The Map-Coloring Game

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1. INTRODUCTION. Suppose that Alice wants to color a planar map using four colors in a *proper* way, that is, so that any two adjacent regions get different colors. Despite the fact that she knows for certain that it is eventually possible, she may fail in her first attempts. Indeed, there are usually many proper partial colorings not extendable to proper colorings of the whole map. Thus, if she is unlucky, she may accidentally create such a *bad* partial coloring.

Now suppose that Alice asks Bob to help her in this task. They color the regions of a map alternately, with Alice going first. Bob agrees to cooperate by respecting the rule of a proper coloring. However, for some reason he does not want the job to be completed—his secret aim is to achieve a bad partial coloring. (For instance, he may wish to start the coloring procedure over and over again just to stay longer in Alice's company.) Is it possible for Alice to complete the coloring somehow, in spite of Bob's insidious plan? If not, then how many additional colors are needed to guarantee that the map can be successfully colored, no matter how clever Bob is?

This *map-coloring game* was invented about twenty-five years ago by Steven J. Brams with the hope of finding a game-theoretic proof of the Four Color Theorem, avoiding perhaps the use of computers. Though this approach has not been successful, at least we are left with a new, intriguing map-coloring problem: *What is the fewest number of colors allowing a guaranteed win for Alice in the map-coloring game in the plane?*

Brams's game was published by Martin Gardner in his "Mathematical Games" column in *Scientific American* in 1981. Surprisingly, it remained unnoticed by the graph-theoretic community until ten years later, when it was reinvented by Hans L. Bodlaender [**1**] in the wider context of general graphs. In this version Alice and Bob play as before by coloring properly the vertices of a graph G. The *game chromatic number* $\chi_g(G)$ of G is the smallest number of colors for which Alice has a winning strategy. As every map is representable by a graph whose edges correspond to adjacent regions of the map, Brams's question is equivalent to determining the game chromatic number of *planar* graphs. Since then the problem has been analyzed in serious combinatorial journals and gained the attention of several experts. In this article we present what is known and what remains unknown about this fascinating game.

2. FOUR COLORS DO NOT SUFFICE. One can see quickly that there are maps demanding more colors in the game than normally. Perhaps the simplest example of this phenomenon is due to Robert High. Consider the map with six regions corresponding to the faces of a cube. Clearly, the map can be 3-colored, but Bob can win even if there are four colors available. Suppose that Alice uses color 1 in her first move. Then Bob answers with color 2 on the opposite face. Thus colors 1 and 2 cannot be used any further, and Alice has to play color 3 (see Figure 1). Then Bob answers with color 4 on the opposite face and wins the game.

Earlier, Lloyd Shapley gave another elegant example based on a regular dodecahedron, showing that even five colors do not suffice. Here Bob's strategy is similar, but this time he replies with the same color on the face opposite to the face that Alice

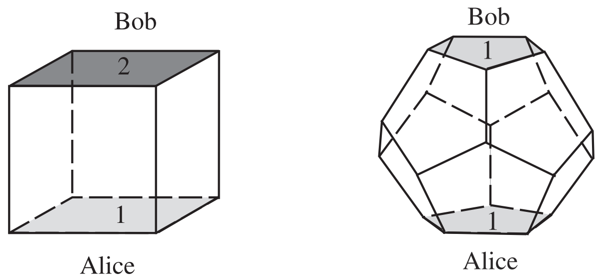

Figure 1. First two moves on the cube and the dodecahedron.

has just colored (see Figure 1). In this way, Alice must use a new color in each of her moves, which leads to a win for Bob after the fifth round.

When Steven Brams and Martin Gardner started to think that perhaps six colors would be the maximum number that Bob could force, Robert High found a map for which Alice needed seven colors to win. Indeed, there is a modification of the dodecahedron map for which a win by Alice requires eight colors. Unfortunately, it is not possible to verify this without a tedious case-by-case analysis. A natural suspicion arose that perhaps there was no finite bound at all, a suspicion that persisted for the thirteen years.

3. POSSIBLE STRATEGIES FOR ALICE. Let us consider the problems posed for Alice in designing an effective strategy for the map-coloring game. Whenever Bob colors a vertex v, he poses a threat to each of its uncolored neighbors. Alice can counter this threat at one of these neighbors w by immediately coloring it, but this in turn creates new threats for the neighbors of w. Moreover, the remaining uncolored neighbors of v are still open to attack, and Bob can strengthen his position on subsequent moves by coloring their uncolored neighbors. Thus Alice should design a strategy that prioritizes responses to threats to uncolored vertices so that she can keep the total threat under control. She must also limit the damage done when she colors a vertex.

