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Rado Numbers for Two Systems of Linear Equations

Anthony Glackin
South Dakota State University, anthony.glackin@jacks.sdstate.edu

RADO NUMBERS FOR TWO SYSTEMS OF LINEAR EQUATIONS

BY

ANTHONY GLACKIN

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THESIS ACCEPTANCE PAGE

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This thesis is approved as a creditable and independent investigation by a candidate for the master's degree and is acceptable for meeting the thesis requirements for this degree. Acceptance of this does not imply that the conclusions reached by the candidate are necessarily the conclusions of the major department.

Donald L. Vestal, Jr
Advisor Date

Department Head Date

Nicole Lounsbery, PhD
Director, Graduate School Date

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Abstract

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For any positive integer n and any equation E of either the form $x_1+x_2+\cdots+x_n = x_0$ or $x_1 + x_2 + n = x_0$, the two-color Rado number $R_2(E)$ is the least integer such that any 2-coloring of the natural numbers 1 through $R_2(E)$ will contain a monochromatic solution to E. Let $\mathcal{E}_k$ be a system of k equations of the aforementioned form, where E_i represents the i^{th} equation in $\mathcal{E}_k$ and the set $\mathcal{I} = \{1, 2, \ldots, k\}$ is the set of indices of these equations. This thesis shows that the two-color Rado number $R_2(\mathcal{E})$ for the system of equations is $R_2(\mathcal{E}) = R_2(E_m)$, where E_m is the equation within $\mathcal{E}$ that has the largest Rado number.

Introduction

In 1916, Issai Schur introduced the concept of a Schur number, which is the least integer $S(r)$, where $r \geq 1$, such that any coloring of the integers 1 through $N \geq S(r)$ using r colors will contain a monochromatic solution to the equation $x + y = z$, meaning that $x + y = z$ and all three of x, y, z are the same color. Though it isn't how Schur initially proved the result, a popular proof uses a method of coloring complete graphs based on work by Frank Plumpton Ramsey. This method is used to prove Theorem 1.

Theorem 1. *For $r \geq 1$, there exists a least natural number $S(r)$ such that for all $N \geq S(r)$, if the integers $\{1, 2, \ldots, N\}$ are colored using r colors, then there exists three numbers $x, y, z \in \{1, 2, \ldots, N\}$ such that x, y, z are all the same color and $x + y = z$.*

For the proof of this theorem, the author introduces the notation $\chi(i)$, where χ represents a function assigning color values to the natural number i. Define also the Ramsey Number $R(r, k)$ to be the least integer such that for any coloring of the edges of a complete graph with r colors, there exists a monochromatic cycle containing k nodes.

Proof. Define $\chi : \mathbb{N} \longrightarrow [1, r]$ to be a coloring of $\mathbb{N}$ using r colors, where $N \geq S(r)$. Define also a corresponding complete graph with vertices $1, 2, \ldots, N$, where $N \geq R(r, 3)$, by coloring edge (i, j) within the graph with $\chi(|i - j|)$. By Ramsey's Theorem, there exists a monochromatic triangle. Call the nodes of this triangle i, j, k where $i < j < k$ and let $x = j - i$, $y = k - j$, and $z = k - i$. Then we have that

$$
\begin{aligned}
x + y &= j - i + k - j \\
&= k - i \\
&= z
\end{aligned}
$$

and that $\chi(x) = \chi(y) = \chi(z)$, giving us a monochromatic solution to $x + y = z$. $\square$

This proof establishes that Schur numbers exist and are bounded above by the corresponding Ramsey numbers. Rado later made contributions to the field [1][2][3].

Definition 1. *For a linear equation $x_1 + x_2 + \cdots + x_n + c = x_0$, the two-color Rado number is the least integer $R(n, c)$ such that any coloring of the integers $\{1, 2, \ldots, R(n, c)\}$ using two colors is guaranteed to contain at least one monochromatic solution to the equation $x_1 + x_2 + \cdots + x_n + c = x_0$.*

Notation 1. *Denote a coloring by C_i, where C represents the color, and i represents the number that is being colored. For example, the string $0_1\ 1_2\ 0_3$ means "color the numbers 1 and 3 with color 0 and color the number 2 with color 1."*

In 1982, Beutelspacher and Brestovansky [4] proved the following result regarding equations with n variables and $c = 0$.

Theorem 2. *For a natural number n and equation $x_1 + x_2 + \cdots + x_n = x_0$, the two-color Rado number $R(n, 0)$ is equal to $n^2 + n - 1$.*

The proof of this type of theorem involves two parts. First, to establish a lower bound on $R(n, 0)$, one must show that there exists a two-coloring of the integers $\{1, 2, \ldots, n^2 + n - 2\}$ that does not contain a monochromatic solution to the given equation. This is done using a three-block coloring pattern wherein the integers are colored using a block of color 0 (red) followed by a block of color 1 (blue), followed by another block of red. Second, to establish an upper bound, it must be shown that any coloring of length $n^2 + n - 1$ must contain a monochromatic solution. This is done using forced colorings. In other words, the color of one of the numbers is assumed without loss of generality, which forces a series of other integers to be colored in a specific manner to avoid monochromatic solutions. Eventually, this results in a contradiction where an integer cannot be colored either blue or red without giving rise to a monochromatic solution. Let us consider a simple example to demonstrate this structure.

Example 0.1. Consider the equation $x_1 + x_2 + x_3 = x_0$ representing the $n = 3$ case. Theorem 2 states the Rado number for this equation should be $R(3, 0) = 3^2 + 3 - 1 = 11$. We will color the relevant integers using the notation from Notation 1, where color 0 is red and color 1 is blue.

