Hat problem on a graph

Submitted by

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I certify that all material in this thesis which is not my own work has been identified and that no material has previously been submitted and approved for the award of a degree by this or any other University.

I dedicate to my Mom

whose son is to be a doctor, but not the kind who cures people.*

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*The idea taken from Professor Randy Pausch**: "After I got my PhD, my mother took great relish in introducing me by saying: «This is my son. He's a doctor, but not the kind who helps people»", R. Pausch, *Last Lecture*, p. 24, Hyperion 2008.

**Randolph Frederick Pausch (October 23, 1960 – July 25, 2008) was an American professor of computer science, human-computer interaction and design at Carnegie Mellon University in Pittsburgh, Pennsylvania, and a best-selling author who achieved worldwide fame for his "The Last Lecture" speech on September 18, 2007 at Carnegie Mellon. The lecture was conceived after Pausch learned, in summer 2007, that his previously known pancreatic cancer was terminal. In May 2008, Pausch was listed by *Time* as one of the World's Top-100 Most Influential People.

ABSTRACT

The topic of this thesis is the hat problem. In this problem, a team of n players enters a room, and a blue or red hat is randomly placed on the head of each player. Every player can see the hats of all of the other players but not his own. Then each player must simultaneously guess the color of his own hat or pass. The team wins if at least one player guesses his hat color correctly and no one guesses his hat color wrong, otherwise the team loses. The aim is to maximize the probability of winning.

This thesis is based on publications, which form the second chapter. In the first chapter we give an overview of the published results.

In Section 1.1 we introduce to the hat problem and the hat problem on a graph, where vertices correspond to players, and a player can see the adjacent players.

To the hat problem on a graph we devote the next few sections. First, we give some fundamental theorems about the problem. Then we solve the hat problem on trees, cycles, and unicyclic graphs. Next we consider the hat problem on graphs with a universal vertex. We also investigate the problem on graphs with a neighborhood-dominated vertex. In addition, we consider the hat problem on disconnected graphs. Next we investigate the problem on graphs such that the only known information are degrees of vertices. We also present Nordhaus-Gaddum type inequalities for the hat problem on a graph.

In Section 1.6 we investigate the hat problem on directed graphs.

The topic of Section 1.7 is the generalized hat problem with $q \geq 2$ colors.

A modified hat problem is considered in Section 1.8. In this problem there are $n \geq 3$ players and two colors. The players do not have to guess their hat colors simultaneously and we modify the way of making a guess. We give an optimal strategy for this problem which guarantees the win.

Applications of the hat problem and its connections to different areas of science are presented in Section 1.9. We also give there a comprehensive list of variations of the hat problem considered in the literature.

Contents

Chapter 1

History and overview of the hat problem

1.1 Introduction

In the hat problem, a team of n players enters a room, and a blue or red hat is randomly placed on the head of each player. Every player can see the hats of all of the other players but not his own. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats, each player must simultaneously guess the color of his own hat or pass. The team wins if at least one player guesses his hat color correctly and no one guesses his hat color wrong, otherwise the team loses. The aim is to maximize the probability of winning.

In our game the team plays against nature, which behaves randomly. Thus all nature conditions are equally probable, that is, the probabilities of getting hats of each color are equal. The aim is to maximize the chance of success. Thus, in terms of game theory, we want to maximize the possible outcome. For additional information on game theory, see [49, 50].

The hat problem with seven players, called "seven prisoners puzzle", was formulated by Todd Ebert in his Ph.D. Thesis [21]. The hat problem was also the subject of articles in The New York Times [55], Die Zeit [8] and abcNews [53]. It was also one of the Berkeley Riddles [6].

In the hat problem, a 50% chance of success is guaranteed by the following strategy. One designated player always guesses he has, let us say, a blue hat, while the remaining players always pass. However, already for $n = 3$ the team can do better. The following strategy gives a 75% chance of success. If a player sees two hats of the same color, then he guesses he has a hat of the other color, otherwise he passes. Let us observe that the team wins when there are two hats of some color and one hat of the other color, and they lose when all three hats have the same color.

Coding theory [56] was inaugurated by Richard Hamming [52]. The authors of [47] showed that strategies for the hat problem with n players are equivalent to binary covering codes [17] of length n and radius one. Optimal strategies for the hat problem are equivalent to minimal binary covering codes. The webpage [37] contains up-to-date information on the best known covering codes. For a comprehensive bibliography on covering radius, see [48]. Covering codes are strongly related to Hamming codes. The hat problem and Hamming codes were the subject of [11, 18].

The hat problem is solved only for special values of n. For $2^k - 1$ players it was solved in [23], and for 2^k players, via extended Hamming codes, in [17]. For other numbers of players, optimal strategies for the hat problem (so also minimal binary covering codes of radius one) are not known if n is larger than nine. The hat problem with n players was investigated in [10].

In this thesis we consider the hat problem on a graph, where vertices correspond to players, and a player can see the adjacent players. We have defined this problem in [40], which is Section 2.1 in this thesis. We also investigate the problem on directed graphs.

Furthermore, we consider the generalized hat problem with n players and q colors. Among other things, we solve this problem for $n = 3$ and $q = 3$.

In addition, we consider a modified hat problem with $n \geq 3$ players and two colors. The players do not have to guess their hat colors simultaneously. Every one of them has two cards with his name and the sentence "I have a blue hat" or "I have a red hat". The players make a guess by coming to the basket and throwing the proper card into it. If someone wants to resign from answering, then he does not do anything. We give an optimal strategy for this problem which guarantees the win.

There are known many variations of the hat problem. In Section 1.9 we give a comprehensive list of them. The hat problem and its variations have many applications and connections to different areas of science, which are also described in Section 1.9.

Investigating our main problem, we first prove some fundamental theorems about the hat problem on a graph. We solve the problem on trees, cycles on at least four vertices, and unicyclic graphs containing a cycle on at least nine vertices. We show that for these graphs the maximum chance of success is $1/2$. Thus, in such graph, an optimal strategy is for example such that one vertex always guesses it is blue, while the remaining vertices always pass. It means that the structure of such graph does not improve the maximum chance of success in the hat problem on a graph comparing to the one-vertex graph. Next we consider the hat problem on graphs with a universal vertex. We also investigate the problem on graphs with a neighborhood-dominated vertex. Then we consider the hat problem on disconnected graphs. In addition, we investigate the hat problem on graphs such that the only known information are degrees of vertices. We also present Nordhaus-Gaddum type inequalities. Furthermore, we investigate the hat problem on directed graphs.

1.2 Preliminaries on the hat problem on a graph

We consider the hat problem on a graph, where vertices correspond to players, and a player can see the adjacent players.

First, let us observe that we can restrict to deterministic strategies (that is, strategies such that the decision of each player is determined uniquely by the colors of the other players). We can do this since for any randomized (that is, nondeterministic) strategy there exists a not worse deterministic one. It is true, because every randomized strategy is a convex combination of some deterministic strategies. The probability of winning is a linear function on the convex polyhedron corresponding to the set of all randomized strategies which can be achieved by combining those deterministic strategies. It is well known that this function achieves its maximum on a vertex of the polyhedron which corresponds to a deterministic strategy.

