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ON THE DIFFERENT DISTANCES DETERMINED BY n POINTS*

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1. Introduction. n distinct points will always determine $\binom{n}{2}$ distances. However, many of these may be the same. We are concerned here with the number of different distances determined. Let $f(n)$ be the least number of different distances determined by n points in a plane. The vertices of an equilateral triangle show that $f(3)=1$ and from the square and the regular pentagon one easily sees that $f(4)=f(5)=2$. P. Erdös** has shown recently that

$$cn/\sqrt{\log n} > f(n) > \sqrt{n-1} - 1, \tag{1}$$

where c is a fixed constant. The upper bound was obtained by considering the points of a square lattice. Erdös conjectured that $f(n) > n^{1-\epsilon}$ for every $\epsilon > 0$ and n sufficiently large. The smallness of the lower bound is therefore rather striking.

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** P. Erdös. On Sets of Distances of n Points. this MONTHLY, pp. 248–250, Vol. 53, 1946.

Nevertheless Erdös wrote that he had long sought to improve it without success. In Section 2 we will prove Theorem 1:

$$f(n) > \frac{n^{2/3}}{2\sqrt[3]{9}} - 1. \tag{2}$$

If the n points form the vertices of a convex polygon, then the least number $f^*(n)$ of different distances determined is much larger. In this case Erdös conjectured that

$$f^*(n) = [n/2], \tag{3}$$

which is strongly suggested by the vertices of the regular polygons, which show that $f^*(n) \leqq [n/2]$. In Section 3 we will prove Theorem II:

$$f^*(n) \geqq [(n+2)/3]. \tag{4}$$

2. Case of n arbitrary points. We first prove two lemmas.

LEMMA 1. *Let r be a positive integer, ϵ a real number, $0 < \epsilon \leqq 1$. Let P be the point $(r+\epsilon, 0)$ and Q and R two points in the first quadrant, equidistant from P, and whose distances from the origin lie between r and $r+1$. Then $\overline{QR} < 2$.* Note Figure 1.

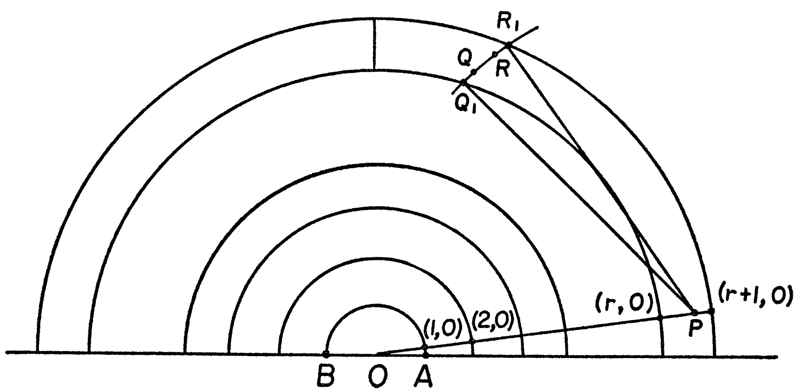

FIG. 1

Proof. Let the circle with center at P and passing through Q and R, cut the circles $x^2+y^2=r^2$ and $x^2+y^2=(r+1)^2$ in $Q_1(x, y)$ and $R_1(x+\Delta x, y+\Delta y)$, respectively. Clearly $\overline{Q_1R_1} \geqq \overline{AR}$ so it will suffice to show that $\overline{Q_1R_1} < 2$ or $\Delta x^2 + \Delta y^2 < 4$. We have

$$x^2 + y^2 = r^2 \tag{5}$$

$$(x+\Delta x)^2 + (y+\Delta y)^2 = (r+1)^2 \tag{6}$$

while $\overline{PQ_1} = \overline{PR_1}$ yields

$$(x - r - \epsilon)^2 + y^2 = (x + \Delta x - r - \epsilon)^2 + (y + \Delta y)^2. \tag{7}$$