To establish a lower bound on $R(3, 0)$, we use the following coloring:

$$0_1\ 0_2\ 1_3\ 1_4\ 1_5\ 1_6\ 1_7\ 1_8\ 0_9\ 0_{10}.$$

To show this coloring contains no monochromatic solutions, one can look at the red blocks first. Since the lowest number colored red is 1, we can plug 1 into our equation, giving $1+1+1=3=x_0$. Likewise, since the highest number colored red in the first block is 2, we can plug 2 into our equation giving $2+2+2=6=x_0$. Since all the integers from 3 through 6 are colored blue, we know there is no monochromatic solution wherein all of x_1, x_2, x_3 are integers in the first red block. We now turn to the blue block. Again, since the lowest number in the blue block is 3, we plug in to arrive at $3+3+3=9=x_0$. Since all integers greater than or equal to 9 are either colored red or not a part of our coloring and any blue solution must contain a number greater than or equal to 9, it follows that there is no monochromatic solution within the blue block. A similar method will show that there is no solution containing only numbers in the second red block. Therefore, we need only to show that there is no solution containing some numbers from the first red block and some from the second red block. Consider that the lowest value of x_0 that can be produced using numbers from both red blocks is $1+1+9=11=x_0$. Since 11 is not included in our coloring and any solution involving numbers from both red blocks will total at least 11, we know there is also no red monochromatic solution here. This provides a lower bound; we can color the integers 1 through 10 while avoiding a monochromatic solution using this three-block coloring.

To establish an upper bound, assume without loss of generality that 1 is colored red. It follows that, to avoid a monochromatic red solution, $1+1+1=1*3=3$ must not be colored red and must instead be colored blue. Thus, 1 is red and 3 is blue. A shorthand way to write this process that will be used henceforth is

$$\chi(1)=0 \wedge 1+1+1=3 \implies \chi(3)=1.$$

Continuing this line of logic gives

$$1)\ \chi(1) = 0 \wedge 1 + 1 + 1 = 3 \implies \chi(3) = 1$$

$$2)\ \chi(3) = 1 \wedge 3 + 3 + 3 = 9 \implies \chi(9) = 0$$

$$3)\ \chi(9) = 0 \wedge \chi(1) = 0 \wedge 1 + 1 + 9 = 11 \implies \chi(11) = 1$$

$$4)\ \chi(9) = 0 \wedge \chi(1) = 0 \wedge 4 + 4 + 1 = 9 \implies \chi(4) = 1$$

$$5)\ \chi(3) = 1 \wedge \chi(11) = 1 \wedge 3 + 4 + 4 = 11 \implies \chi(4) = 0$$

Statements 4 and 5 above contradict each other. We cannot color 4 red without forming a red monochromatic solution to our equation, but we cannot color 4 blue without forming a blue monochromatic solution to our equation. Therefore, within the set $\{1, 3, 4, 9, 11\}$, we must have a monochromatic solution. This establishes the upper bound on $R(3, 0)$ as 11. Because the upper bound on $R(3, 0)$ is 11 and we have demonstrated a coloring of length 10 that does not contain a monochromatic solution, we know that $R(3, 0) = 11$.

The general proof for this type of problem follows the same pattern, beginning by establishing a lower bound and then establishing an upper bound by finding a set that must contain a monochromatic solution. Recall that Theorem 2 states that the two-color Rado number for $x_1 + x_2 + \cdots + x_n = x_0$ is $R(n, 0) = n^2 + n - 1$.

Proof of theorem 2. Let $n \geq 1$ and define $\chi : [1, n^2 + n - 2] \longrightarrow \{0, 1\}$ as a coloring function on the integers such that $\chi(i) = 0$ means integer i is red and $\chi(i) = 1$ means integer i is blue. We will exhibit a coloring of the integers 1 through $n^2 + n - 2$ that does not include a monochromatic solution to $x_1 + x_2 + \cdots + x_n = x_0$, thus establishing a lower bound on $R(n, 0)$. We propose the following coloring with no monochromatic solutions:

$$0_1\ 0_2 \ldots 0_{n-1}\ 1_n\ 1_{n+1} \ldots 1_{n^2-1}\ 0_{n^2}\ 0_{n^2+1} \ldots 0_{n^2+n-2}.$$

As in the example, we first consider the first red block. Using all 1's gives $1 + 1 + \cdots + 1 = 1 * n = n = x_0$. Likewise, using all $n - 1$'s gives $(n-1) + (n-1) + \cdots + (n-1) = n(n-1) = n^2 - n = x_0$. Since all numbers within the block n through $n^2 - 1$ are colored blue, it follows that there is no monochromatic solution using only integers from the first red block. Looking next at the blue block, we see that a monochromatic blue solution must contain an integer at least as large as $n+n+\cdots+n = n*n = n^2 = x_0$. Since all integers greater than or equal to n^2 are either red or uncolored, it follows that there is no monochromatic blue solution. The same process easily shows that there are no monochromatic solutions containing only integers from the second red block. Finally, we use 1 and n^2 to establish the lowest possible value of x_0 resulting from numbers in both red blocks, giving $n^2+1+1+\cdots+1 = n^2+n-1$. Since all integers greater than or equal to n^2+n-1 are uncolored in our coloring, there is no monochromatic solution containing integers from both red blocks. This gives a lower bound coloring with no monochromatic solutions.