The following idea allows Alice to avoid creating new points of vulnerability when she colors a vertex. Let $\chi(G)$ denote the usual *chromatic number* of a graph G, that is, the least number of colors in a proper coloring of the vertices of G. Suppose that $f : V(G) \to \{c_1, \dots, c_t\}$ is an optimal coloring, with $t = \chi(G)$, and that the game is played on a set of kt colors of the form $\{c_j^i : i \in [k], j \in [t]\}$, where we use the standard abbreviation $[n]$ for $\{1, 2, \dots, n\}$. We refer to c_j^i as a *shade* of c_j. We say that a color is *correct* for a vertex v if it is a shade of $f(v)$; otherwise it is incorrect. Alice should always try to color a vertex v with a correct color. If she is successful, then her early moves will never conflict with her later moves. This is possible as long as she can play in such a way that at any time any uncolored vertex is adjacent to fewer than k vertices incorrectly colored by Bob.

This leaves Alice with the problem of limiting the number of neighbors colored incorrectly by Bob that an uncolored vertex can have. In ordinary (noncompetitive) graph-coloring there is a well-known strategy for managing such threats. The *coloring number* $\mathrm{col}(G)$ of a graph G is the smallest integer k such that every subgraph of G has a vertex with degree less than k. For example, it is a simple consequence of Euler's formula that the coloring number of an arbitrary planar graph is at most six. The vertices of a graph with coloring number at most k can be ordered as $v_1, \dots, v_n$ so that each vertex v_i has fewer than k neighbors that precede it in the ordering: we simply construct the ordering from right to left, always choosing a vertex whose degree among the unchosen vertices is less than k. It is also clear that the existence of such an ordering implies that $\mathrm{col}(G) \leq k$. Now, if Alice colors the vertices from left to right, she will need at most k colors, since each uncolored vertex will have fewer than k colored neighbors. In other words, $\chi(G) \leq \mathrm{col}(G)$ for all graphs G.

Suppose that Alice attempts to modify this strategy for the map-coloring game. She might naturally try to color the left-most (in the ordering) neighbor of the last vertex colored by Bob. This vertex will have fewer than k backward (left) neighbors colored by Bob, but the number of its forward (right) neighbors colored by Bob is unbounded. This was the situation with the problem in the early nineties when help came from an unexpected source.

4. RAMSEY NUMBERS AND ARRANGEABILITY. A *simple graph G* consists of a vertex set $V(G)$ and an edge set $E(G)$ where each edge is an unordered pair of distinct vertices. A *complete graph* or *clique* is a simple graph in which every pair of vertices forms an edge. A clique on t vertices is denoted by K_t. According to Ramsey's famous theorem, any graph G has a *Ramsey number* $r(G)$ defined as the smallest integer t such that for any red-blue coloring of the edges of K_t there is a monochromatic copy of G in K_t. For a class $\mathcal{C}$ of graphs define $r_{\mathcal{C}}(n)$ by $r_{\mathcal{C}}(n) = \max\{r(G) : G \in \mathcal{C}, |V(G)| = n\}$. Burr and Erdős conjectured that $r_{\mathcal{C}}(n)$ is linear in n for any class of graphs satisfying $|E(G)| = O(|V(G)|)$. This conjecture remains open in general, although some special cases have been proved. Chvátal, Rödl, Szemerédi, and Trotter [**5**] verified the conjecture for the special case of graphs with bounded degree by using Szemerédi's regularity lemma. The proof involved constructing an embedding of G into a monochromatic part of K_n one vertex at a time in an order $v_1, \dots, v_n$. It was crucial to their argument that each vertex have a bounded number of backward and forward neighbors. If the argument had required only that the backward degree be bounded, they would have established the whole conjecture. Notice that planar graphs satisfy $|E(G)| = O(|V(G)|)$ but may have arbitrarily large degree.

Chen and Schelp made the brilliant observation that something less was required: the number of vertices v_h with $h < i$ that are adjacent to v_i or to a *forward* neighbor v_j of v_i should be bounded (see Figure 2). This idea, in one form or another, has turned out to be absolutely crucial for the theory of coloring games.

