Finitizing Mileti's Canonical Ramsey Theorem Proof

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1 Abstract

Imagine that Alice colors every natural number one of two colors, either red or blue. For example, Alice can color every even number red and every odd number blue. Using that coloring, Bob could find an infinite subset of the natural numbers that is all one color (either an infinite set of all even numbers or all odd numbers would work in the above example). This is called an infinite homogenous set.

Now imagine that instead of coloring each natural number either red or blue, Alice can color them with an infinite number of colors (each natural number is colored with another natural number). For example, Alice could color x as $\left\lfloor \frac{x}{2} \right\rfloor$. In this case, if Bob considered the set of even natural numbers, each would have a distinct coloring. Any set like this one, where every natural number is colored differently, is called a rainbow set.

Consider if instead of coloring each natural number, Alice colored unordered pairs of distinct natural numbers. For example, given (x, y) such that $x < y$, Alice could color the pair as 0 if $x + y$ is even, and 1 if $x + y$ is odd. In this example, if Bob took only the set of even natural numbers, the sum of any two even numbers is even as well. This means that

every pair of even numbers would be colored as 0. This type of set can be considered as infinite homogenous, and is analogous to how we define a homogenous set for a coloring of natural numbers.

Similarly, given a set of natural numbers such that any two pairs are colored differently, we can consider that set to be rainbow. An example of this given (x, y) where $x < y$ is to color the pair as the product of the x^{th} and y^{th} prime numbers. In this example, the set of all natural numbers is rainbow, since no two pairs have the same coloring.

Whether coloring natural numbers or pairs of natural numbers, if we only use a finite number of colors, we are always guaranteed an infinite homogenous set. The theorem that guarantees this is called Ramsey's Theorem.

On the other hand, rainbow sets are not guaranteed to exist. However, the Canonical Ramsey Theorem guarantees one of four possible options, including an infinite rainbow set and an infinite homogenous set. There are three proofs of this theorem, two of which use use a mathematical object known as a hypergraph. The third one, Mileti's proof, avoids using these objects, and instead only uses ordinary graphs.

The Ramsey Theorem and the Canonical Ramsey Theorem were both initially created for colorings of countably infinite sets. However, they can be adapted for finite sets. For the Canonical Ramsey Theorem to guarantee a set of size k, the finitization of the two hypergraph-based proofs have bounds of $4^{4^{O(k^3)}}$ and $16^{16^{16^{O(k)}}}$ respectively. We present a clear finite version of Mileti's proof for Canonical Ramsey with a bound of $k^{2^{O(k^4)}}$.

2 Definitions

Notation 2.1

1. If $n \in \mathsf{N}$ then $[n]$ is $\{1, \ldots, n\}$

2. If A is a set and $k \in \mathsf{N}$ then $\binom{A}{k}$ is the set of k-elements subsets of A.

3. (Combining the above two notations) If $n \in \mathsf{N}$ then $\binom{[n]}{k}$ is the set of all k-subsets of $\{1, \ldots, k\}$.

Notation 2.2 When we have v_1, v_2 it is assumed $v_1 < v_2$. Same for u_1, u_2.

Notation 2.3 When we have vertex v and color c, $\deg_c(v)$ is the number of edges of color c that include v as one of their vertices.

Def 2.4 Let $\mathrm{COL}\colon \mathsf{N} \to \omega$. Let $H \subseteq \mathsf{N}$.

1) H is homogenous (henceforth homog) if

$$(\exists c \in \omega)(\forall v \in H)[\mathrm{COL}(v) = c].$$

2) H is rainbow if

$$(\forall v_1, v_2 \in H)[COL(v_1) = COL(v_2) \leftrightarrow v_1 = v_2].$$

Def 2.5 Let $\mathrm{COL}\colon \binom{\mathsf{N}}{2} \to \omega$. Let $H \subseteq \mathsf{N}$.

1) H is homog if

$$(\exists c \in \omega)(\forall v_1, v_2 \in H)[COL(v_1, v_2) = c].$$

2) H is min-homog if

$$(\forall v_1, v_2, u_1, u_2 \in H)[COL(v_1, v_2) = COL(u_1, u_2) \leftrightarrow v_1 = u_1].$$

3) H is max-homog if

$$(\forall v_1, v_2, u_1, u_2 \in H)[COL(v_1, v_2) = COL(u_1, u_2) \leftrightarrow v_2 = u_2].$$

4) H is rainbow if

$$(\forall v_1, v_2, u_1, u_2 \in H)[COL(v_1, v_2) = COL(u_1, u_2) \leftrightarrow (v_1 = u_1) \wedge (v_2 = u_2)].$$

(So every edge is a different color.)