We now turn our attention to an upper bound using forced colorings. We may assume without loss of generality that $\chi(1) = 0$. To avoid a monochromatic solution using 1, $1+1+\cdots+1 = n = x_0$, n must be colored with blue. We follow this process again using $n+n+\cdots+n = n^2 = x_0$ to show $\chi(n^2) = 0$. Thus, so far we know

$$1)\ \chi(1) = 0 \wedge 1+1+\cdots+1 = n \implies \chi(n) = 1$$

$$2)\ \chi(n) = 1 \wedge n+n+\cdots+n = n^2 \implies \chi(n^2) = 0.$$

Using a combination of n^2 and a string of 1's, both of which are colored red, we see $n^2 + 1+1+\cdots+1 = n^2+n-1 = x_0$, implying that $\chi(n^2+n-1) = 1$. Using n^2 and 1 in a different manner, with n^2 on the right hand side of the equation, we see $1+(n+1)+(n+1)+\cdots+(n+1) = n^2$. Therefore, $\chi(n+1) = 1$, resulting in the following forced coloring so far

$$1)\ \chi(1) = 0 \wedge 1+1+\cdots+1 = n \implies \chi(n) = 1$$

$$2)\ \chi(n) = 1 \wedge n+n+\cdots+n = n^2 \implies \chi(n^2) = 0$$

$$3)\ \chi(n^2) = 0 \wedge \chi(1) = 0 \wedge n^2+1+1+\cdots+1 = n^2+n-1 \implies \chi(n^2+n-1) = 1$$

$$4)\ \chi(n^2) = 0 \wedge \chi(1) = 0 \wedge 1+(n+1)+(n+1)+\cdots+(n+1) = n^2 \implies \chi(n+1) = 1.$$

However, the fact that $\chi(n) = 1$ and $\chi(n^2+n-1) = 1$ gives that n and n^2+n-1 can be plugged in with $n+1$ as well, $n+(n+1)+(n+1)+\cdots+(n+1) = n^2+n-1$, to show that $\chi(n+1) = 0$. Thus, we have

1) $\chi(1) = 0 \wedge 1 + 1 + \cdots + 1 = n \implies \chi(n) = 1$

2) $\chi(n) = 1 \wedge n + n + \cdots + n = n^2 \implies \chi(n^2) = 0$

3) $\chi(n^2) = 0 \wedge \chi(1) = 0 \wedge n^2 + 1 + 1 + \cdots + 1 = n^2 + n - 1 \implies \chi(n^2 + n - 1) = 1$

4) $\chi(n^2) = 0 \wedge \chi(1) = 0 \wedge 1 + (n+1) + (n+1) + \cdots + (n+1) = n^2 \implies \chi(n+1) = 1$

5) $\chi(n) = 1 \wedge \chi(n^2 + n - 1) = 1 \wedge n + (n+1) + (n+1) + \cdots + (n+1) = n^2 + n - 1$

$\implies \chi(n+1) = 0.$

Statements 4 and 5 cannot both be true, thus proving that within the set of values $\{1, n, n+1, n^2, n^2+n-1\}$ there must exist a monochromatic solution to the equation $x_1+x_2+\cdots+x_n = x_0$. □

A similar proof structure was used by Burr and Loo [5] to prove another important result, this time regarding equations with 2 variables and a constant.

Theorem 3. *For a natural number c and equation $x_1 + x_2 + c = x_0$, the two-color Rado number $R(2, c)$ is equal to $4c + 5$.*

A simple example will help demonstrate this result.

Example 0.2. Consider the equation $x_1+x_2+1 = x_0$, representing the $c = 1$ case. Theorem 3 states that the Rado number for this equation should be $R(2, 1) = 4(1) + 5 = 9$. Recall the notation from Notation 1, where color 0 is red and color 1 is blue. To establish a lower bound, consider the following three-block coloring:

$$0_1\ 0_2\ 1_3\ 1_4\ 1_5\ 1_6\ 0_7\ 0_8.$$

Once again, to show this contains no monochromatic solutions, we look at each block. Once again, $1 + 1 + 1 = 3 = x_0$ and $2 + 2 + 1 = 5 = x_0$ along with the fact that the integers between 3 and 6 are all colored blue gives no solutions from the first red block. Also, $3 + 3 + 1 = 7 = x_0$ and the fact that the integers greater than or equal to 7 are not blue gives no solution from the blue block. The second red block follows the same process, and finally the solution $1+7+1=9=x_0$ along with the fact that the integers greater than or equal to 9 are not part of our coloring give that there is no solution using both red blocks. Thus, this coloring establishes a lower bound. Therefore, the Rado number $R(2,1)$ is at least 9.

To arrive at an upper bound, we again turn to forced colorings. Assume without loss of generality that $\chi(1)=0$. Plugging 1 into the given equation will force $1+1+1=3$ to be colored using blue. Substituting 3 in the same manner yields $3+3+1=7$ and therefore $\chi(7)=0$. Now, using 1 and 7, both of which are colored red, we see that $1+7+1=9$ and therefore that $\chi(9)=1$. With this information and the fact that $\chi(3)=1$, our equation gives $3+5+1=9$ and thus that $\chi(5)=0$. However, we can see from $\chi(1)=0$, $\chi(7)=0$, and $1+5+1=7$ that $\chi(5)=1$. Summarizing this with the notation defined in Example 1.1, we have

$$1)\ \chi(1)=0 \wedge 1+1+1=3 \implies \chi(3)=1$$

$$2)\ \chi(3)=1 \wedge 3+3+1=7 \implies \chi(7)=0$$

$$3)\ \chi(1)=0 \wedge \chi(7)=0 \wedge 1+7+1=9 \implies \chi(9)=1$$

$$4)\ \chi(1)=0 \wedge \chi(7)=0 \wedge 1+5+1=7 \implies \chi(5)=1$$

$$5)\ \chi(3)=1 \wedge \chi(9)=1 \wedge 3+5+1=9 \implies \chi(5)=0.$$

Once again, statements 4 and 5 cannot both be true at once. It follows that somewhere within the set $\{1,3,5,7,9\}$ there is a monochromatic solution to $x_1+x_2+1=x_0$. This shows that the upper bound on $R(2,1)$ is 9. From our upper and lower bounds we can gather that $R(2,1)=9$.

The proof of Theorem 3 will follow the same structure. Recall that Theorem 3 claims that for $c \in \mathbb{N}$ the Rado number for the equation $x_1+x_2+c=x_0$ is $R(2,c)=5c+4$.