The following concepts are defined similarly as in [40], which is Section 2.1 in this thesis.

A *graph* is an ordered pair $G = (V, E)$, where a set V is called the *set of vertices* and E is the *set of edges*, which are 2-element subsets of V. We say that two vertices $u, v \in V(G)$ are *adjacent* if the edge $\{u, v\}$ (for short, we write uv) belongs to the set $E(G)$. By *complement* of G, denoted by $\overline{G}$, we mean a graph which has the same vertices as G, and two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. We say that H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Then we write $H \subseteq G$. Let $v \in V(G)$. The *set of neighbors* (called the *open neighborhood*) of v, that is $\{x \in V(G) \colon vx \in E(G)\}$, we denote by $N_G(v)$. The *closed neighborhood* of v, that is $N_G(v) \cup \{v\}$, we denote by $N_G[v]$. We say that a vertex v is *universal* if $N_G[v] = V(G)$. A graph is *complete* if all its vertices are universal. By a *leaf* we mean a vertex having exactly one neighbor. We say that a vertex v of a graph G is *neighborhood-dominated* if there is some other vertex $w \in V(G)$ such that $N_G(v) \subseteq N_G(w)$. The *degree* of vertex v, that is, the number of its neighbors, is denoted by $d_G(v)$. Thus $d_G(v) = |N_G(v)|$. A *path* in a graph is a sequence of pairwise distinct vertices (possibly except for the first one and the last one) such that every two consecutive vertices are adjacent. We say that a graph is *connected* if for every pair of vertices there is a path between them. A graph is *disconnected* if it is not connected. A *cycle* in a graph is a path in which the first vertex is the same as the last vertex. A *cycle* is also a connected graph in which every vertex has degree two. We say that a graph is *unicyclic* if it contains exactly one cycle as a subgraph. A *tree* is a connected graph such that the number of edges is one less than the number of vertices. A *path* is a tree in which every vertex has degree at most two. A path (cycle, complete graph, respectively) on n vertices we denote by P_n (C_n, K_n, respectively).

Let $f \colon X \to Y$ be a function. If $Z \subseteq X$, then the restriction of f to Z we denote by $f|_Z$. Let $y \in Y$. If for every $x \in X$ we have $f(x) = y$, then we write $f \equiv y$.

Let $V(G) = \{v_1, v_2, \dots, v_n\}$. By $Sc = \{1, 2\}$ we denote the set of colors, where 1 corresponds to the blue color and 2 corresponds to the red color. In Section 1.7 we consider sets of more than two colors.

By a *case* for a graph G we mean a function $c\colon V(G) \to \{1,2\}$, where $c(v_i)$ means the color of vertex v_i. The set of all possible cases for the graph G we denote by $C(G)$, obviously $|C(G)| = 2^{|V(G)|}$. If $c \in C(G)$, then to simplify the notation we write $c = c(v_1)c(v_2)\dots c(v_n)$ instead of $c = \{(v_1, c(v_1)), (v_2, c(v_2)), \dots, (v_n, c(v_n))\}$. For example, if for some graph G with four vertices a case c is such that $c(v_1) = 2$, $c(v_2) = 1$, $c(v_3) = 1$ and $c(v_4) = 2$, then we write $c = 2112$.

By a *situation* of a vertex v_i we mean a function $s_i\colon V(G) \to Sc \cup \{0\} = \{0,1,2\}$, where $s_i(v_j) = c(v_j) \in \{1,2\}$ if v_i and v_j are adjacent, and $s_i(v_j) = 0$ otherwise. The set of all possible situations of v_i in the graph G we denote by $St_i(G)$, obviously $|St_i(G)| = 2^{d_G(v_i)}$. If $s_i \in St_i(G)$, then for simplicity of notation we write $s_i = s_i(v_1)s_i(v_2)\dots s_i(v_n)$ instead of $s_i = \{(v_1, s_i(v_1)), (v_2, s_i(v_2)), \dots, (v_n, s_i(v_n))\}$. For example, if $s_3 \in St_3(C_4)$ is such that $s_3(v_2) = 2$ and $s_3(v_4) = 1$, then we write $s_3 = 0201$.

We say that a case c for the graph G *corresponds* to a situation s_i of vertex v_i if $c(v_j) = s_i(v_j)$, for every v_j adjacent to v_i. This implies that a case corresponds to a situation of v_i if every neighbor of v_i in the case has the same color as in the situation. Obviously, to every situation of vertex v_i correspond exactly $2^{|V(G)|-d_G(v_i)}$ cases.

Let G and H be graphs such that $H \subseteq G$. We say that a case c for the graph G *corresponds* to a case d for the graph H if $c|_{V(H)} = d$, that is, every vertex of H in both cases c and d has the same color. Obviously, to every case for the graph H correspond $2^{|V(G)|-|V(H)|}$ cases for the graph G.

For a vertex $v_i \in V(H)$, we say that its situation s_i in the graph G *corresponds* to its situation t_i in the graph H if $s_i|_{V(H)} = t_i$, that is, every neighbor of v_i in the graph H in both situations t_i and s_i has the same color.

By a *guessing instruction* of a vertex $v_i \in V(G)$ we mean a function $g_i\colon St_i(G) \to Sc \cup \{0\} = \{0,1,2\}$, which for a given situation outputs the color v_i guesses it is, or outputs 0 if v_i passes. Thus a guessing instruction is a rule determining the behavior of a vertex in every situation. We say that v_i *never guesses* its color if v_i passes in every situation, that is $g_i \equiv 0$. We say that v_i *always guesses* its color if v_i guesses its color in every situation, that is, for every $s_i \in St_i(G)$ we have $g_i(s_i) \in \{1,2\}$ ($g_i(s_i) \neq 0$, equivalently).

Let c be a case, and let s_i be the situation (of vertex v_i) corresponding to this case. The guess of v_i in the case c is *correct* (*wrong*, respectively) if $g_i(s_i) = c(v_i)$ ($0 \neq g_i(s_i) \neq c(v_i)$, respectively). By *result* of the case c we mean a win if at least one vertex guesses its color correctly, and no vertex guesses its color wrong, that is, $g_i(s_i) = c(v_i)$ (for some i) and there is no j such that $0 \neq g_j(s_j) \neq c(v_j)$. Otherwise the result of the case c is a loss.

By a *strategy* for the graph G we mean a sequence $(g_1, g_2, \ldots, g_n)$, where g_i is the guessing instruction of vertex v_i. The family of all strategies for a graph G we denote by $\mathcal{F}(G)$.