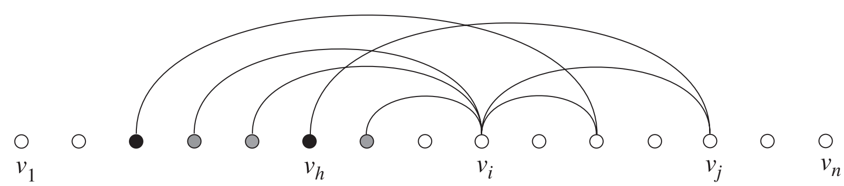

Figure 2. The number $\mathrm{col}_2(G)$ bounds the number of predecessors of v_i that are adjacent to v_i (grey nodes) or to a forward neighbor of v_i (black nodes).

Assume that the vertices of a graph G are ordered as $v_1, v_2, \dots, v_n$. A vertex v_h is called a *loose backward neighbor* of v_i if $h < i$ and either (a) v_h is adjacent to v_i or (b) v_h is adjacent to v_j and v_j is adjacent to v_i for some j with $j > i$ (see Figure 2). We define the 2-*coloring number* $\mathrm{col}_2(G)$ of a graph G to be the least integer k such that there is an ordering of $V(G)$ for which each vertex has fewer than k loose backward neighbors. This is an inessential variant of the parameter that Chen and Schelp called *arrangeability* in [**3**], where it is proved that the conjecture of Burr and Erdős is true for the family of graphs with bounded 2-coloring numbers. They then proved the Burr-Erdős conjecture for planar graphs by showing that the 2-coloring number of any planar graph is at most 761.

Kierstead and Trotter [**14**] realized that this was the missing tool for bounding the game chromatic number $\chi_g(G)$ of planar graphs G. They proved the following theorem:

Theorem 1. *Every graph G satisfies* $\chi_g(G) \le \chi(G)(1 + \mathrm{col}_2(G))$.

Proof. Let $\chi(G) = t$ and $\mathrm{col}_2(G) = k$. To prove the assertion we describe a strategy for Alice using at most $t(k + 1)$ colors. Alice should begin by fixing an optimal coloring f of G using the colors $c_1, \dots, c_t$ and an ordering $v_1, \dots, v_n$ of G that satisfies $\mathrm{col}_2(G) = k$. She prioritizes responses to threats according to this ordering. Whenever Bob colors a vertex v, Alice colors the least-indexed uncolored backward neighbor u of v with a correct color; if v has no such backward neighbor then Alice correctly colors the least-indexed uncolored vertex. To be certain that u has a correct color available (i.e., a color not used by any of its colored neighbors), it suffices to show that u has at most k neighbors that are colored by Bob. Assume that Bob has colored s loose backward neighbors of u and that Alice has colored s' loose backward neighbors of u. In accordance with the stated strategy, each time Bob colors a forward neighbor z of u, Alice colors the least-indexed uncolored backward neighbor u' of z. Since u is an uncolored backward neighbor of z, u' precedes u, so u' is a loose backward neighbor of u. As Alice has colored s' loose backward neighbors of u, we conclude that Bob has colored at most s' forward neighbors of u. Therefore the number of neighbors of u colored by Bob is at most $s + s'$ $(< k)$. Note that this counts only the number of colored neighbors of u at the end of Alice's previous move. After that move, Bob may have colored another neighbor of u, meaning that u now has at most k neighbors colored by Bob. Thus u has an available correct color, and Alice can win the game with $t(k + 1)$ colors. ∎

This reasoning implies that $\chi_g(G) \le 3044$ for a planar graph G. Kierstead and Trotter obtained much tighter bounds on the game chromatic number of planar graphs. They demonstrated that $\mathrm{col}_2(G) \le 10$ holds for each planar graph G (which gives $\chi_g(G) \le 44$) and provided an example of a planar graph G with $\mathrm{col}_2(G) = 8$. They further improved their bound on the game chromatic number of planar graphs to $\chi_g(G) \le 33$ by using a more carefully designed parameter than 2-coloring number.

5. ACYCLIC COLORING. A *cycle* on vertices $v_1, \dots, v_n$ is a graph whose edges are $v_1v_2, v_2v_3, \dots, v_nv_1$. A *forest* is a graph without cycles. A coloring of the vertices of a graph G is *acyclic* if it is a proper coloring such that no cycle of G is 2-colored. In other words, a subgraph of G whose vertex set is a union of any two color classes is a forest. The minimum number of colors needed is the *acyclic chromatic number* of G, denoted by $a(G)$. This notion was introduced by Grünbaum, who conjectured that $a(G) \le 5$ holds for any planar graph G. The conjecture was turned into a theorem by Borodin in 1979 [**2**].