Proof of theorem 3. Let $c \in \mathbb{N}$ and consider the proposed lower bound coloring

$$0_1\ 0_2 \ldots 0_{c+1}\ 1_{c+2}\ 1_{c+3} \ldots 1_{3c+3}\ 0_{3c+4}\ 0_{3c+5} \ldots 0_{4c+4}.$$

To show this contains no monochromatic solutions, we examine each block in the three-block coloring. The smallest integer that can be reached using only integers colored red in the first block is $1+1+c=c+2$, while the largest is $(c+1)+(c+1)+c=3c+2$. Since the integers between $c+2$ and $3c+2$ are all blue, it follows that there is no monochromatic solution created using only the first block of red integers. Likewise, since the smallest integer value that can be created using blue integers is $(c+2)+(c+2)+c=3c+4$ and all integers larger than $3c+4$ are either red or uncolored, we know there is no solution containing only the integers from the blue block. The second red block has a minimum output of $(3c+4)+(3c+4)+c=7c+8$, which is outside the bounds of our coloring. Therefore, there is no monochromatic solution containing only elements of the second red block. Finally, we consider a solution created using one input from the first red block and one from the second red block, which has a minimum output of $1+(3c+5)+c=4c+6$. Since this is again outside of the bounds of our coloring, we know there is no monochromatic solution created using both the first and second red blocks. Therefore, this coloring contains no monochromatic solutions.

To arrive at an upper bound, we again turn to forced colorings. Assume without loss of generality that $\chi(1)=0$. Plugging 1 into the given equation will force $1+1+c=c+2$ to be colored blue. Substituting $c+2$ in the same manner yields $(c+2)+(c+2)+c=3c+4$ and therefore $\chi(3c+4)=0$. Now, using 1 and $3c+4$, both of which were colored red, we see that $1+(3c+4)+c=4c+5$ and therefore that $\chi(4c+5)=1$. With this information and the fact that $\chi(c+2)=1$, our equation gives $(c+2)+(2c+3)+c=4c+5$ and thus that $\chi(2c+3)=0$. However, we can see from $\chi(1)=0$, $\chi(3c+4)=0$, and $1+(2c+3)+c=3c+4$ that $\chi(2c+3)=1$. Summarizing this with the notation defined in Example 1.1, we have

1) $\chi(1)=0 \wedge 1+1+c=c+2 \implies \chi(c+2)=1$

2) $\chi(c+2)=1 \wedge (c+2)+(c+2)+c=3c+4 \implies \chi(3c+4)=0$

3) $\chi(1)=0 \wedge \chi(3c+4)=0 \wedge 1+(3c+4)+c=4c+5 \implies \chi(4c+5)=1$

4) $\chi(1)=0 \wedge \chi(3c+4)=0 \wedge 1+(2c+3)+c=3c+4 \implies \chi(2c+3)=1$

5) $\chi(c+2)=1 \wedge \chi(4c+5)=1 \wedge (c+2)+(2c+3)+c=4c+5 \implies \chi(2c+3)=0.$

The contradiction in statements 4 and 5 above shows that somewhere in the set $\{1, c+2, 2c+3, 3c+4, 4c+5\}$, there must exist a monochromatic solution. This indicates that the upper bound on $R(2,c)$ is $4c+5$. Since we have established that $R(2,c) \geq 4c+5$ and $R(2,c) \leq 4c+5$, we know that for this type of equation $R(2,c) = 4c+5$. □

The results from Theorem 2 and Theorem 3, in particular the sets generated, will be used in the following sections to prove that for linear systems of equations taking the form described in Definition 1, the Rado number for the system will be the same as largest Rado number for any individual equation in the system.

Main Results

Two Disjoint Equations of the Form $x_1 + x_2 + \cdots + x_n = x_0$

To begin, recall the set established in Theorem 2, which will henceforth be referred to as the Beutelspacher-Brestovansky sufficient set (B-B sufficient set). Upon close examination, one can conclude that for any $n \in \mathbb{N}$, one of the following solutions created within the B-B sufficient set must be monochromatic.

1) $1 + 1 + \cdots + 1 = n$

2) $n + n + \cdots + n = n^2$

3) $n^2 + 1 + 1 + \cdots + 1 = n^2 + n - 1$

4) $n + (n+1) + (n+1) + \cdots + (n+1) = n^2 + n - 1$

5) $1 + (n+1) + (n+1) + \cdots + (n+1) = n^2$

These five solution options will form the basis for the main result regarding two-equation systems, but first some new notation and definitions must be introduced.

Notation 2. *Let $\mathcal{E}_{n,m}$ denote a system of two linear disjoint equations, $x_1+x_2+\cdots+x_n = x_0$ and $y_1 + y_2 + \cdots + y_m = y_0$, where $n \geq m$. In such a system, denote $x_1 + x_2 + \cdots + x_n = x_0$ as E_n and $y_1 + y_2 + \cdots + y_m = y_0$ as E_m.*

Definition 2. *A monochromatic solution exists to a system of two linear disjoint equations* $\mathcal{E}_{n,m}$ *if there exists a monochromatic solution to both* E_n *and* E_m *and both solutions are monochromatic in the same color.*

Notation 3. *Let* $R(\mathcal{E}_{n,m})$ *denote the two-color Rado number of a system of two linear disjoint equations* $\mathcal{E}_{n,m}$*.*

Having established the notation and definitions that will be used, it is now time to introduce the primary theorem for this section. The theorem will then be proved using a three-block coloring to find a lower bound and forced colorings to find an upper bound. This process will involve separate cases for each of the five monochromatic solutions to the B-B sufficient set.

Theorem 4. *Let* $\mathcal{E}_{n,m}$ *be a system of two disjoint linear equations such that* $n \geq m$*. Then* $R(\mathcal{E}_{n,m}) = R(n, 0) = n^2 + n - 1$*.*

Proof. Consider some system of disjoint linear equations $\mathcal{E}_{n,m}$ where $n \geq m$. We begin by finding a lower bound coloring, recalling that color 0 is red and color 1 is blue. To do this, note that, since a coloring of the integers 1 through $n^2 + n - 2$ exists that avoids a monochromatic solution to $x_1 + x_2 + \cdots + x_n = x_0$, therefore it is possible to color those integers without a monochromatic solution to $\mathcal{E}_{n,m}$. Thus, it is clear that $R(\mathcal{E}_{n,m}) \geq R(n) = n^2 + n - 1$.