If $S \in \mathcal{F}(G)$, then the set of cases for the graph G for which the team wins (loses, respectively) using the strategy S we denote by $W(S)$ ($L(S)$, respectively). The set of cases for which the team loses while some vertex guesses its color (no vertex guesses its color, respectively) we denote by $Ls(S)$ ($Ln(S)$, respectively). By the *chance of success* of the strategy S we mean the number $p(S) = |W(S)|/|C(G)|$. By the *hat number* of the graph G we mean the number $h(G) = \max\{p(S)\colon S \in \mathcal{F}(G)\}$. We say that a strategy S is *optimal* for the graph G if $p(S) = h(G)$. The family of all optimal strategies for the graph G we denote by $\mathcal{F}^0(G)$.

Let $t, m_1, m_2, \ldots, m_t \in \{1, 2, \ldots, n\}$, where $m_j \neq m_k$ for every $j \neq k$. Let $c_{m_1}, c_{m_2}, \ldots, c_{m_t} \in \{1,2\}$. The set of cases c for the graph G such that $c(v_{m_j}) = c_{m_j}$ we denote by $C(G, v_{m_1}^{c_{m_1}}, v_{m_2}^{c_{m_2}}, \ldots, v_{m_t}^{c_{m_t}})$. Let $S \in \mathcal{F}(G)$. The set of cases $c \in C(G, v_{m_1}^{c_{m_1}}, v_{m_2}^{c_{m_2}}, \ldots, v_{m_t}^{c_{m_t}})$ for which the team wins (loses, respectively) using the strategy S we denote by $W(S, v_{m_1}^{c_{m_1}}, v_{m_2}^{c_{m_2}}, \ldots, v_{m_t}^{c_{m_t}})$ ($L(S, v_{m_1}^{c_{m_1}}, v_{m_2}^{c_{m_2}}, \ldots, v_{m_t}^{c_{m_t}})$, respectively).

By *solving* the hat problem on a given graph we mean determining its hat number.

Now we give an example of the notation for the hat problem on the graph K_3. The vertices are denoted by v_1, v_2 and v_3. Obviously, there are $2^3 = 8$ possible cases for K_3. Assume for example that in a case c the vertices v_1 and v_3 have the first color and the vertex v_2 has the second color. Thus $c(v_1) = c(v_3) = 1$ and $c(v_2) = 2$. Now let us consider situations of some vertex, say v_1. The vertex v_1 can see that

v_2 has the second color and v_3 has the first color. Obviously, the vertex v_1 cannot see its own color. Thus $s_1(v_1) = 0$, $s_1(v_2) = 2$ and $s_1(v_3) = 1$. A case corresponds to this situation if in the case each neighbor of v_1 has the same color as in the situation. It is easy to observe that the case in which v_1 and v_2 have the second color and v_3 has the first color corresponds to that situation. These are the only two such cases, as $2^{|V(K_3)|-d_{K_3}(v_1)} = 2$. Now let us consider a guessing instruction of some vertex, say v_2. Assume for example that the vertex v_2 guesses it has the second color when v_1 and v_3 have the first color; it guesses it has the first color when v_1 and v_3 have the second color; otherwise it passes. We have $g_2(101) = 2$, $g_2(202) = 1$ and $g_2(102) = g_2(201) = 0$. If a case c is such that $c(v_1) = c(v_3) = 1$ and $c(v_2) = 2$, then the guess of v_2 is correct as $g_2(101) = 2 = c(v_2)$.

1.3 Fundamental theorems

Now we summarize a part of [40], which is Section 2.1 in this thesis.

We present a relation between the hat number of a graph and the hat number of its any subgraph. We characterize the number of cases in which the loss of the team is caused by a guess of a vertex. We also give a sufficient condition for removing a vertex of a graph without changing its hat number. Let us observe that in a case in which some vertex already guesses its color, a guess of any other vertex is unnecessary.

Theorem 1. *Let G be a graph.*

- If H is a subgraph of G, then $h(H) \leq h(G)$.
- We have $h(G) \geq 1/2$.
- If $S \in \mathcal{F}^0(G)$, then $p(S) \geq 1/2$.
- Let $H \subseteq G$ and $S \in \mathcal{F}^0(G)$. If there exists a strategy S' for the graph H such that $p(S') = p(S)$, then $S' \in \mathcal{F}^0(H)$.
- Let v_i be a vertex of G. If v_i guesses its color in a situation, then the team loses for at least half of cases corresponding to this situation.

- We have $h(G) < 1$.

- Let v be a vertex of G. If a strategy $S \in \mathcal{F}(G)$ is such that v always guesses its color, then $p(S) \leq 1/2$.

- Let v be a vertex of G. If there is a strategy $S \in \mathcal{F}^0(G)$ such that v always guesses its color, then $h(G) = 1/2$.

- Let v be a vertex of G. If there is a strategy $S \in \mathcal{F}^0(G)$ such that v never guesses its color, then $h(G) = h(G - v)$.

- Let c be a case in which some vertex guesses its color. Then a guess of any other vertex cannot improve the result of the case c.

1.4 Main results

Now we present the main results concerning the hat problem on a graph.

Hat problem on a tree

The following solution of the hat problem on paths is a part of [40], which is Section 2.1 in this thesis.

Lemma 2. *For every path P_n we have $h(P_n) = 1/2$.*

In Section 2.1 in this thesis [40] we use the above lemma to solve the hat problem on trees.

Theorem 3. *For every tree T we have $h(T) = 1/2$.*

Hat problem on cycles on four or at least nine vertices

The following solution of the hat problem on the cycle on four vertices has been published as [42], which is Section 2.2 in this thesis.

Theorem 4. $h(C_4) = 1/2$.

One can observe that Theorem 1 and Lemma 2 together with the inequality $(7/8)^6 < 1/2$ imply that $h(C_n) = 1/2$ for integers $n \geq 18$.

Theorem 5. *For every integer $n \geq 18$ we have $h(C_n) = 1/2$.*

Proof. We assume that $E(C_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$. Let S be an optimal strategy for C_n. If some vertex, say v_i, never guesses its color, then by Theorem 1 we have $h(C_n) = h(C_n - v_i)$. Since $C_n - v_i = P_{n-1}$ and $h(P_{n-1}) = 1/2$ (by Lemma 2), we get $h(C_n) = 1/2$. Now assume that every vertex guesses its color (rather than passing) in some situation. A single guess of a vertex v_i is wrong in exactly 1/8 of all cases. Since the closed neighborhoods of the vertices $v_2, v_5, v_8, v_{11}, v_{14}, v_{17}$ are pairwise disjoint, in at most $(7/8)^6$ of all cases no one of these vertices guesses its color wrong. Thus the team can win for at most $(7/8)^6$ of all cases. Since $(7/8)^6$ $< 1/2$, we have $p(S) < 1/2$. Now we get $h(C_n) = p(S) < 1/2$, a contradiction. □

Now we proceed to solve the hat problem on cycles on at least nine vertices. The following results are from [46], which is Section 2.3 in this thesis.