Just as the chromatic number of a graph G is bounded by its coloring number, its acyclic chromatic number is bounded by its 2-coloring number. Let $v_1, \dots, v_n$ be an ordering of the vertices of G that realizes its 2-coloring number. Color vertices recursively so that no vertex is colored the same color as a loose backward neighbor. This yields a proper coloring using at most $\mathrm{col}_2(G)$ colors. We still must check that each cycle C in G receives at least three colors. The largest-indexed vertex v_j in C has two backward neighbors v_h and v_i with $h < i < j$. Since v_h is a loose backward neighbor of v_i, they have different colors. Since v_j is adjacent to both v_i and v_h, it has a third color.

Twenty years after Grünbaum introduced acyclic coloring, Dinski and Zhu [**6**] applied the acyclic chromatic number to the Brams game for arbitrary graphs. They proved the following theorem, which implies that thirty colors suffice for Alice to win on any planar map:

Theorem 2. *Every graph G satisfies $\chi_g(G) \leq a(G)(a(G)+1)$.*

Proof. The argument is similar to the proof of Theorem 1. Suppose that G has an acyclic coloring using the color set $\{c_1, \dots, c_t\}$, and let $V_1, \dots, V_t$ be the corresponding color classes of vertices. When $i \neq j$, the subgraph F_{ij} of G consisting of all edges with ends in $V_i \cup V_j$ is a forest. Fix an orientation of each F_{ij} so that every vertex has outdegree at most one. Since every edge of G appears in exactly one of the forests F_{ij}, this provides an orientation of G such that every vertex has at most one outneighbor in each color class. Now a vertex v in V_i is *endangered* if it is as yet uncolored but there is an inneighbor u of v colored with a shade of c_i (see Figure 3). Since Alice always colors with a correct color, an endangered vertex can be created only by Bob. Clearly, in a single move Bob can create at most one endangered vertex v. Alice should color this vertex immediately after it is created by Bob. This is always possible, for there are at most t neighbors of v colored with shades of c_i at this moment (at most $t-1$ outneighbors and only one inneighbor u that has just been colored by Bob). ∎

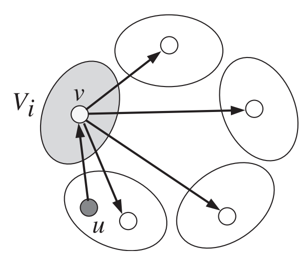

Figure 3. An endangered vertex v will be colored by Alice in her next move.

6. DALTONISM MAY HELP.

Both strategies for Alice considered earlier were local. Alice responded to any threat posed when Bob colored a vertex by coloring one of its neighbors. As any Go player knows, global strategies can be more powerful. Our next aim is to develop a global strategy for Alice.

Curiously, further progress in bounding the game chromatic number was achieved by making the game more difficult for Alice. Suppose Alice suffers from full daltonism (colorblindness): she just cannot distinguish between colors. Because of her disease she proposes to modify the rules of the coloring game so that she can still have some fun playing it. Rather than coloring vertices, the players create an ordering $<_g$ of vertices by taking turns choosing the next vertex in the ordering. For a fixed positive integer k Alice's goal is to have $<_g$ demonstrate that $\mathrm{col}(G) \leq k$. Bob, wishing to remain in her company, tries to thwart her. For the purposes of actual play it is convenient for the players to indicate that they have chosen a vertex by coloring it gray. Bob wins if at some time some uncolored vertex has k gray neighbors, and Alice wins if this never occurs. We denote the least number k allowing success for Alice by $\mathrm{col}_g(G)$ and call it the *game coloring number* of G. Clearly, if Alice can win this *ordering* game for some integer k, she can also win the coloring game with k colors (even as a completely colorblind person!). In other words, $\chi_g(G) \leq \mathrm{col}_g(G)$ is true for every graph G.