Proceeding to the upper bound, first note that in the case where $n = m$, the proof is trivial. Apply the monochromatic solution formed in the B-B sufficient set for $x_1 + x_2 + \cdots + x_n = x_0$ to the equation $y_1 + y_2 + \cdots + y_m = y_0$ to form a monochromatic solution to the entire system. Thus the following will assume that $n > m$ and will show that for a coloring of length $n^2 + n - 1$, there exists a monochromatic solution to $\mathcal{E}_{n,m}$.

Assume there is a coloring of the integers 1 through $n^2 + n - 1$. Note that, since $n > m$, it is true that $n^2 + n - 1 > m^2 + m - 1$. From Theorem 2, it then follows that there exists a monochromatic solution, without loss of generality in red, to E_m and that this monochromatic solution exists in the B-B sufficient set for E_m. Also from Theorem 2, it is clear that there exists a monochromatic solution to E_n in the integers from 1 through

n^2+n-1. If these two monochromatic solutions are in the same color the theorem statement holds, so we will assume that the solution to E_n is monochromatic in blue. Therefore, to avoid a monochromatic solution to the system, one must avoid either a monochromatic blue solution to E_m or a red monochromatic solution to E_n. The following cases are based on which monochromatic solution to E_m exists.

Case 1: The monochromatic red solution to E_m is $1+1+\cdots+1=m$.

In this case, 1 and m are both colored red. If we plug 1 into E_n, which yields $1+1+\cdots+1=n$, it becomes apparent that one must color n blue to avoid a red monochromatic solution to E_n and thus to the system of equations. Following the same process, but with m instead of 1, we have that $m+m+\cdots+m=mn$, thus implying that mn must be colored blue as well. However, plugging n into E_m gives $n+n+\cdots+n=nm$, thus implying that either mn is colored red or there exists a blue monochromatic solution to E_m. Therefore, within the set $\{1,n,m,mn\}$, there exists either a red solution to E_n or a blue solution to E_m. In either case, a monochromatic solution to $\mathcal{E}_{n,m}$ exists, thus validating the theorem statement. Note that this argument is symmetric and thus forces 1 and n to be colored differently to avoid a monochromatic solution. For this case, a set has been found, $\{1,m,n,mn\}$ that guarantees a monochromatic solution to the system of equations.

Case 2: The monochromatic red solution to E_m is $m+m+\cdots+m=m^2$.

First, note that case 1 gives that in this scenario, 1 must be blue and n must be red. Using m within E_n gives $m+m+\cdots+m=mn$, which forces mn to be colored blue. A similar process with n will show that $n+n+\cdots+n=n^2$ forces n^2 to be blue. Now, plug n^2 and mn, respectively, into E_m along with a string of ones to give $1+1+\cdots+1+mn=mn+m-1$ and $1+1+\cdots+1+n^2=n^2+m-1$, thus forcing both $mn+m-1$ and n^2+m-1 to be red. Now, using E_n and plugging in $m-1$ m's and $n-m+1$ $m+1$'s along with $mn+m-1$ gives $m+m+\cdots+m+(m+1)+\cdots+(m+1)=mn+m-1$ which means $m+1$ must be colored blue. Following the same process with n, $n+1$, and n^2+m-1 gives $n+n+\cdots+n+(n+1)+\cdots+(n+1)=n^2+m-1$ and thus implies that $n+1$ is blue. Using the fact that $n+1$ is blue will result in the following substitution for E_m: $(n+1)+\cdots+(n+1)=mn+m$. This implies that $mn+m$ must be colored red. Finally, use m, $2m$, and $mn+m$ in E_n to give $m+\cdots+m+2m=mn+m$ and to force $2m$ to be blue. However, $1+1+\cdots+1+(m+1)=2m$ implies that $2m$ must be red. Thus, within the set $\{1, m, m+1, n, n+1, 2m, m^2, mn, mn+m-1, mn+m, n^2, n^2+m-1\}$, there must exist either a blue monochromatic solution to E_m or a red monochromatic solution to E_n, either of which yields a monochromatic solution to the system.

Case 3: The monochromatic red solution to E_m is $m^2+1+1+\cdots+1=m^2+m-1$.

Given that in this scenario 1 is red, it follows from case 1 that m and n must be blue. Likewise, because both m and n are blue, it is clear that mn must be colored red to avoid a monochromatic solution to the system. Now consider the following substitutions for E_m: $m+n+n+\cdots+n = m+n(m-1) = mn+m-n$. To avoid a blue solution to E_m, $mn+m-n$ must be colored red. Now using this fact, plug into E_n with 1, giving $1+1+\cdots+1+(mn+m-n) = 1(n-1)+mn+m-n = mn+m-1$ and thus implying that to avoid a monochromatic solution in red, $mn+m-1$ must be colored blue. In conjunction with the fact that n is colored blue, this implies that we can plug into E_m to give $n+(n+1)+(n+1)+\cdots+(n+1) = n+(n+1)(m-1) = mn+m-1$ and forcing us to color $n+1$ red to avoid a monochromatic solution to our system. However, if we use 1, mn, and $n+1$ in E_n, we see that $1+1+\cdots+1+(n+1)+(n+1)+\cdots+(n+1) = 1(n-m+1)+(n+1)(m-1) = mn$ which would give us a monochromatic red solution to E_n and thus to our system. Again, it has been shown that there exists a set, in this case $\{1, m, n, n+1, m^2, mn, mn+m-n, m^2+m-1, mn+m-1\}$, that is guaranteed to contain a monochromatic solution to the system.