Simplification of (7), using (5) and (6), yields

$$\Delta x = 1 + \frac{1 - 2\epsilon}{2r + 2\epsilon}, \tag{8}$$

so that

$$1 - \frac{1}{2r} < \Delta x < 1 + \frac{1}{2r} \leqq 3/2. \tag{9}$$

It is interesting to note that Δx is independent of $\overline{PR}$. From (5) and (6) we have

$$2x\cdot\Delta x + 2y\cdot\Delta y + \Delta x^2 + \Delta y^2 = 2r + 1. \tag{10}$$

Further, we may assume $\Delta x^2 + \Delta x^2 > 3$, for otherwise our lemma is proved. Hence

$$x\cdot\Delta x + y\cdot\Delta y < r - 1. \tag{11}$$

Now, using the left hand side of (9) to estimate Δx and the fact that $x \leqq r$ and $x \geqq 0$, we obtain from (11)

$$\Delta y < \frac{r - 1 - x\cdot\Delta x}{y} < \frac{r - x}{y} = \frac{r - x}{\sqrt{r^2 - x^2}} = \sqrt{\frac{r - x}{r + x}} \leqq 1. \tag{12}$$

The lemma now follows from (12) and the right hand side of (9).

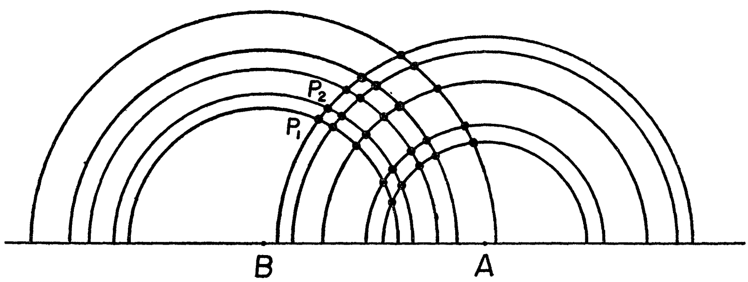

FIG. 2

LEMMA 2. *Given two points A and B and n other points $P_1, P_2, \cdots, P_n$, lying on, or to one side of the line AB. Of the distances $AP_1, AP_2, \cdots, AP_n$, $BP_1, BP_2, \cdots, BP_n$, at least $\sqrt{n}$ are distinct.*

Proof. Consider all semicircles to one side of AB having centers at A and B passing through the points $P_1, P_2, \cdots, P_n$ (Fig. 2). Let the number of distinct semicircles with centers at A and B be a and b, respectively. If max $\{a, b\} \geqq \sqrt{n}$ the lemma is clearly true. If, however, $a < \sqrt{n}$ and $b < \sqrt{n}$ then the number of intersection points is at most $a\cdot b < \sqrt{n}\cdot\sqrt{n} = n$, since any two of the semicircles intersect at most once. But this yields a contradiction since each of the n distinct points $P_1, P_2, \cdots, P_n$ is an intersection point of two of these semicircles.

Hence the lemma is proved.

We note that this lemma is best possible in that, in it, it is not possible in general to replace $\sqrt{n}$ by a larger number. Further, we note that by choosing A and B to be consecutive points on the convex cover of the set of n points the lemma yields

$$f(n) \geqq \sqrt{n-2}, \tag{13}$$

which is already a little better than the right hand side of (1). Indeed, from (13) and the regular heptagon we may deduce that $f(7)=3$.

We proceed to the proof of Theorem I. Let A and B be two points determining the minimum distance in the set of n points. Let us denote this distance by 2. One of the half planes determined by AB, including the points on the line AB, contains at least $n/2$ points. Henceforth we will deal only with these points. Let O be the midpoint of AB. With center at O we construct semicircles of radii $1, 2, 3, \cdots$ cutting the half plane into half-annular boxes which we make open on the inside and closed on the outside (Fig. 1). Let s be a number, not necessarily an integer which for the present we restrict only by $1 \leqq s \leqq n$. We consider 2 cases.