We assume that $E(C_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$. Let S be a strategy for C_n such that every vertex guesses its color (rather than passing) in exactly one situation. Let $\alpha_i(S), \beta_i(S), \gamma_i(S)$ (we write $\alpha_i, \beta_i, \gamma_i$) be such that the guess of v_i is wrong when $c(v_{i-1}) = \alpha_i$, $c(v_i) = \beta_i$ and $c(v_{i+1}) = \gamma_i$ (for $i \in \{2, 3, \ldots, n-1\}$), the guess of v_1 is wrong when $c(v_n) = \alpha_1$, $c(v_1) = \beta_1$ and $c(v_2) = \gamma_1$, and the guess of v_n is wrong when $c(v_{n-1}) = \alpha_n$, $c(v_n) = \beta_n$ and $c(v_1) = \gamma_n$. For example, if the vertex v_2 guesses it has the second color when v_1 has the first color and v_3 has the second color, then it follows that the vertex v_2 guesses its color wrong when $c(v_1) = c(v_2) = 1$ and $c(v_3) = 2$. Therefore $\alpha(v_2) = \beta(v_2) = 1$ and $\gamma(v_2) = 2$.

Let us consider strategies such that every vertex guesses its color (rather than passing) in exactly one situation. In the following lemma we give such strategy for which the number of cases in which some vertex guesses its color wrong is as small as possible.

Lemma 6. *Let us consider the strategies for C_n such that every vertex guesses its color (rather than passing) in exactly one situation. The number of cases in which* *some vertex guesses its color wrong is minimal for strategies S such that $\gamma_{i-1} = \beta_i = \alpha_{i+1}$ (for $i \in \{2, 3, \ldots, n-1\}$), $\gamma_{n-1} = \beta_n = \alpha_1$ and $\gamma_n = \beta_1 = \alpha_2$.*

For integers $n \geq 3$, let

$$A_n = \{c \in C(C_n)\colon c(v_{i-1}) = c(v_i) = c(v_{i+1}) = 1, \text{ for some } i \in \{2, 3, \ldots, n-1\}\}.$$

Thus A_n is the set of cases for C_n in which there are three vertices of the first color the indices of which are consecutive integers. Let a sequence $\{a_n\}_{n=0}^{\infty}$ be such that $a_n = |A_n|$ (for $n \geq 3$) and $a_0 = a_1 = a_2 = 0$.

In the following lemma we give a recursive formula for a_n.

Lemma 7. *If $n \geq 3$ is an integer, then $a_n = 2^{n-3} + a_{n-3} + a_{n-2} + a_{n-1}$.*

For integers $n \geq 3$, let

$$B_n = \{c \in C(C_n)\colon c(v_{i-1}) = c(v_i) = c(v_{i+1}) = 1 \text{ (for some } i \in \{2, 3, \ldots, n-1\})$$
$$\text{or } c(v_{n-1}) = c(v_n) = c(v_1) = 1 \text{ or } c(v_n) = c(v_1) = c(v_2) = 1\}.$$

Thus B_n is the set of cases for C_n in which there are three consecutive vertices of the first color. Let a sequence $\{b_n\}_{n=3}^{\infty}$ be such that $b_n = |B_n|$.

Now we give a relation between the number b_n (where $n \geq 6$) and the elements of the sequence $\{a_n\}_{n=0}^{\infty}$.

Lemma 8. *If $n \geq 6$ is an integer, then $b_n = 5 \cdot 2^{n-6} + a_n - 2a_{n-5} - a_{n-6}$.*

Now we give a lower bound on the number b_n for $n \geq 9$.

Lemma 9. *For every integer $n \geq 9$ we have $b_n > 2^{n-1}$.*

Next we solve the hat problem on cycles on at least nine vertices.

Theorem 10. *For every integer $n \geq 9$ we have $h(C_n) = 1/2$.*

The above approach does not succeed in solving the hat problem on any cycle on less than nine vertices, because the inequality $b_n > 2^{n-1}$ (see Lemma 9) holds only for integers $n \geq 9$.

Hat problem on a unicyclic graph

We say that a graph is unicyclic if it contains exactly one cycle as a subgraph.

Now we present a result from Section 2.4 in this thesis [44], where we solve the hat problem on unicyclic graphs containing a cycle on at least nine vertices.

Theorem 11. *If G is a unicyclic graph containing a cycle C_k for some $k \geq 9$, then $h(G) = 1/2$.*

Hat problem on a graph with a universal vertex

Now we consider the hat problem on graphs G with a universal vertex, that is, a vertex v such that $N_G[v] = V(G)$. The following results are from [44], which is Section 2.4 in this thesis.

Optimal strategies for graphs with a universal vertex have the following property.

Fact 12. *Let G be a graph, and let v be a universal vertex of G. If $S \in \mathcal{F}^0(G)$, then for every situation of v, in at least one of the two cases corresponding to this situation some vertex guesses its color.*

Now let us consider a strategy for a graph with a universal vertex such that there are two cases corresponding to the same situation of a universal vertex, and in one of them some vertex guesses its color while in the other one no vertex guesses its color. In the following lemma we give a method of designing a strategy, which is not worse than that.

Lemma 13. *Let G be a graph, and let v be a universal vertex of G. Let c and d be cases corresponding to the same situation of v. Assume that a strategy $S \in \mathcal{F}(G)$ is such that in the case c no vertex guesses its color and in the case d some vertex guesses its color. Let the strategy S' for the graph G differ from S only in that in the situation to which correspond the cases c and d the vertex v guesses it has the color which it has in the case c. Then $p(S') \geq p(S)$.*

One can prove that if a graph has a universal vertex, then there exists an optimal strategy such that in every case some vertex guesses its color. This implies that to solve the hat problem on a graph with a universal vertex, it suffices to examine only strategies such that in every case some vertex guesses its color. Thus, if in some case of a strategy no vertex guesses its color, then we can cease further examining this strategy.

Theorem 14. *If a graph G has a universal vertex, then there is a strategy $S \in \mathcal{F}^0(G)$ such that $|Ln(S)| = 0$.*

There exists a graph with a universal vertex for which there is an optimal strategy such that in some case no vertex guesses its color.

Fact 15. *There exists a strategy $S \in \mathcal{F}^0(K_2)$ such that $|Ln(S)| > 0$.*

Hat problem on graphs with neighborhood-dominated vertex

We say that a vertex v of a graph G is neighborhood-dominated if there is some other vertex $w \in V(G)$ such that $N_G(v) \subseteq N_G(w)$.

Now we present results from Section 2.4 in this thesis [44], where we consider the hat problem on graphs with a neighborhood-dominated vertex.

First, we investigate optimal strategies for such graphs.

Theorem 16. *Let G be a graph, and let v_1 and v_2 be vertices of G. If $N_G(v_1) \subseteq N_G(v_2)$, then there exists an optimal strategy for the graph G such that there is no case in which both vertices v_1 and v_2 guess their colors.*

Corollary 17. *Let G be a graph, and let $v_1, v_2, \ldots, v_k$ be vertices of G such that $N_G(v_1) = N_G(v_2) = \ldots = N_G(v_k)$. Then there exists an optimal strategy for the graph G such that in every situation at most one of the vertices $v_1, v_2, \ldots, v_k$ guesses its color.*

There exists a graph having two vertices with the same open neighborhood for which there is an optimal strategy such that in some situation both these vertices guess their colors.