Case 4: The monochromatic red solution to E_m is $m+(m+1)+(m+1)+\cdots+(m+1) = m^2+m-1$.

Since m is colored red in this case, it follows from Case 1 that 1 must be colored blue. From plugging m into equation E_n to get $m+m+\cdots+m = mn$, it also follows that mn must be colored blue. Using 1 and mn in E_m to give $mn+1+1+\cdots+1 = mn+m-1$ then implies that one must color $mn+m-1$ red. However, using m and $m+1$ in E_n shows that $m+m+\cdots+m+(m+1+(m+1)+\cdots+(m+1) = m(n-m+1)+(m+1)(m-1) = mn+m-1$, thus forcing a monochromatic solution to E_n and therefore to $\mathcal{E}_{n,m}$.

Case 5: The monochromatic red solution to E_m is $1+(m+1)+(m+1)+\cdots+(m+1) = m^2$.

Once again, the fact that 1 is red means that both m and n must be blue from case 1.

Using m and n in E_m then yields $m+\cdots+m+n = m^2+n-m$, implying that m^2+n-m is red. However, using 1 and $m+1$ in E_n yields $1+\cdots+1+(m+1)+\cdots+(m+1) = m^2+n-m$, thus giving a red solution to E_n and therefore a monochromatic solution to the system.

We see that, in each case, a monochromatic solution is forced to the system of equations. Thus it has been shown that, regardless of which red solution exists to E_m, there exists a monochromatic solution to the system of equations. $\square$

It should be noted that, while this section specifically deals with disjoint equations, it appears from preliminary investigations using computer programs applied to concrete examples that the results from this section may apply to non-disjoint equations as well. The author offers the following conjecture to this effect.

Conjecture 1. *The two-color Rado number for a system E of two linear equations $x_1 + x_2 + \cdots + x_n = z$ and $y_1 + y_2 + \cdots + y_m = z$ where $n \geq m$ is $R(E) = R(n, 0) = n^2 + n - 1$.*

Two Disjoint Equations of the Form $x_1 + x_2 + c = x_0$

Begin by recalling the set established in Theorem 3 which, for ease of reference, will be referred to as the Burr-Loo sufficient set (B-L sufficient set). As with the B-B sufficient set, the B-L sufficient set produces a list of potential monochromatic solutions for a given $c \in \mathbb{N}$. This set of solutions is as follows.

1) $1 + 1 + c = c + 2$

2) $(c + 2) + (c + 2) + c = 3c + 4$

3) $1 + (3c + 4) + c = 4c + 5$

4) $(c + 2) + (2c + 3) = 4c + 5$

5) $1 + (2c + 3) + c = 3c + 4$

The supremum of the values within the B-L sufficient set, $4c + 5$, will help inform the cases for the main result regarding two-equation systems of this particular form. However, the primary component in the main result will be the coloring of 1 and the coloring of whatever solution from the B-L sufficient set exists. Once again, some notation must be defined before proceeding.

Notation 4. *Let $\mathcal{L}_{c,k}$ denote a system of two linear disjoint equations, $x_1+x_2+c=x_0$ and $y_1+y_2+k=y_0$, where $c,k \in \mathbb{N}$, c,k have the same parity, and $c>k$. In such a system, denote $x_1+x_2+c=x_0$ as L_c and $y_1+y_2+k=y_0$ as L_k.*

The definition of a monochromatic solution to this system is synonymous with the definition given in Definition 2. The notation for the two-color Rado number to this system will be the same as the notation described in Notation 4. The remainder of this section will be designated towards proving that the Rado number for a system of two equations, L_c and L_k, of the given form will, under specific circumstances, be the same as the larger of the individual Rado numbers for the equations within the system.

The circumstances mentioned above pertain to the parity of c and k. Note that if c and k have different parity, for instance if c is odd and k is even, it is possible to color the integers in such a way that a monochromatic solution to the system is never reached. This will be explained in greater detail following the main results in this section, which will now be formally stated and proved.

Theorem 5. *The Rado number for a system of linear equations like that defined in Notation 4 is $R(\mathcal{L}_{c,k})=R(L_c)=4c+5$ provided that c,k have the same parity.*

To establish this result it is first necessary to introduce two lemmas, which will be used to establish an upper bound on the Rado number for the given system. Lemma 1 will show what must happen if the integer 1 is the same color as the solution to L_k. Lemma 2 will establish what happens if the integer 1 is a different color than the solution to L_k. Before proving these lemmas, we will introduce some new notation for the sake of brevity in the proofs.

Notation 5. *If $z^* \in \mathbb{N}$ and the color of z^* has been established, the notation $2z+k=z^*$ or $2z+c=z^*$ will be employed where $z=y_1=y_2$ or $z=x_1=x_2$ depending on the equation being examined. Here, the goal is to use z^* on the right hand side of the equation to force a number smaller than z^* to be colored in a particular way.*

Lemma 1. *If $\mathcal{L}_{c,k}$ is defined as in Notation 4 and $\chi(1)$ is the same as the color of the solution to L_k, then there exists a monochromatic solution to $\mathcal{L}_{c,k}$ in the integers from 1 through $4c+5$.*

Proof. Assume without loss of generality that $\chi(1) = 0$. Assume also that the monochromatic solution to L_k as forced by Theorem 3 is red. Note at this point that if we color out to $4c+5$ there must exist a monochromatic solution within the B-L sufficient set for L_c. If this solution is also red, the lemma statement holds. Thus, we will assume the solution to L_c is monochromatic in blue. Therefore, we will seek to show a forced blue solution to L_k or a forced red solution to L_c, either of which will give a monochromatic solution to the system $\mathcal{L}_{c,k}$.