3 1-ary Finite Can Ramsey

Theorem 3.1 *Let $k \in \mathsf{N}$. Let $f(k) = (k-1)^2 + 1$. For any coloring* $\mathrm{COL} : [f(k)] \to \omega$ *one of the following occurs:*

1. *There exists a homog set of size $\geq k$.*
2. *There exists a rainbow set of size $\geq k$.*

Proof: For $c \in \omega$ let

$$A_c = \{v \in A : \mathrm{COL}(v) = c\}.$$

There are two cases:

Case 1 $(\exists c \in \omega)[|A_c| \geq k]$. Then A_c is a homog set of size $\geq k$.

Case 2 $(\forall c \in \omega)[|A_c| \leq k-1]$. Assume that L colors were used on $[f(k)]$. By renumbering we take those colors to be $1, \ldots, L$. Then

$$[f(k)] = A_1 \cup \cdots \cup A_L.$$

Hence

$$f(k) = |A_1| + \cdots + |A_L| \le L(k-1).$$

Hence

$$L \ge \left\lceil \frac{f(k)}{(k-1} \right\rceil = \left\lceil \frac{(k-1)^2+1}{k-1} \right\rceil = k$$

Therefore at least k colors are used. Let H be a set that uses one of each color. Then $|H| \ge k$ and H is rainbow.

▮

The following corollary will be useful.

Corollary 3.2 *Let* $n \in \mathsf{N}$*. For any coloring* $\mathrm{COL}\colon [n] \to \omega$ *one of the following occurs.*

1. *There exists a homog set of size* $\ge \sqrt{n}$*.*
2. *There exists a rainbow set of size* $\ge \sqrt{n}$*.*

Proof: Let k be the largest number such that $n \ge (k-1)^2+1$. Restrict COL to $[(k-1)^2+1$. By Theorem 3.1 there exists either a homog or rainbow set of size k. Hence we need a lower bound on k.

We show that $k \ge \sqrt{n}$:

$(\sqrt{n}-1)^2 = n - 2\sqrt{n} + 1 < n$ ▮

4 2-ary Finite Can Ramsey

Lemma 4.1 *Let* $k \in N$*, let* $f(k) = 2k^3$*. For* $\mathrm{COL}\colon \binom{[f(k)]}{2} \to \omega$ *such that* $\forall v \in [f(k)]$*,* $\deg_c(v) \le 2$*, there exists a rainbow set of size* $\ge k$*.*

Proof: Let $V = [f(k)]$, let $R \subseteq V$ be the maximal rainbow set, let $U = V - R$, let $C = \{\text{COL}(r_1, r_2) : r_1, r_2 \in R\}$. Assume $|R| \leq k - 1$. This means that $|C| \leq \binom{k-1}{2}$.

For each $u \in U$, map it to (r, c) such that $r \in R$, $c \in C$, and $\text{COL}(r, u) = c$. There are $|R| * |C| \leq (k-1)\binom{k-1}{2}$ distinct combinations for (r, c).

It can be shown that

$$2k^3 > (k-1) + 2(k-1)\binom{k-1}{2}.$$

This means that

$$|U| > 2(k-1)\binom{k-1}{2} \geq 2|R||C|.$$

Since we know that there are more than double the vertices in U than there are combinations of (r, c), we know that there is at least one (r, c) such that three vertices in U map to it.

However, this is impossible, as $deg_c(v) \leq 2$, so this is a contradiction. Thus, our assumption that $|R| \leq k - 1$ cannot be true, and we know that $|R| > k - 1$, or $|R| \geq k$. As such, we are guaranteed a rainbow set of at least size k.

▮

Lemma 4.2 *Let $a \geq 1$ and $b \geq 1$ such that $n = ab$. For all* $\text{COL} : [n] \to \omega$*, at least one of the following occurs.*

1. *There exists a rainbow set of size $\geq a$.*

2. *There exists a homog set of size $\geq b$.*

Proof: Let $V = [n]$. Let $C = \{\text{COL}(v) : v \in V\}$.

There are two cases:

Case 1 $|C| \geq a$. Then there is at least one vertex with each color $c \in C$, so there are $\geq a$ vertices all of different colors.

Case 2 $|C| \leq a$. Then we have at most a colors, so by 1-dimensional Ramsey Theory, there is a homog set of size $\geq \frac{n}{|C|} \geq \frac{n}{a} = b$.

■

Notation 4.3 Let $k \in N$. We define the following sequence.