Alice is now forced to use a strategy that does not rely on shades of correct colors to control threats created by her moves. Suppose that she again selects a specific ordering $v_1, \ldots, v_n$ (not to be confused with $<_g$) of the vertex set for the purpose of prioritizing responses to threats. Suppose also that no vertex has too many backward neighbors in this order. When Bob chooses a vertex v (colors it gray), Alice faces conflicting requirements. On the one hand, she should choose a backward neighbor x of v, addressing the threat to x posed by v. But this in turn produces a threat from x to its backward neighbors. Alternatively, she could put off choosing x and instead choose a backward neighbor y of x so that y would already be protected when later she finally did choose x. Or should she, perhaps, choose a backward neighbor of y? Here we propose a compromise plan: the first time Alice contemplates choosing x she should move on to considering y, but the second time she considers choosing x she should actually do so.

Here are the details of Alice's *activation strategy*. Imagine that the vertices of G are made of light bulbs that shine a warning when activated. Let $N^+(v)$ and $N^-(v)$ denote the sets of backward and forward neighbors of v, respectively. On her first move Alice activates vertex v_1 and colors it. The basic step of the strategy is to activate a vertex (leaving a warning that it is in danger) and then jump to its least-indexed uncolored backward neighbor. Suppose that Bob has just colored a vertex v. Then Alice starts by activating v (provided it has not been activated hitherto) and jumps to the uncolored vertex x of smallest index in $N^+(v)$. If x is already active, then it is endangered, so Alice stops and colors it. Otherwise she repeats the activation step for x, that is, she activates x and jumps to the least-indexed uncolored vertex y in $N^+(x)$. This process goes on until she stops at some vertex u, either because u is active or because there are no uncolored vertices in $N^+(u)$. In each case she activates and colors u. If it happens that $N^+(v)$ already contains no uncolored vertices, then she picks the uncolored vertex of smallest index, activates it (provided it has not yet been activated), and colors it. Notice that after Alice's move all gray vertices are active, whereas after Bob's move there can be only one gray inactive vertex.

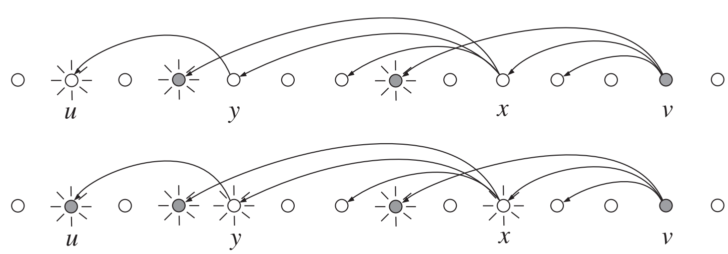

Figure 4. Basic step of the activation procedure.

For example, let's see how this strategy works for trees (see Figure 5). Any tree T can be ordered so that each vertex has at most one backward neighbor. Suppose that Bob has colored a vertex v, and let x be its unique backward neighbor. Then Alice should activate all vertices on the unique backward path starting from v and ending at some active vertex u. Notice that Alice can jump to the same vertex only twice (the first time she activates it, the second time she colors it). It follows that if x stayed uncolored after her move, it could have no more than two active neighbors (one backward and one forward). Hence, after Bob's move there are at most three colored vertices around x, which gives $\mathrm{col}_g(G) \leq 4$. This result was proved by Faigle, Kern, Kierstead, and

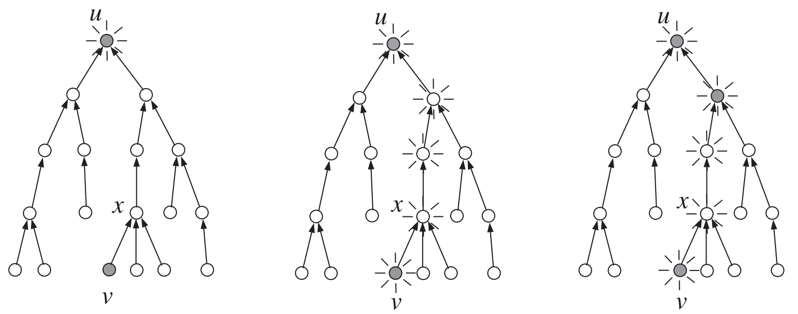

Figure 5. Activation on a tree.

Trotter in [**8**]. The following result reveals the information provided by the activation strategy for general graphs:

Theorem 3. *Every graph G satisfies* $\mathrm{col}_g(G) \leq 3\,\mathrm{col}_2(G) - 1$.

