An Exposition of Two Proofs of Can VDW

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1 Introduction

We need the following terinology.

Notation 1.1 Let $W \in \mathsf{N} \cup \omega$. Let $k, c \in \mathsf{N}$.

1. A *monochromatic k-AP* is an arithmetic sequence of length k where all of the elements of it are the same color. Henceforth we use the abbreviation *mono k-AP*.

2. A *rainbow k-AP* is an arithmetic sequence of length k where all of the elements have different colors. Henceforth we use the abbreviation *RB k-AP*.

Recall VDW's Theorem:

Theorem 1.2 *For all k, c there exists $W = W(k, c)$ such that for all $\mathrm{COL}\colon [W] \to [c]$ there is a mono k-AP.*

What if there is no bound on the number of colors? Clearly there is a coloring of ω without a mono k-AP: $\mathrm{COL}(x) = x$. However, in that coloring, you have a k-RB. This counterexample motivates the Can VDW Theorem:

Theorem 1.3 *For all k there exists $C = W(k$ such that for all* $\text{COL}\colon [C] \rightarrow \mathsf{N}$ *there is either a mono k-AP or a RB k-AP.*

We present several proofs of the Can VDW theorem and discuss their pros and cons.

2 Proof that Does Not Use Gallai-Witt

2.1 The Set Up

We will prove a generalization of the Can VDW theorem which effectively loads the induction hypothesis. There are several changes we make:
1) We replace the range N with the range N^m. This may seem trivial; however, the next item will use the new range in an interesting way.

Notation 2.1 Let $C, m \in \mathsf{N}$. Let $\text{COL}\colon [C] \rightarrow \mathsf{N}^m$. Then $\text{COL}_i\colon [C] \rightarrow \mathsf{N}^m$ is the function that, on input x, outputs the ith element of $\text{COL}(x)$.

2) We will **redefine** mono k-AP and RB k-AP.

Def 2.2 Let $k, m, C \in \mathsf{N}$. Let $\text{COL}\colon [C] \rightarrow \mathsf{N}^m$.

1. A *mono k-AP* is a sequence $a, a+d, \ldots, a+(k-1)d$ such that

   $$(\exists 1 \le i \le m)[\text{COL}_i(a) = \text{COL}_i(a+d) = \cdots = \text{COL}_i(a+(k-1)d)].$$

   Note that this new definition of mono k-AP is weaker then the usual definition of mono k-AP which would insist that

   $$(\forall 1 \le i \le m)[\text{COL}_i(a) = \text{COL}_i(a+d) = \cdots = \text{COL}_i(a+(k-1)d)].$$

Also note that if $m = 1$ then the old and new definition of mono k-AP are the same.

2. A *RB k-AP* is a sequence $a, a+d, \ldots, a+(k-1)d$ such that

$$(\forall 0 \le x < y \le k-1)(\forall 1 \le i < j \le m)[\mathrm{COL}_i(a+xd) \ne \mathrm{COL}_j(a+yd)].$$

For example, we have.

$$\{\mathrm{COL}_1(a+2d), \mathrm{COL}_2(a+2d), \ldots, \mathrm{COL}_2(a+(k-1)d)\} \bigcap$$

$$\{\mathrm{COL}_1(a+7d), \mathrm{COL}_2(a+7d), \ldots, \mathrm{COL}_7(a+(k-1)d)\} = \emptyset$$

Note that this new definition of RB k-AP is stronger then the usual definition of RB k-AP which would only requires

$$(\forall 0 \le x < y \le k-1)(\exists 1 \le i \le m)[\mathrm{COL}_i(a+xd) \ne \mathrm{COL}_i(a+yd)].$$

Also note that if $m = 1$ then the old and new definition of RB k-AP are the same.

3) We will prove the following asymmetric version of the Can VDW: For all $k, t, m \in \mathsf{N}$ there exists C such that, for all $\mathrm{COL}\colon [C] \to [\mathsf{N}]^m$, one of the following occurs:

- There is a mono k-AP.
- There is a RB t-AP.

This theorem implies the Can VDW theorem by setting $t = k$ and $m = 1$.

2.2 The Proof

Theorem 2.3 *For all $k, t, m \in \mathsf{N}$ there exists C such that, for all* $\mathrm{COL}\colon [C] \rightarrow [\mathsf{N}]^m$, *one of the following occurs:*

- *There is a mono k-AP.*
- *There is a RB t-AP.*

Proof:

We prove this by induction on t.

Base Case: $t = 1$. For any k, m can take $C(k, 1, m) = 1$.

The $t = 1$ case is so trivial that we do the $t = 2$ case.

Base Case: $t = 2$. Let $\mathrm{COL}\colon [C] \rightarrow \mathsf{N}^m$ where we determine C later.