Case 1. Some box contains at least s points.

Case 2. No box contains as many as s points.

In Case 1, we take a box containing at least s points. This we cut into two equal parts by a line through O, and retain only one half which contain in the interior or on its boundary at least $s/2$ points. Let P be a point in this region (Fig. 1) making the angle AOP as small as possible. Suppose there are two other points Q and R in the region equidistant from P. By Lemma 1 we have $QR<2$, which contradicts the fact that 2 is the minimum distance determined. Hence, no two points in the region are equidistant from P. We obtain therefore at least $s/2-1$ different distances determined from P so that, in Case 1, we have

$$f(n) \geqq \frac{s}{2} - 1. \tag{14}$$

In Case 2, we divide the half-annular boxes into 3 classes, putting a box in class i if its outer radius $r \equiv i \pmod 3$, $i=1, 2, 3$. Now if A and B are excluded, at least one of the classes will contain at least the following number of points:

$$\frac{(n-2)}{2} \cdot \frac{1}{3} = \frac{n-2}{6}.$$

We retain now only A and B and the points of such a class of boxes. If d is any distance determined between A or B and any of the points of our class in a box outer radius r, then,

$$r-2 < d \leqq r+1,$$

while for a distance d' determined between A or B and a point of a neighboring box of the class, say the one of outer radius $r+3$,

$$r + 1 < d' \leqq r + 4.$$

Hence, there will be no overlap of distances determined between A and B and the points of the retained ith and jth boxes, respectively, for i. e. $i \neq j$. Suppose now that there remain A and B and t non-empty boxes containing $n_1, n_2, \cdots, n_t$ points. Then by our Lemma 2, and the above remarks, we have

$$f(n) \geqq \sqrt{n_1} + \sqrt{n_2} + \cdots + \sqrt{n_t}, \tag{15}$$

$$\frac{n-2}{6} \leqq n_1 + n_2 + \cdots + n_t, \tag{16}$$

moreover

$$s > n_i, \qquad i = 1, 2, 3, \cdots, t, \tag{17}$$

since no box contains as many as s points.

Now, by a well known inequality,

$$\sqrt{n_i} + \sqrt{n_j} \leqq 2\sqrt{\frac{n_i + n_j}{2}}, \tag{18}$$

so that the right hand side of (15) is not increased by replacing n_i and n_j by their arithmetic mean. Hence, in seeking a lower bound for $f(n)$, we may take

$$n_1 = n_2 = \cdots = n_t = \frac{n-2}{6t} \cdot \tag{19}$$

By (15) and (19), we have

$$f(n) \geqq t\sqrt{\frac{n-2}{6t}} = \sqrt{\frac{(n+2)\cdot t}{6}} \cdot \tag{20}$$

But now (17) and (19) give

$$t \geqq \frac{n-2}{6s}, \tag{21}$$

so that, by (20) and (21), we have

$$f(n) \geqq \frac{n-2}{6\sqrt{s}} > \frac{n}{6\sqrt{s}} - 1. \tag{22}$$

If we now combine (14) and (22), we obtain, in any case,

$$f(n) \geqq \min\left\{\frac{s}{2} - 1, \frac{n}{6\sqrt{s}} - 1\right\}, \qquad 1 \leqq s \leqq n. \tag{23}$$

To obtain the best value for s we let $s/2=n/6\sqrt{s}$ and find $s=n^{2/3}/\sqrt[3]{9}$. This value of s in (23) yields (2). Thus, we have completed the proof of Theorem I:

$$f(n) \geqq \frac{n^{2/3}}{2\sqrt[3]{9}} - 1.$$

This is better than the right hand side of (1) for $n>5184$.