Proof. Fix an order of the vertices $v_1, \dots, v_n$ realizing $\mathrm{col}_2(G) = k$, and assume that Alice applies the activation strategy with respect to this order. For each vertex v_j let $S(v_j)$ be the set of loose backward neighbors of v_j. By the definition of $\mathrm{col}_2(G)$, the size of $S(v_j)$ is at most $k - 1$. We count the number of active forward neighbors that v_j could have before it gets colored. From each active vertex in $N^-(v_j)$ Alice jumps to the least-indexed uncolored backward neighbor in $S(v_j)$ or possibly to v_j. If she jumps to v_j, then she immediately colors it or immediately jumps to another vertex in $S(v_j)$. Since she can jump only twice to a vertex, there can be at most $2(k - 1)$ active vertices in $N^-(v_j)$ after Alice's move. Hence, there are at most $3(k - 1)$ active vertices around v_j after Alice's move. Counting Bob's last move, there are at most $3k - 2$ colored vertices around v_j when it gets colored. ■

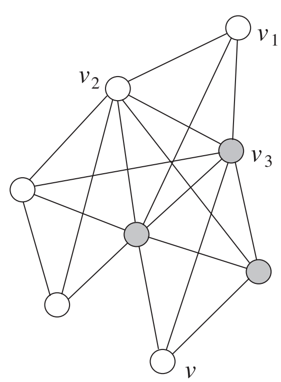

Figure 6. Construction of a 3-tree.

There is a nice class of graphs for which the bound in Theorem 3 is optimal (as long as we assume that Alice cannot distinguish between colors). Fix a number k and start with a clique on k vertices $v_1, \dots, v_k$. Then in each subsequent step add a new vertex v and join it to k vertices of any clique in the graph constructed so far. In this recursive manner we produce graphs called *k-trees* (see Figure 6). The resulting ordering of the vertices shows that $\mathrm{col}_2(G) = k + 1$, which gives $\mathrm{col}_g(G) \leq 3k + 2$ for any k-tree G.

It can be proved that this bound is optimal for each k (> 1). For planar graphs the theorem gives $\chi_g(G) \leq 29$, which is not bad but is still not optimal.

7. SIX DOLLAR ORDER. We now demonstrate that the estimate $\chi_g(G) \leq 18$ holds for every planar graph G, which is almost the best upper bound discovered so far. This time we take more care in ordering the vertices $v_1, \dots, v_n$ so as to make the activation strategy more efficient. We define this order inductively (in reverse order), starting with any vertex v_n of degree at most five. Suppose that we have already picked vertices $v_n, v_{n-1}, \dots, v_{i+1}$ and are looking for a candidate for v_i. Partition the vertex set V into two parts, $C = \{v_n, v_{n-1}, \dots, v_{i+1}\}$ and $U = V \setminus C$, and construct a new graph H as follows:

1. Delete each edge between pairs of vertices in C.
2. Delete each vertex v in C with at most three neighbors in U.
3. For each deleted vertex v in C add edges between its neighbors in U so that these neighbors form a clique.

It can be verified that the new graph H is planar: the new edges in H can be drawn near deleted edges incident to deleted vertices of C. By Euler's formula H satisfies the inequality $|E(H)| \leq 3|V(H)| - 6$.

Now suppose that each edge of H is assigned \$2 (in half-dollar coins). These coins are distributed to the vertices of H as follows: if an edge e links two vertices in U, then e gives \$1 to each of the two vertices; if e links a vertex x in C and a vertex y in U, then e gives \$1.50 to x and \$0.50 to y (see Figure 7). As the total amount of dollars is equal to $2|E|$ ($< 6|V(H)|$), there is a vertex v that receives less than \$6. This will be the ith vertex v_i in our order. Notice that v_i cannot belong to C, for each remaining vertex in C is incident to at least four edges of H, hence receives at least \$6.

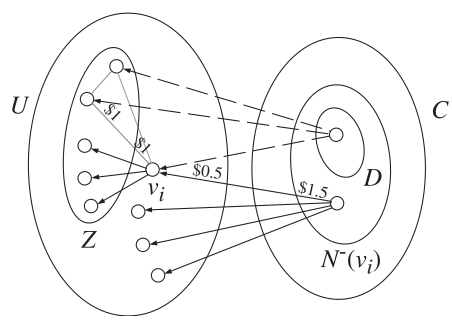

Figure 7. Ordering the set of vertices of G by finding a poor vertex in a graph H.

Now we are ready to prove the result stated at the beginning of this section:

Theorem 4. *Every planar graph G satisfies* $\text{col}_g(G) \leq 18$.