Fact 18. *There exists a strategy $S \in \mathcal{F}^0(P_3)$ such that in some situation both leaves guess their colors.*

Let G be a graph, and let $A_1, A_2, \dots, A_k$ be a partition of the set of vertices of G such that the open neighborhoods of the vertices in each set A_i can be linearly ordered by inclusion.

Now we give an upper bound on the chance of success of any strategy for the hat problem on a graph with neighborhood-dominated vertices.

Theorem 19. *Let G be a graph and let k mean the minimum number of sets to which $V(G)$ can be partitioned in a way described above. Then $h(G) \le k/(k+1)$.*

Next we use the previous theorem to solve the hat problem on the graph H given in Figure 1. This graph is obtained from K_4 by the subdivision of one edge.

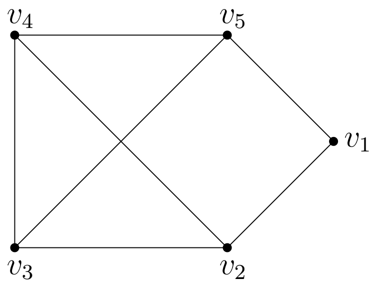


Figure 1: The graph H

Fact 20. $h(H) = 3/4$.

Hat problem on a disconnected graph

Now we present results from Section 2.5 in this thesis [45], where we consider the hat problem on disconnected graphs.

Let G and H be vertex-disjoint graphs, and let $S_1 \in \mathcal{F}(G)$ and $S_2 \in \mathcal{F}(H)$. By the union of the strategies S_1 and S_2 we mean the strategy $S \in \mathcal{F}(G \cup H)$ such that every vertex of G behaves in the same way as in S_1 and every vertex of H behaves in the same way as in S_2. If S is the union of S_1 and S_2, then we write $S = S_1 \cup S_2$.

From now to the end of this subsection, writing that G and H are graphs we assume that they are vertex-disjoint.

In the following theorem we give a sufficient condition for that the union of two strategies gives worse chance of success than some component of the union.

Theorem 21. *Let G and H be graphs, and let $S = S_1 \cup S_2$, where $S_1 \in \mathcal{F}(G)$ and $S_2 \in \mathcal{F}(H)$. Assume that $p(S_1) > 0$ and $p(S_2) > 0$. If $|Ln(S_1)| \cdot |Ln(S_2)| < |Ls(S_1)| \cdot |Ls(S_2)|$, then $p(S) < \max\{p(S_1), p(S_2)\}$.*

Corollary 22. *Let G and H be graphs, and let $S = S_1 \cup S_2$, where $S_1 \in \mathcal{F}(G)$ and $S_2 \in \mathcal{F}(H)$. Assume that $p(S_1) > 0$ and $p(S_2) > 0$. If $|Ln(S_1)| = 0$ or $|Ln(S_2)| = 0$, then $p(S) < \max\{p(S_1), p(S_2)\}$.*

From now to the end of this subsection, writing $S_1 \in \mathcal{F}(G)$ and $S_2 \in \mathcal{F}(H)$ we assume that $p(S_1) > 0$, $p(S_2) > 0$ and $|Ln(S_1)| \cdot |Ln(S_2)| \geq |Ls(S_1)| \cdot |Ls(S_2)|$.

The next theorem determines when the union of two strategies gives at least the same chance of success as each component of the union.

Theorem 23. *If G and H are graphs and $S = S_1 \cup S_2$, where $S_1 \in \mathcal{F}(G)$ and $S_2 \in \mathcal{F}(H)$, then*

$$p(S) \geq \max\{p(S_1), p(S_2)\} \Leftrightarrow \frac{|W(S_1)|}{|W(S_2)|} \in \left[\frac{|Ls(S_1)|}{|Ln(S_2)|}; \frac{|Ln(S_1)|}{|Ls(S_2)|}\right].$$

Corollary 24. *If G and H are graphs and $S = S_1 \cup S_2$, where $S_1 \in \mathcal{F}(G)$ and $S_2 \in \mathcal{F}(H)$, then*

$$p(S) < \max\{p(S_1), p(S_2)\} \Leftrightarrow \frac{|W(S_1)|}{|W(S_2)|} \notin \left[\frac{|Ls(S_1)|}{|Ln(S_2)|}; \frac{|Ln(S_1)|}{|Ls(S_2)|}\right].$$

The following theorem determines when the union of two strategies gives a chance of success better than each component of the union.

Theorem 25. *If G and H are graphs and $S = S_1 \cup S_2$, where $S_1 \in \mathcal{F}(G)$ and $S_2 \in \mathcal{F}(H)$, then*

$$p(S) > \max\{p(S_1), p(S_2)\} \Leftrightarrow \frac{|W(S_1)|}{|W(S_2)|} \in \left(\frac{|Ls(S_1)|}{|Ln(S_2)|}; \frac{|Ln(S_1)|}{|Ls(S_2)|}\right).$$

The next theorem determines when the union of two strategies gives the same chance of success as some component of the union.

Theorem 26. *If G and H are graphs and $S = S_1 \cup S_2$, where $S_1 \in \mathcal{F}(G)$ and $S_2 \in \mathcal{F}(H)$, then*

$$p(S) = p(S_1) \Leftrightarrow \frac{|W(S_1)|}{|W(S_2)|} = \frac{|Ln(S_1)|}{|Ls(S_2)|}$$

and

$$p(S) = p(S_2) \Leftrightarrow \frac{|W(S_1)|}{|W(S_2)|} = \frac{|Ls(S_1)|}{|Ln(S_2)|}.$$

Corollary 27. *Assume that G and H are graphs and $S = S_1 \cup S_2$, where $S_1 \in \mathcal{F}(G)$ and $S_2 \in \mathcal{F}(H)$. Let $i \in \{1,2\}$ be such that $p(S_i) = \max\{p(S_1), p(S_2)\}$, and let $j \in \{1,2\}$, $j \neq i$. Then*

$$p(S) = \max\{p(S_1), p(S_2)\} \Leftrightarrow \frac{|W(S_i)|}{|W(S_j)|} = \frac{|Ln(S_i)|}{|Ls(S_j)|}.$$

There exists a disconnected graph for which there is an optimal strategy such that every vertex guesses its color.

Fact 28. *There exists a strategy $S \in \mathcal{F}^0(K_2 \cup K_2)$ such that every vertex guesses its color.*

Hat problem on a graph when are known only degrees of vertices

Now we consider the hat problem on a graph such that the only known information are degrees of vertices. A major part of material presented here is from [40], which is Section 2.1 in this thesis.

In the following theorem we give an upper bound on the chance of success of any strategy for a graph which is based only on the degrees of vertices.