If there is a RB 2-AP then we are done. So assume there is no RB 2-AP. Let

$\mathrm{COL}(1) = (c_1, \ldots, c_m)$.

For all $x \geq 2$, there has to be an i, j such that $\mathrm{COL}_i(x) = c_j$. We define a coloring using this fact.

$\mathrm{COL}'\colon [C] - \{1\} \rightarrow [m] \times [m]$ is defined by

$\mathrm{COL}'(x) = (i, j)$ where $\mathrm{COL}_i(x) = c_j$.

We want to use VDW's theorem on COL'. Let $C = W(k, m) + 1$. Then there exists a, d, i, j such that

$$\mathrm{COL}'(a) = \mathrm{COL}'(a + d) = \cdots = \mathrm{COL}'(a + (k - 1)d) = (i, j).$$

Note that

$$\mathrm{COL}_i(a) = \mathrm{COL}_i(a + d) = \cdot = \mathrm{COL}_i(a + (k - 1)d) = c_j.$$

Hence we get a mono k-AP.

Case that contains most of the ideas: $t = 3$, $m = 1$ We can assume that, for all k, m, $C(k, 2, m)$ exists.

Let COL: $[C] \rightarrow \mathsf{N}$ where we determine C later.

Break $[C]$ into (suggestively named, TBD) C' blocks of size (suggestively named, TBD) m. Call the blocks $B_1, B_2, \ldots, B_{C'}$.

We define a coloring

COL$'$: $[C'] \rightarrow \mathsf{N}^m$ by

COL$'(x) = (COL_1(B_x), COL_2(B_x), \ldots, COL_m(B_x))$

We take $C' \geq C(k, 2, m)$.

Inductively we have two cases.

Case 1: There is a mono k-AP, in which case we are done (overdone really since we would have a mono k-AP under the old definition)

Case 2: There is a RB 2-AP x, y. Let z be such that x, y, z is a 3-AP. In order for z to be in the domain of COL$'$. we need $C' \geq 2C(k, 2, m)$.

Let A, D, M be such that

- $B_x = (A, A + 1, \ldots, M - 2, M - 1, M)$
- $B_y = (A + D, A + D + 1 \ldots, M + D - 2, M + D - 1, M + D)$
- $B_z = (A + 2D, a + 2D + 1 \ldots, M + 2D - 2, M + 2D - 1, M + 2D)$

Since x, y is a RB 2-AP we have

(1) $\{\mathrm{COL}(A), \mathrm{COL}(A{+}1), \mathrm{COL}(A{+}2), \ldots, \mathrm{COL}(M)\} \cap \{\mathrm{COL}(A{+}D), \mathrm{COL}(A{+}D{+}1), \mathrm{COL}(A{+}2), \ldots,$

We consider COL$(M + 2d)$. Look at each 3-AP (written backwards)

$M + 2D, M + D, M$

$M + 2D, M + D - 1, M - 2$

$M + 2D, M + D - 2, M - 4$

$M + 2D, M + D - 3, M - 6$

$\vdots$

$M + 2D, M + D - x, M - 2x$

$\vdots$

$M + 2D, M + D - \frac{M-A}{2}, A$

Lets look at $M+2D, M+D-x, M-2x$. We know that $\mathrm{COL}(M+D-x) \neq \mathrm{COL}(M-2x)$ by (1). If $\mathrm{COL}(M + 2D) \notin \mathrm{COL}(M + d - x) \neq \mathrm{COL}(M - 2x)$ then we would have a RB 3-AP. This observation motivates the following coloring.

$\mathrm{COL}''\colon B_y \to [2]$

$$\mathrm{COL}''(w) = \begin{cases} 1 & if \mathrm{COL}''(w) = \mathrm{COL}(M + 2D) \\ 2 & if \mathrm{COL}''(w) \neq \mathrm{COL}(M + 2D) \end{cases} \quad (1)$$

We will take M (the size of the blocks) to be at least $W(2, k)$. Hence we apply VDWs Theorem to COL' and get two cases.

Case 2.1: There is a mono k-AP with color 1. The mono k-AP actually a mono k-AP with color COL.

Case 2.2: There is a mono k-AP with color 2. So we have

$\mathrm{COL}(M + 2D) \neq \mathrm{COL}(M + D)$ and $\mathrm{COL}(M + D) \neq \mathrm{COL}(M)$. If $\mathrm{COL}(M + 2D) \neq \mathrm{COL}(M)$ then we have a RB 3-AP hence we can assume $\mathrm{COL}(M + 2D) = \mathrm{COL}(M)$.

Similarly $\mathrm{COL}(M + 2D) = \mathrm{COL}(M - 2)$.

$\vdots$

Similarly, for $1 \leq x \leq 0$, $\mathrm{COL}(M + 2D) = \mathrm{COL}(M + D - x)$.

Hence we have the following mono k-AP.

$$A, A+1, \ldots, M$$

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