Since 1 is red, it then follows from $1+1+c = c+2$ that $c+2$ must be colored blue. Plugging this into the right hand side of L_k using the notation from Notation 6 gives that $2z+k = c+2$ and thus that $z = \frac{c-k}{2}+1$. Since c,k have the same parity, it follows that $z \in \mathbb{N}$ and therefore that, $\chi(\frac{c-k}{2}+1) = 0$. Using this result in L_c yields $(\frac{c-k}{2}+1)+(\frac{c-k}{2}+1)+c = 2c-k+2$, which forces $2c-k+2$ to be colored blue. Now, plug $c+2$ and $2c-k+2$ into L_k to give $(c+2)+(2c-k+2)+k = 3c+4$ and make $\chi(3c+4) = 0$. From this, we can see $1+(3c+4)+c = 4c+5$, implying that $\chi(4c+5) = 1$. Finally, we use 1 with $3c+4$ in L_c and we use $2c-k+2$ with $4c+5$ in L_k to give $1+(2c+3)+c = 3c+4$ and $(2c-k+2)+(2c+3)+k = 4c+5$. This means that $2c+3$ cannot be colored either blue or red. Therefore, within the set $\{1, \frac{c-k}{2}+1, c+2, 2c-k+2, 2c+3, 3c+4, 4c+5\}$ there must be a monochromatic solution to either L_c in red or L_k in blue, giving us a monochromatic solution to the system $\mathcal{L}_{c,k}$. □

Lemma 2. *If $\mathcal{L}_{c,k}$ is defined as in Notation 4 and $\chi(1)$ is not the same as the color of the solution to L_k, then there exists a monochromatic solution to $\mathcal{L}_{c,k}$ in the integers from 1 through $4c+5$.*

Proof. Assume without loss of generality that $\chi(1) = 0$. Assume also that the monochromatic solution to L_k as forced by Theorem 3 is blue. As established in the previous lemma, coloring to $4c+5$ will yield a monochromatic solution to L_c. We will this time assume that solution is red and thus we must show a forced red solution to L_k or a forced blue solution to L_c to give a monochromatic solution to the system.

Because 1 is colored red, it follows from $1+1+k=k+2$ that $k+2$ must be colored blue. Plugging this result into L_c gives $(k+2)+(k+2)+c=c+2k+4$ and forces $\chi(c+2k+4)=0$. Using this along with 1 in L_k yields $1+(c+2k+4)+k=c+3k+5$ and thus forces $c+3k+5$ to be blue. Plugging in $c+2k+4$ on the right hand side of L_k so that $1+y_2+k=c+2k+4$ gives that $y_2=c+k+3$ and thus forces $c+k+3$ to be colored blue as well. We now substitute $c+3k+5$ along with $k+2$ into L_c, giving $(k+2)+x_2+c=c+3k+5$. Therefore, we must color $x_2=2k+3$ red. Using $c+k+3$ and $k+2$ in L_c forces $(c+k+3)+(k+2)+c=2c+2k+5$ to be colored red as well, which can be used along with 1 in L_k to give $y_1+1+k=2c+2k+5$ and to force $y_1=2c+k+4$ to be colored blue. Using $k+2$ on the left hand side and plugging in $2c+k+4$ on the right hand side of L_c gives $x_1+(k+2)+c=2c+k+4$. Therefore, we must color $x_1=c+2$ red. Using this in L_k gives $2z+k=c+2$ and forces $z=\frac{c-k}{2}+1$ to be colored blue. Plugging this result into L_c results in $(\frac{c-k}{2}+1)+(\frac{c-k}{2}+1)+c=2c-k+2$ being colored red. However, we can plug this into L_k along with $2k+3$ to give $(2c-k+2)+(2k+3)+k=2c+2k+5$. This implies that $\chi(2c+2k+5)=1$, but we previously established that $2c+2k+5$ cannot be colored blue without resulting in a blue solution to L_c. Therefore, within the set $\{1,k+2,c+2k+4,c+3k+5,c+k+3,2k+3,2c+2k+5,2c+k+4,c+2,\frac{c-k}{2}+1,2c-k+2\}$, there exists a monochromatic solution to either L_k in red or L_c in blue. Either solution results in a monochromatic solution to the system $\mathcal{L}_{c,k}$. □

We now turn our attention to the proof of Theorem 5. This proof will use Lemma 1 and Lemma 2 to establish an upper bound on the Rado number. It will also reference Theorem 3 to establish the lower bound. Recall that Theorem 5 states that for a system of two disjoint linear equations $\mathcal{L}_{c,k}$ where L_c is $x_1+x_2+c=x_0$ and L_k is $y_1+y_2+k=y_0$, where $c,k\in\mathbb{N}$ and $c>k$, the Rado number for the system is $R(\mathcal{L}_{c,k})=R(L_c)=4c+5$.

Proof of Theorem 5. The result from Burr and Loo demonstrated in Theorem 3 gives a lower bound coloring of the integers 1 through $4c+4$ that avoids a monochromatic solution to L_c. Using this coloring, one avoids a monochromatic solution to the system $\mathcal{L}_{c,k}$. Therefore, it is true that $R(\mathcal{L}_{c,k}) \geq 4c + 5$.

To establish an upper bound, first note that when coloring the integers 1 through $4c+5$, Theorem 3 shows that there is a monochromatic solution within the B-L sufficient set for both L_c and L_k. If these monochromatic solutions to the individual equations are the same color, the theorem statement holds. Therefore, we must consider only the case where the solutions are in different colors. We further break this case down into two sub-cases based on how the color of the solution to L_k corresponds to $\chi(1)$. We may assume without loss of generality that $\chi(1) = 0$. If the color of the solution to L_k is also red, we may apply Lemma 1 to show that there is a monochromatic solution to our system within the set $A = \{1, \frac{c-k}{2} + 1, c + 2, 2c - k + 2, 2c + 3, 3c + 4, 4c + 5\}$. If the color of the solution to L_k is instead blue, we apply Lemma 2 to show that there is a monochromatic solution to our system within the set $B = \{1, k+2, c+2k+4, c+3k+5, c+k+3, 2k+3, 2c+2k+5, 2c+k+4, c+2, \frac{c-k}{2}+1, 2c-k+2\}$. Since $4c + 5 \geq i$ for all $i \in A \cup B$, it is true that in either case we have a monochromatic solution to our system in the integers from 1 through $4c+5$. Therefore, it is true that $R(\mathcal{L}_{c,k}) \leq 4c + 5$. Since $R(\mathcal{L}_{c,k}) \geq 4c + 5$ and $R(\mathcal{L}_{c,k}) \leq 4c + 5$, it follows that $R(\mathcal{L}_{c,k}) = R(L_c) = 4c + 5$. □