Theorem 29. *Let G be a graph, and let S be any strategy for this graph. Then*

$$|W(S)| \leq \sum_{v \in V(G)} \left\lfloor 2^{d_G(v)+1} - \frac{|W(S)|}{2^{|V(G)|-d_G(v)-1}} \right\rfloor \cdot 2^{|V(G)|-d_G(v)-1}.$$

We use the previous theorem to solve the hat problem on complete graphs on two, three and four vertices.

Fact 30. $h(K_2) = 1/2$.

Fact 31. $h(K_3) = 3/4$.

Fact 32. $h(K_4) = 3/4$.

Next we solve the hat problem on the graph $K_3 \cup K_2$.

Fact 33. $h(K_3 \cup K_2) = 3/4$.

Nordhaus-Gaddum type inequalities

A Nordhaus-Gaddum type result is a lower or upper bound on the sum or product of a parameter of a graph and its complement. In 1956 Nordhaus and Gaddum [51] proved the following inequalities for the chromatic number of a graph G and its complement: $2\sqrt{n} \le \chi(G) + \chi(\overline{G}) \le n + 1$ and $n \le \chi(G)\chi(\overline{G}) \le (n + 1)^2/4$.

In Section 2.4 in this thesis [44] we give Nordhaus-Gaddum type inequalities for the hat number.

Using Theorem 1 we immediately get the following lower and upper bounds on the sum and product of the hat numbers of a graph and its complement.

Fact 34. *For every graph G we have $1 \le h(G)+h(\overline{G}) < 2$ and $1/4 \le h(G)h(\overline{G}) < 1$.*

We show that for every number smaller than two there exists a graph for which the sum of its hat number and the hat number of its complement is greater than that number. We also show that for every number smaller than one there exists a graph for which the product of its hat number and the hat number of its complement is greater than that number.

Theorem 35.

- For every $\alpha < 2$ there is a graph G such that $h(G) + h(\overline{G}) > \alpha$.
- For every $\alpha < 1$ there is a graph G such that $h(G)h(\overline{G}) > \alpha$.

1.5 Recent results and state of the art

In this section we review recent results concerning the hat problem on a graph.

We say that a vertex v of a graph G is neighborhood-dominated if there is some other vertex $w \in V(G)$ such that $N_G(v) \subseteq N_G(w)$.

After publishing the results reviewed in the previous sections, Uriel Feige [25] has proved the following property of graphs with a neighborhood-dominated vertex.

Lemma 36. *Let G be a graph. If v is a neighborhood-dominated vertex of G, then $h(G) = h(G - v)$.*

The above result implies that the hat number of every bipartite graph is 1/2. In particular, this solves the hat problem on trees and cycles of even length. One can also observe that Lemma 36 implies that $h(G) \leq h(K_{\chi(G)})$, where $\chi(G)$ means the chromatic number of G. The lemma also implies that $h(G) = h(K_{\omega(G)})$ for graphs such that $\chi(G) = \omega(G)$, where $\omega(G)$ means the clique number of G. Since $h(K_4)$ $= 3/4$, one can conclude that the hat number of every planar graph containing a triangle equals 3/4.

Feige [25] has proved that the problem of computing $h(G)$ is NP-hard. He has also proved that for every disjoint graphs G and H we have $h(G \cup H) = \max\{h(G), h(H)\}$.

Hat problem on odd cycles

Now we present results from Section 2.6 in this thesis [41], where we solve the hat problem on cycles of odd length. Obviously, $h(C_3) = 3/4$.

Uriel Feige [25] has conjectured that the hat number of any graph equals the hat number of its maximum clique. He has proved this for graphs with equal chromatic and clique numbers. A well known class of such graphs is that of perfect graphs (where the equality holds not only for the graph, but also for all its subgraphs). Thus Feige has solved the hat problem for perfect graphs. By the strong perfect graph theorem [16], every graph such that neither it nor its complement contains an induced odd cycle of length at least five is perfect. Thus a next step to prove or refute the conjecture could be to solve the hat problem on odd cycles. We prove that the hat number of every odd cycle on at least five vertices is 1/2, which is consistent with the conjecture of Feige.

First we solve the hat problem on the cycle on five vertices.

Lemma 37. *$h(C_5) = 1/2$.*

Next we prove the main result.

Theorem 38. *For every odd integer $n \geq 5$ we have $h(C_n) = 1/2$.*

Since cycles of even length are bipartite, we conclude that the hat number of every cycle on at least four vertices equals 1/2.

1.6 Hat problem on a directed graph

This section contains the results of the joint work [34], which is Section 2.7 in this thesis. Here we consider the hat problem on a directed graph. If there is an arc from u to v, then the vertex u can see the vertex v. Still we can restrict to deterministic strategies.

Previous works focused on the problem on undirected graphs. It was solved for some classes of graphs, leading Uriel Feige to conjecture that the hat number of any graph equals the hat number of its maximum clique.

We show that the conjecture does not hold for directed graphs. Moreover, for every value of the maximum clique size, we provide a tight characterization of the range of possible values of the hat number. We construct families of directed graphs with a fixed clique number the hat number of which is asymptotically optimal. We also determine the hat number of tournaments to be one half.

A *directed graph* (called a *digraph*) is an ordered pair $D = (V, A)$, where a set V is called the *set of vertices* and E is the *set of arcs*, which are ordered pairs of vertices. We say that E is a *subgraph* of a digraph D if $V(E) \subseteq V(D)$ and $A(E) \subseteq A(D)$. Then we write $E \subseteq D$. A *tournament* is a directed graph such that for every two vertices there is exactly one arc between them. By the *skeleton* of a digraph D, denoted by $\text{skel}(D)$, we mean an undirected graph with the vertex set V in which x and y are adjacent if both arcs between them are present in D. By the *clique number* of a digraph D we mean the clique number of its skeleton, that is, $\omega(D) = \omega(\text{skel}(D))$.

Given two disjoint digraphs C and D, we define the *directed union* of C and D, denoted by $C \to D$, to be the union of these two digraphs with additional arcs from all vertices of C to all vertices of D. Notice that this operator is associative, that is, $(C \to D) \to E = C \to (D \to E)$, for any three digraphs C, D and E. Thus the notation $C \to D \to E$ is unambiguous. The directed union of n disjoint copies of a digraph D, that is $\underbrace{D \to D \to \ldots \to D}_{n}$, we denote by $D^{\to n}$.

All concepts regarding the hat problem we define similarly as when considering the hat problem on undirected graphs.

Let us observe that the following statements regarding the hat problem on a directed graph, which are generalizations of those for undirected graphs, are also true.

Theorem 39. *Let D be a digraph.*

- If $E \subseteq D$, then $h(E) \leq h(D)$.
- We have $h(D) \geq 1/2$.
- Let v be a vertex of D. If there is a strategy $S \in \mathcal{F}^0(D)$ such that v always guesses its color, then $h(D) = 1/2$.
- Let v be a vertex of D. If there is a strategy $S \in \mathcal{F}^0(D)$ such that v never guesses its color, then $h(D) = h(D - v)$.
- Let c be a case in which some vertex guesses its color. Then a guess of any other vertex cannot improve the result of the case c.
- Let v be a vertex of D. If v has no outgoing arcs, then $h(D) = h(D - v)$.