$a_0 = (4k^4)^{2^{4k^4}-2}$

$a_{i+1} = \left\lceil \frac{\sqrt{a_i - 1}}{i+1} \right\rceil$

Lemma 4.4 *Let $k \in N$. For all a_i such that $0 \leq i \leq 4k^4 - 2$, $a_i \geq 1$.*

Proof: Define sequence b_i such that

$b_0 = (4k^4)^{2^{4k^4}-2}$

$b_{i+1} = \frac{\sqrt{b_i}}{4k^4}$

It can be seen that $b_0 = a_0$ and $b_i = (4k^4)^{2^{4k^4-i}-2}$. Thus, $b_{4k^4-1} = 1$.

We want to show that

$$\forall(0 \leq i \leq 4k^4 - 1)[a_i \geq 1]$$

As both a_i and b_i are decreasing, it is sufficient to show that $a_i \geq b_i > b_{4k^4-1} = 1$.

Assume $a_i \geq b_i$ and $b_{i+1} > a_{i+1}$. It can be seen that

$$a_{i+1} \geq \frac{\sqrt{a_i - 1}}{i+1} \geq \frac{\sqrt{a_i - 1}}{4k^4 - 1} > \frac{\sqrt{a_i - 1}}{4k^4}.$$

This means that $a_i < (4k^4 a_{i+1})^2 + 1$, so $a_i \leq (4k^4 a_{i+1})^2$ since $a_i, a_{i+1} \in N$.

At the same time, we can see that

$$b_i = (4k^4 b_{i+1})^2 > (4k^4 a_{i+1})^2 \geq a_i.$$

This means that $b_i > a_i$, which contradicts $a_i \geq b_i$. As such,

$$a_i \geq b_i \rightarrow a_{i+1} \geq b_{i+1}.$$

Since $a_0 = b_0$, we then know that $\forall(0 \leq i \leq 4k^4 - 2)[a_i \geq b_i]$. Because b_i is a decreasing sequence and $b_{4k^4-1} = 1$, this is sufficient to conclude that

$$\forall(0 \leq i \leq 4k^4 - 2)[a_i \geq 1].$$

$\blacksquare$

Theorem 4.5 *Let $k \in \mathsf{N}$. Let $f(k) = (4k^4)^{2^{4k^4}-2}$. Note that $f(k) = k^{2^{O(k^4)}}$. For all* COL: $\binom{[f(k)]}{2} \rightarrow \omega$ *one of the following occurs.*

1. *There exists a homog set of size $\geq k$.*
2. *There exists a min-homog set of size $\geq k$.*
3. *There exists a max-homog set of size $\geq k$.*
4. *There exists a rainbow set of size $\geq k$.*

Proof:

We construct a sequence of vertices $x_1, \ldots, x_{4k^4-1}$, a sequence of sets $V_0, \ldots, V_{4k^4-1}$, and a coloring COL* of $\{x_1, \ldots, x_{4k^4-1}\}$.

CONSTRUCTION

Stage 0

$V_0 = [f(k)]$. Note that $|V_0| = a_0$.

x_1 is the least element of V_0.

Stage i

Assume that V_{i-1} is defined and $|V_{i-1}| \geq a_{i-1}$, and let x_i be the least element of V_{i-1}. By Lemma 4.4, we know that $|V_{i-1}| \geq a_{i-1} \geq 1$, so there is always some least element $x_i \in V_{i-1}$.

$Y = V_{i-1} - \{x_i\}$. (We will remove many elements from Y and what is left will be V_i.)

We will define $\mathrm{COL}^*(x_i)$ and define V_i.

$\mathrm{COL}'\colon Y \to \omega$ via $\mathrm{COL}'(y) = \mathrm{COL}(x_i, y)$.

Apply Corollary 3.2 to COL'. One of the following occurs.

Case 1 If we get a homog set of color c, then $\mathrm{COL}^*(x_i) = (H, c)$. We call the homog set Y_c. Note that

$$|Y_c| \geq \sqrt{|Y|} = \sqrt{|V_{i-1}| - 1} \geq \sqrt{a_{i-1} - 1} \geq \left\lceil \frac{\sqrt{a_{i-1} - 1}}{i} \right\rceil = a_i.$$

Let $V_i = Y_c$.

Case 2 If we get a rainbow set then we call that set Y_ω. Note that

$$|Y_\omega| \geq \sqrt{|Y|} = \sqrt{|V_{i-1}| - 1} \geq \sqrt{a_{i-1} - 1}.$$

For now let $V_i = Y_\omega$, though we will do a lot more thining out of V_i to get to our final V_i.

We define a coloring $\mathrm{COL}''\colon V_i \to [i]$ as follows:

1. Input $y \in V_i$.
2. We now iterate through x_j's from x_1 to x_{i-1}.
   (a) If $\mathrm{COL}^*(x_j) = (H, c)$ then goto the next j.
   (b) Let $\mathrm{COL}^*(x_j) = (RB, k_j)$. If $\mathrm{COL}(x_j, y) = \mathrm{COL}(x_i, y)$ then $\mathrm{COL}''(y) = (RB, k_j)$. (Technically the color is not in $[i]$ but the number of colors is still i so we ignore

this point.)

