigning the colour of x_i in the solution of eqn. (1a), (1b) to $x_i - 1$, we note that the range of c_1 and c_2 translate to

$$-c_1' + a' + 1 \leq -c_2' \leq -(a'+2)c_1'.$$

We use $\chi(n) = 0$ for $0 \leq n \leq a$, and prove that $\chi(n) = 0$ for $a+1 \leq n \leq -c_1' - 1$.

Let $t+1 = \min\{n : \chi(n) = 1\}$; we have shown that $t+1 > a$. By way of contradiction, we may assume $t+1 \leq -c_1' - 1$. By Proposition 4, the expression $\sum_{i=1}^{m-2} x_i + ax_{m-1}$ assumes every value in the interval $[0, (a'+1)t]$ as each x_i runs over the set $\{0, \ldots, t\}$. Under the same range for the x_i's, the expression

$$\sum_{i=1}^{m-2} x_i + ax_{m-1} - c_2' = x_m$$

assumes every value in the interval $\mathcal{I} = [-c_1' + a' + 1, (a'+1)t - (a'+2)c_1']$. So in order to avoid a monochromatic solution to eqn. (2b), we must have $\chi(n) = 1$ for each $n \in \mathcal{I}$.

Now choosing $x_i = t+1$, $1 \leq i \leq m-1$ in eqn. (2a) forces $\chi\big((a'+1)(t+1) - c_1'\big) = 0$ in order to avoid a mononchromatic solution. But $(a'+1)(t+1) - c_1'$ lies within $[-c_1' + a' + 1, (a'+1)t - (a'+2)c_1']$, and this is a contradiction to the conclusion from the previous paragraph. Therefore we have the claim that $\chi(n) = 0$ for $0 \leq n \leq -c_1' - 1$.

From the above argument for $t = -c_1' - 1$, the expression

$$\sum_{i=1}^{m-2} x_i + ax_{m-1} - c_2' = x_m$$

assumes every value in the interval $\mathcal{J} = [-c_1' + a' + 1, -(a'+1)(c_1'+1) - (a'+2)c_1']$. Since $-(a'+2)c_1' \in \mathcal{J}$, there exist $x_1, \ldots, x_{m-1}$, with each $x_i \in \{0, \ldots, -c_1' - 1\}$, such that $\sum_{i=1}^{m-2} x_i + ax_{m-1} - c_2' = -(a'+2)c_1'$. This gives a monochromatic solution to eqn. (2b), since $\chi\big(-(a'+2)c_1'\big) = 0$ by Table 2.

Case (iii) $\big((a'+3)c_1 - a'(a'+2) \leq c_2 < (a'+2)c_1 - a'(a'+1)\big)$

By assigning the colour of x_i in the solution of eqn. (1a), (1b) to $x_i - 1$, we note that the range of c_1 and c_2 translate to

$$-(a'+2)c_1' < -c_2' \leq -(a'+3)c_1' - 1.$$

Recall that $\chi(n) = 0$ for $0 \le n \le \lfloor \frac{-2c_1'}{a'} \rfloor$. Each step in the following sequence forces a colour on some number in the given range in order to avoid a mononchromatic solution to eqn. (2a).

- $x_i = -c_1'$, $2 \le i \le m-1$, $x_m = -c_2' \Rightarrow \chi\big((a'+1)c_1' - c_2'\big) = 0$.
- $x_1 = (a'+1)c_1' - c_2'$, $x_i = 0$, $2 \le i \le m-1 \Rightarrow \chi\big(a'c_1' - c_2'\big) = 1$.

We capture this information as Table 3.

0	**1**
0	$-c_1'$
$-(a'+2)c_1'$	$-c_2'$
$(a'+1)c_1' - c_2'$	$-(a'+1)c_1'$

Table 3. Some initial colourings

Arguing as in Case (ii), with $t = K$, the expression

$$\sum_{i=1}^{m-2} x_i + ax_{m-1} - c_2' = x_m$$

assumes every value in the interval $\mathcal{K} = [-c_2', (a'+1)K - c_2']$. Since $(a'+1)c_1' - c_2' \in \mathcal{K}$, there exist $x_1, \ldots, x_{m-1}$, with each $x_i \in \{0, \ldots, K\}$, such that $\sum_{i=1}^{m-2} x_i + ax_{m-1} - c_2' = (a'+1)c_1' - c_2'$. This gives a monochromatic solution to eqn. (2b), since $\chi\big((a'+1)c_1' - c_2'\big) = 0$ by Table 3. This completes the proof of Theorem 6. ∎

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