Since the hat number of the complete graph K_m is known to be $m/(m+1)$ when $m + 1$ is a power of two [23], we have the following lower bound on the hat number of any digraph.

Lemma 40. *For every digraph D we have $h(D) \geq \omega(D) / (\omega(D) + 2)$.*

For an undirected graph G, it is known that if G contains a triangle, then $h(G) \geq 3/4$, and in [25] it is conjectured that if G is triangle-free, then $h(G) = 1/2$. We show that directed graphs introduce something in between. Let us consider the hat problem on the digraph D_1 given in Figure 2.

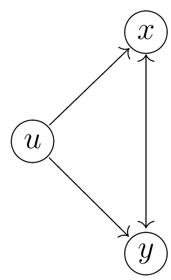
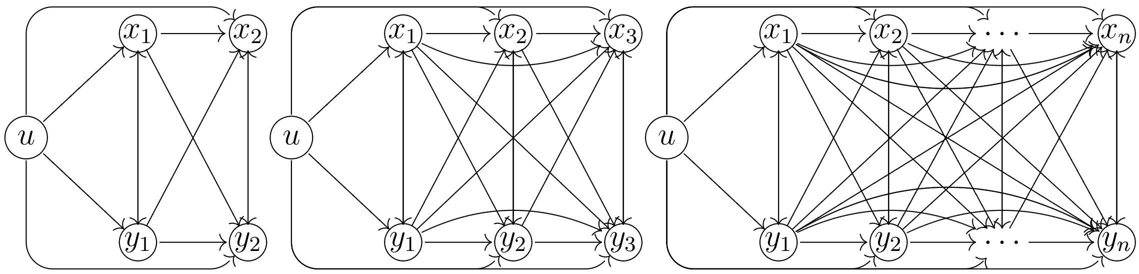


Figure 2: The directed graph D_1

Fact 41. $h(D_1) = 5/8$.

We extend D_1 to a sequence of digraphs that asymptotically achieve the hat number $2/3$, with the property that $\omega(D_n) = 2$. Let $D_n = K_1 \to K_2^{\to n}$. Note that the family $\{D_n\}_{n=1}^{\infty}$ satisfies the recurrence relation $D_{n+1} = D_n \to K_2$. In Figure 3 we give examples of D_n for $n = 2$, $n = 3$, and a general n.



(a) $n = 2$ (b) $n = 3$ (c) General n

Figure 3: The directed graphs D_2, D_3 and D_n. All vertical arcs have antiparallel counterparts. The remaining arcs are rightwards

We proceed to compute the hat numbers of the digraphs of the family $\{D_n\}_{n=1}^{\infty}$. First we prove an upper bound.

Lemma 42. *For every digraph D we have $h(D \to K_2) \leq \max\{h(D), 1/2+h(D)/4\}$.*

Next we prove a lower bound.

Lemma 43. *For every digraph D we have $h(D \to K_2) \geq 1/2 + h(D)/4$.*

In the next lemma we give a lower bound for a more general setting.

Lemma 44. *For every positive integer m there exists $c \geq 1/2$ such that for any digraph D we have $h(D \to K_m) \geq cm/(m+1) + (1-c) \cdot h(D)$. Moreover, if $m = 2$, then $c = 3/4$ satisfies the inequality.*

We use Lemmas 42 and 43 to calculate the hat number of D_n.

Fact 45. *For every positive integer n we have*

$$h(D_n) = \frac{2}{3} - \frac{1}{6} \cdot \frac{1}{4^n}.$$

Corollary 46. *For every $\varepsilon > 0$ there exists a digraph D satisfying $\omega(D) = 2$ such that $h(D) > 2/3 - \varepsilon$.*

We generalize the previous result to an arbitrary clique number m.

Theorem 47. *For every $\varepsilon > 0$ there exists a digraph D satisfying $\omega(D) = m$ such that $h(D) > m/(m+1) - \varepsilon$.*

A natural question is whether a chance of success better than $m/(m+1)$ is possible for such digraphs. It turns out that $m/(m+1)$ is asymptotically optimal for digraphs with clique number m. Feige [25] proved that for every undirected graph G we have $h(G) \leq \omega(G)/(\omega(G)+1)$. We refine his proof to show that the same holds for digraphs.

Theorem 48. *For every digraph D we have $h(D) \leq \omega(D)/(\omega(D)+1)$.*

Observe that for any digraph D the hat number $h(D)$ is always a rational number whose denominator is a power of two. Therefore $h(D) < \omega(D) / (\omega(D)+1)$ unless $\omega(D)$ is a power of two decreased by one. If $\omega(D) = 2^k - 1$, then the upper bound is met by a complete graph K_{2^k-1} as $h(K_{2^k-1}) = (2^k-1)/2^k$.

Corollary 49. *For every tournament T we have $h(T) = 1/2$.*

Søren Riis [54] defined for directed graphs a guessing game with $q \geq 2$ colors, which differs from the hat problem in that the team wins only if every vertex guesses its color correctly. A guessing number of a digraph D equals k if there is a strategy such that the team wins with probability $(1/s)^{n-k}$. The aim is to determine the maximum guessing number of a given graph. We have shown that the hat number of a directed union of two digraphs may be different from the hat number of their union. It turns out that it is not the case concerning the guessing game. Gadouleau and Riis [28] proved that the guessing number of a directed union of two graphs always equals the guessing number of their union. Therefore the additional arcs are superfluous for the guessing game. By Corollary 49, the hat number of every tournament equals one half, while the maximum guessing number of tournaments is of the same order as the number of players [28]. Thus here the arcs are useful for the guessing game, but not for the hat problem.

1.7 Hat problem with q colors

In this section, which is similar to [39] (Section 2.8 in this thesis), we generalize the hat problem to $q \geq 2$ colors. Let us observe that we can restrict to deterministic strategies due to the same reasons as for the hat problem on a graph.

All concepts regarding the hat problem with q colors we define similarly as when considering the problem with two colors. The family of all strategies for the hat problem with n players and q colors is denoted by $\mathcal{F}(n, q)$. We define $h(n, q)$ to be the maximum chance of success for this problem.

First we investigate the hat problem with three colors. Our main result is the solution for three players. Obviously, if there is only one player, then the chance of success is $1/3$. As an example, we show that already two players can do better.

Let us consider the following strategy for the hat problem with two players and three colors.

Strategy 50. *Let $S = (g_1, g_2) \in \mathcal{F}(2, 3)$ be a strategy such that*

$$g_1(s_1) = \begin{cases} 1 & \text{if } s_1(v_2) \neq 3, \\ 0 & \text{otherwise;} \end{cases}$$

$$g_2(s_2) = \begin{cases} 3 & \text{if } s_2(v_1) \neq 1, \\ 0 & \text{otherwise.} \end{cases}$$

It means that the players proceed as follows.