Proof. Let $v_1, \dots, v_n$ be the ordering of the vertices of G obtained in the manner just described. It suffices to show that if Alice applies the activation strategy, then at the end of each of her turns there will be at most sixteen active neighbors around any uncolored vertex. Let $v = v_i$ be any uncolored vertex, let H, C, and U be as indicated in the discussion preceding Theorem 4 (notice that all of them depend on v_i), let D be the set of those forward neighbors of v (in G) that were deleted while constructing H, and let Z be the set of those vertices of U that are adjacent to v in H. Since we didn't delete any edges between vertices of U, we see that $N^+(v)$ is contained in Z. Also Z includes all vertices, other than v, to which Alice can jump after activating any vertex in the set D (see Figure 7). Hence, arguing as in the previous proof, there can be at most

$$3|Z| + |N^-(v) \setminus D|$$

active vertices around v at the end of any move by Alice. Since v received strictly less than \$6 in the ordering procedure,

$$1 \times |Z| + 0.5 \times |N^-(v) \setminus D| < 6.$$

This implies that $3|Z| + |N^-(v) \setminus D| \leq 16$, which completes the proof. ∎

8. VARIANTS OF THE GAME. Currently, the best upper bounds for a planar graph G are $\chi_g(G) \leq \mathrm{col}_g(G) \leq 17$, as proved in [**20**]. The proof is based on a modification of the activation strategy. At present, the best lower bounds for planar graphs are $8 \leq \chi_g(G)$ and $11 \leq \mathrm{col}_g(G)$. There is evidence that the upper bound on the game coloring number $\mathrm{col}_g(G)$ may be sharp. However, we cannot even anticipate what the correct answer is for the game chromatic number $\chi_g(G)$ of planar graphs.

This situation probably exists because of the rather strange general behavior of the parameter $\chi_g(G)$. For instance, $\chi_g(G)$ is already unbounded for graphs satisfying $\chi(G) = 2$ and is also not monotone with respect to subgraphs. Indeed, let $U = \{u_1, \dots, u_n\}$ and $V = \{v_1, \dots, v_n\}$ be disjoint sets of vertices, and let $K_{n,n}$ be the graph on these vertices obtained by joining each vertex in U with all vertices in V. Let M be a *perfect matching* in $K_{n,n}$ (i.e., a set of n edges no two of which are incident with the same vertex; for instance, $M = \{u_1v_1, \dots, u_nv_n\}$). Clearly we have $\chi_g(K_{n,n}) = 3$, but $\chi_g(K_{n,n} - M) = n$. Bob's strategy is to copy Alice's moves on the other ends of the missing matching (see Figure 8).

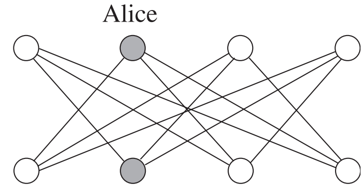

Figure 8. Copy cat strategy allows Bob to win on $K_{n,n} - M$ if fewer than n colors are available.

More surprisingly, it is not clear what influence increasing the number of colors has on the coloring game. Suppose that Alice wins with k colors on a graph G. Then it would seem trivial that she should also win on G with $k + 1$ colors. But can you prove it? As yet, nobody has been able to do so!

In view of such pathologies it is natural to look for more predictable variants of the coloring game. For instance, in a modification considered in [**3**] Bob is not allowed to use a new color until it is absolutely necessary, that is, unless all uncolored vertices are surrounded by all previously used colors. Chen, Schelp, and Shreve prove there that $\chi_g^*(T) \leq 3$ holds for any tree T, where $\chi_g^*(G)$ is the related game chromatic number. However, no better bounds for arbitrary planar graphs are provided. In this setting it is clear that increasing the number of colors cannot help Bob. However, we are still left with the anomaly that $\chi_g^*(K_{n,n} - M) = n$.

There are many natural variations of the coloring and ordering games. One variation is to allow Alice more moves so as to limit Bob's ability to interfere [**12**], [**13**], [**17**]. Other variants consider oriented coloring [**15**], [**16**], [**18**] and d-relaxed coloring [**4**], [**7**]. It is remarkable that the activation strategy is still effective in these diverse settings.

ACKNOWLEDGMENT. We would like to thank Steven Brams for useful materials and comments on the origin of the map-coloring game. The research of the fourth author is supported in part by the National Science Council under grant NSC95-2115-M-110-013-MY3.

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