Having proved the main result from this section, we will now address one specific case wherein the prior result would not hold. If c, k have different parity, the Rado number will be infinite. Consider as an example the case where $k = 1$ and $c = 2$. Thus, we look for solutions to the system where L_c is $x_1 + x_2 + 2 = x_0$ and L_k is $y_1 + y_2 + 1 = y_0$. For this system, the following coloring of infinite length would avoid monochromatic solutions to the system: Define $\chi : [1, \infty) \to \{0, 1\}$ so that $\chi(i) = 0$ if $i = 2n$ for some $n \in \mathbb{N}$ and $\chi(i) = 1$ if $i = 2m - 1$ for some $m \in \mathbb{N}$. Using this method results in an alternating color scheme where all even numbers are red and all odd numbers are blue.

Given this coloring, consider trying to create a red solution to the system. Plugging $i_1 = 2n_1$ and $i_2 = 2n_2$, where $n_1, n_2 \in \mathbb{N}$, into L_k gives $2n_1 + 2n_2 + 1 = 2(n_1 + n_2) + 1$. Since $n_1, n_2 \in \mathbb{N}$, it follows that $2n_1 + 2n_2 + 1 = 2z + 1$ where $z \in \mathbb{N}$. Since $2z + 1$ is odd, it follows that $\chi(2z+1) = 1$ and thus that it is not possible to create a red solution to L_k and therefore to our system. A similar process will show that it is impossible to create a blue solution to L_c. Therefore, there cannot be a monochromatic solution to the entire system, regardless of the length of the coloring.

Systems of Disjoint Linear Equations

We will now seek to show that the results from sections 2 and 3 can be expanded to cover systems involving more than 2 equations of the same type. This will be done using induction, but first we must introduce some new notation.

Notation 6. *Let $\mathcal{S}_n$ denote a system of disjoint linear equations $E_1, E_2, \ldots, E_n$. The Rado number for this system, if it exists, is the smallest positive integer $R(\mathcal{S}_n)$ such that any coloring of the integers 1 through $R(\mathcal{S}_n)$, there exists a monochromatic solution to each equation in the system $\mathcal{S}_n$ and these solutions are the same color.*

Notation 7. *Let $\{E_i, E_j\}$ denote a pairing of two equations E_i and E_j forming a subsystem of $\mathcal{S}_n$.*

Theorem 6. *Suppose that $\mathcal{S}_n$ is a system of equations $E_1, E_2, \ldots, E_n$ where R_k denotes the Rado number for E_k. Furthermore, suppose that for any pair of positive integers $i, j \leq n$, the 2-color Rado number for the subsystem $\{E_i, E_j\}$ is equal to the max of R_i and R_j. Then the Rado number for the system is $R(\mathcal{S}_n) = max\{R_1, R_2, \ldots, R_n\}$.*

Proof. Consider each possible subsystem of the form $\{E_i, E_j\}$. We know that each subsystem of this form contains a monochromatic solution to the subsystem of equations in the integers 1 through $\max\{R_1, R_2, \ldots, R_n\}$ by the definition of our system. If each of these solutions are monochromatic in one color, without loss of generality red, then the lemma statement holds. Therefore, assume that all subsystems of $\mathcal{S}_n$ are monochromatic in red except for some $\{E_{k_1}, E_{k_2}\}$ which does not yield a monochromatic red solution. This implies that at least one of E_{k_1} or E_{k_2} does not have a monochromatic red solution; assume without loss of generality that E_{k_1} is the equation that meets this criteria. We then know that E_{k_1} must have a monochromatic blue solution. Therefore, since each subsystem of our equation yields a monochromatic solution, we know that there must be a blue solution to $\{E_{k_1}, E_j\}$ for all $1 \leq j \leq n$, giving us a monochromatic blue solution to each equation in our system. Therefore, there must be a monochromatic solution to the system of equations in the integers from 1 through $R(E_N)$. □

From this, we know the following corollary statements must be true. Note that these corollary statements apply our theorem to specific systems involving equations of the B-B and B-L form.

Corollary 1. *For a system of disjoint linear equations $\mathcal{S}_n$ where each equation is of the form $x_1 + x_2 + \cdots + x_m = x_0$ for some $m \in \mathbb{N}$ and where R_k represents the Rado number for equation k, the Rado number for the system is $R(\mathcal{S}_n) = max\{R_1, R_2, \ldots, R_n\}$.*

Corollary 2. *For a system of disjoint linear equations $\mathcal{S}_n$ where each equation is of the form $x_1 + x_2 + c = x_0$ with $c \in \mathbb{N}$, all distinct values of c have the same parity, and R_k represents the Rado number for equation k, the Rado number for the system is $R(\mathcal{S}_n) = max\{R_1, R_2, \ldots, R_n\}$.*

We have now established that the results for a two-equation linear system extend to systems of n equations of like types, both in the case of equations of the form $x_1 + x_2 + \cdots + x_n = x_0$ and for equations of the form $x_1 + x_2 + c = x_0$. It should be noted that these systems were addressed separately and no result was established for a mixed system, although the author believes a result could be established for such systems. The reason for this will be explained further in the Future Work section.

Future Work

Future problems that can be investigated include similar results for non-disjoint systems as discussed in Conjecture 1. One could also investigate similar results for different types of equations that do not fit the forms defined in this paper.

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