It can be shown that if there exists some $1 \le a \le i-1$ such that $a \ne j$ and $\mathrm{COL}^*(x_a) = (RB, k_a)$, then either $k_a = k_j$ or $\mathrm{COL}(x_a, y) \ne \mathrm{COL}(x_j, y)$ and so $\mathrm{COL}(x_a, y) \ne \mathrm{COL}(x_i, y)$. This means that iterating further through the x_j's will not change $COL''(y) = (RB, k_j)$, so we can stop iterating.

3. If you got here then $\nexists 1 \le j \le i-1$ such that $\mathrm{COL}(x_j, y) = \mathrm{COL}(x_i, y)$.

   This means that $(\forall 1 \le j \le i-1)[\mathrm{COL}(x_j, y) \ne \mathrm{COL}(x_i, y)]$.

   In this case $\mathrm{COL}''(y) = (RB, k)$ where k is the least number that is not any of the k_j's.

Note that there are only i colors since, in the worst case, for $1 \le j \le i-1$ each x_i has a different rainbow color, and you may also have to use the case where none of them agree.

Hence we apply the 1-dimensional homog Ramsey Theorey (not 1-dimensional Can Ramsey) on V_i using the COL'' coloring. Since $|V_i| = ia_i$, and we use i colors, there is a homog set of size $\ge \left\lceil \frac{\sqrt{a_{i-1}-1}}{i} \right\rceil = a_i$. Let this homog set be V_i. Let $\mathrm{COL}^*(x_i)$ be the color of the homog set using COL''. Note that it will be of the form (RB, j).

END OF CONSTRUCTION

We now have a sequence

$$x_1, x_2, \ldots, x_{4k^4-1}.$$

There are cases depending on COL^*.

Case 1 There exists $2k^4$ vertices with color of the form $(H, -)$. By Corollary 3.2 one of the following must occur.

Case 1.1 There exists c such that $\ge \sqrt{2}k^2$ of these vertices are colored (H, c). Those vertices form a homog set of size $\ge k$ where every pair has color c.

Case 1.2 There exists $\geq \sqrt{2}k^2$ vertices with all different c's. Those vertices form a min-homog set of size $\geq k$. We leave the proof to the reader.

Case 2 There exists $2k^4$ vertices with color of the form $(RB, -)$. By Lemma 4.2 one of the following must occur.

Case 2.1 There exists c such that $\geq k$ of these vertices are colored (RB, c). Those vertices form a max-homog set of size $\geq k$. We leave the proof to the reader.

Case 2.2 There exists $\geq 2k^3$ vertices with all different c's. We show that those vertices form a psuedo-rainbow set of size $\geq k$. Let $v_1 < v_2$ and $u_1 < u_2$ be four of the vertices. We look at all possibilities of equalities and inequalities among the v_1, v_2, u_1, u_2.

If $v_1 < u_1$ or $u_1 < v_1$ and $v_2 = u_2$ then $\mathrm{COL}(v_1, v_2) \neq \mathrm{COL}(u_1, u_2)$ since $\mathrm{COL}^*(v_1)$ and $\mathrm{COL}^*(u_1)$ are different rainbow colors. We shorten this to say

$$((v_1 \neq u_1) \wedge (v_2 = u_2)) \implies \mathrm{COL}(v_1, v_2) \neq \mathrm{COL}(u_1, u_2).$$

In addition, because each vertex is colored $(RB, -)$, given $v_1 = u_1$ and $v_2 < u_2$ or $u_2 < v_2$, then $COL(v_1, v_2) \neq COL(u_1, u_2)$ because only vertices with edges with v_1 that form a rainbow set are selected after it. We shorten this to say

$$((v_1 = u_1) \wedge (v_2 \neq u_2)) \implies \mathrm{COL}(v_1, v_2) \neq COL(u_1, u_2).$$

These two statements together mean that there can be at most two edges from a given vertex of any color, so for any vertex v, $\deg_c(v) \leq 2$. This means that we can apply Lemma 4.1 to construct a rainbow set of size $\geq k$.

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5 Future Research

This paper creates a finite version of Mileti's proof of the 2-ary Canonical Ramsey Theorem, which states that any infinite graph has either an infinite rainbow, homog, min-homog or max-homog set of vertices. This can be generalized as the a-ary Canonical Ramsey Theorem, which states that for COL : $\binom{N}{a} \to \omega$, there exists an infinite I-homog set for $I \subseteq [a]$. Further research could include the creation of a finite proof in the style of the 2-ary Mileti proof for the a-ary Canonical Ramsey Theorem.

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References

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