3. Convex Case. We now turn to the case where the n points form the vertices of a convex polygon, our object being to prove (4). First we consider the following

LEMMA 3. *Let A and B be two points at the vertices of a sector of a circle less than or equal to a semicircle. Let $P_1, P_2, \cdots, P_m$ be m other points inside or on the sector. If all the points considered form the vertices of a convex polygon, then from A (and from B) exactly $m+1$ distances are determined.*

Proof. If $\overline{AP_i}=\overline{AP_j}$ then clearly, $AP_i \cdot BP_j$ cannot be part of a convex polygon. Note Figure 3.

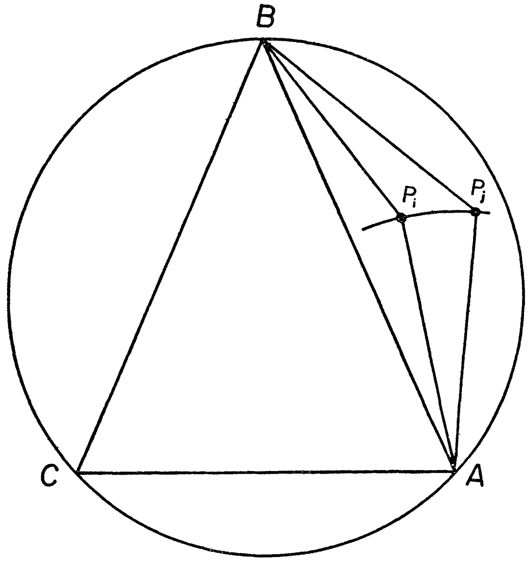

FIG. 3

Now consider the smallest circle containing our n points. If only two points lie on this circle, then they must be diametrically opposite. In this case, one of the semicircles will contain at least $(n+1)/2$ points in or on it, and Lemma 3 will yield

$$f^*(n) > \frac{n}{2} \geqq \left[\frac{n+2}{3}\right], \qquad (n \geqq 3).$$

Suppose then, that at least 3 points lie on the circle. We can then find three points A, B, C, on it which form a triangle having no angle greater than $\pi/2$, for otherwise the circle could be further contracted by shifting it in the direction of the angle greater than $\pi/2$ (since the center of the circumscribed circle is in the exterior of the triangle). Since our points are the vertices of a convex polygon, there are no points inside the triangle ABC, and hence at least one of the sectors determined by AB, AC, BC, will contain at least $[(n+5)/3]$ points in or on it, so that by Lemma 3 we obtain Theorem II:

$$f^*(n) \geqq \left[\frac{n+2}{3}\right].$$

NOTE ON A SECOND FUNCTIONAL EQUATION, CONNECTED WITH THE FUNCTION $\wp(z)$

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It is known* that the Weierstrass function $\wp(z)$ satisfies the functional equation:

$$f(x+y) - f(x-y) = -\frac{f'(x)f'(y)}{\{f(x)-f(y)\}^2}. \tag{1}$$

We propose to determine all the analytic† solutions, having at most poles in the finite plane. If $f(x)$ satisfies (1), so does $f(x)+c$ where c is an arbitrary constant. To within this additive constant we shall show that the only solution is $\wp(x)$.

It is clear that $f(x)$ must have a pole at $x=0$, since otherwise as $x \to y$, the L.S. of (1) would remain finite for general y, whereas the R.S. would become infinite. Hence

$$f(x) = \frac{g(x)}{x^n}, \tag{2}$$

where n is a positive integer and $g(x)$ is regular and not zero at $x=0$.

Divide (1) by $2y$ and let $y \to 0$. We obtain

$$f'(x) = -\frac{1}{2}\cdot f'(x) \times \lim_{y\to 0}\left[\frac{f'(y)}{y\{f(x)-f(y)\}^2}\right];$$

and on using (2) (for the variable y) it is easily calculated that, in order that the limit on the right may exist and have the right value, we must have $n=2$

* See Whittaker & Watson, Modern Analysis, (1915), Ex 1, p. 449.

† On clearing (1) of fractions, the new functional equation has the solution $f(x)=$const.; but this trivial case is ruled out.