- **The player** v_1. If v_2 has a hat of the first or the second color, then he guesses he has a hat of the first color, otherwise he passes.
- **The player** v_2. If v_1 has a hat of the second or the third color, then he guesses he has a hat of the third color, otherwise he passes.

Analyzing all cases, we make the following observation.

Observation 51. *Using Strategy 50 the team wins for 4 of 9 cases.*

Now we solve the hat problem with two players and three colors.

Fact 52. $h(2, 3) = 4/9$.

Proof. Since using Strategy 50 the team wins for 4 of 9 cases, we have $h(2,3) \geq 4/9$. Suppose that $h(2,3) > 4/9$, that is, there exists a strategy S for the hat problem with two players and three colors such that the team wins for more than 4 cases. Any guess made by any player in any situation is wrong in exactly two cases, because to any situation of any player correspond three cases, and in exactly two of them this player has a hat of a color differing from the one he guesses. In the strategy S every player guesses his hat color in at most 2 situations, because if some player guesses his hat color in at least 3 situations, then the team loses for at least 6 cases, and wins for at most 3 cases, a contradiction. Any guess made by any player in any situation is correct in exactly one case, because to any situation of any player correspond three cases, and in exactly one of them this player has a hat of the color he guesses. There are two players, every one of them guesses his hat color in at most two cases, and every guess is correct in exactly one case. Therefore using the strategy S the team wins for at most 4 cases, a contradiction. □

Now we proceed to solve the hat problem with three players and three colors.

We say that a strategy is *symmetric* if every player makes his decision on the basis of only numbers of hats of each color seen by him, and all players behave in the same way. A strategy is *nonsymmetric* if it is not symmetric.

The authors of [31] solved the hat problem with three players and three colors by giving a symmetric strategy found by computer, and proving that it is optimal.

We solve this problem by proving the optimality of a nonsymmetric strategy found without using a computer.

Let us consider the following strategy for the hat problem with three players and three colors.

Strategy 53. *Let $S = (g_1, g_2, g_3) \in \mathcal{F}(3,3)$ be a strategy such that*

$$g_1(s_1) = \begin{cases} s_1(v_3) & \textit{if } s_1(v_2) \neq s_1(v_3), \\ 0 & \textit{otherwise;} \end{cases}$$

$$g_2(s_2) = \begin{cases} s_2(v_3) & \textit{if } s_2(v_1) \neq s_2(v_3), \\ 0 & \textit{otherwise;} \end{cases}$$

$$g_3(s_3) = \begin{cases} s_3(v_1) & \textit{if } s_3(v_1) = s_3(v_2), \\ 0 & \textit{otherwise.} \end{cases}$$

It means that the players proceed as follows.

- **The player** v_1. If v_2 and v_3 have hats of different colors, then he guesses he has a hat of the color v_3 has, otherwise he passes.

- **The player** v_2. If v_1 and v_3 have hats of different colors, then he guesses he has a hat of the color v_3 has, otherwise he passes.

- **The player** v_3. If v_1 and v_2 have hats of the same color, then he guesses he has a hat of the color they have, otherwise he passes.

Analyzing all cases, we make the following observation.

Observation 54. *Using Strategy 53 the team wins for* 15 *of* 27 *cases.*

Next we solve the hat problem with three players and three colors.

Fact 55. $h(3,3) = 5/9$.

The hat problem with three colors and more than three players remains unsolved.

Now, we investigate the hat problem with n players and q colors. First we prove an upper bound on the number of cases for which the team wins using any strategy for the problem.

Theorem 56. *For every strategy* $S \in \mathcal{F}(n,q)$ *we have*

$$|W(S)| \leq n \left\lfloor \frac{q^n - |W(S)|}{q-1} \right\rfloor.$$

Now we formulate an equivalent upper bound on the chance of success of any strategy for the hat problem with n players and q colors.

Theorem 57. *For every strategy* $S \in \mathcal{F}(n,q)$ *we have*

$$p(S) \leq \frac{n}{q^n} \left\lfloor \frac{q^n - q^n \cdot p(S)}{q-1} \right\rfloor.$$

Let us observe that Facts 52 and 55 follow from Theorem 56 as well as from Theorem 57.

Next we prove a weaker theorem (following from Theorem 56 or Theorem 57), which is an explicit upper bound on the chance of success of any strategy for the hat problem with n players and q colors. This bound was previously proved as Proposition 3 in [47].

Theorem 58. *For every strategy $S \in \mathcal{F}(n, q)$ we have*

$$p(S) \le \frac{n}{n+q-1}.$$

Then we show that Theorem 56 does not follow from Theorem 58.

Now let us consider the hat problem with two colors ($q = 2$), and any strategy S for this problem. Using Theorem 58 we get the bound $p(S) \le n/(n+1)$ previously given in [23], which is sharp for $n = 2^k - 1$.

Noga Alon [2] proved that for the hat problem with n players and q colors there exists a strategy such that the chance of success is at least

$$1 - \frac{(q-1)\log n + 1}{n} - \left(1 - \frac{1}{q}\right)^n.$$

Now, we consider the number of strategies the verification of which suffices to solve the hat problem and the generalized hat problem with q colors.

First we count all possible strategies for the hat problem. We have n players, there are 2^{n-1} possible situations of each one of them, and in every situation there are three possibilities of behavior (to guess the first color, to guess the second color, or to pass). This implies that the number of possible strategies equals

$$\left(3^{2^{n-1}}\right)^n.$$

Next we prove that it is not necessary to examine every strategy to solve the hat problem.

Fact 59. *To solve the hat problem with n players, it suffices to examine*

$$\left(3^{2^{n-1}-2}\right)^n = \frac{1}{9^n} \cdot \left(3^{2^{n-1}}\right)^n$$

strategies.

Now, we count all possible strategies for the generalized hat problem with q colors. We have n players, there are q^{n-1} possible situations of each one of them, and in every situation there are $q+1$ possibilities of behavior (to guess one of the q colors, or to pass). This implies that the number of possible strategies equals

$$\left((q+1)^{q^{n-1}}\right)^n.$$

Next we prove that it is not necessary to examine every strategy to solve the generalized hat problem with q colors.

Fact 60. *To solve the hat problem with n players and q colors, it suffices to examine*

$$\left((q+1)^{q^{n-1}-1}\right)^n = \frac{1}{(q+1)^n} \cdot \left((q+1)^{q^{n-1}}\right)^n$$

strategies.

1.8 Modified hat problem

This section contains material from [38], which is Section 2.9 in this thesis. Let us consider the hat problem with $n \geq 3$ players and two colors, in which the players do not have to guess their hat colors simultaneously. Every player has two cards with his name and the sentence "I have a blue hat" or "I have a red hat". The players make a guess by coming to the basket and throwing the proper card into it. If someone wants to resign from answering, then he does not do anything. Let us consider the following strategy for this problem.

Strategy 61. *Players proceed as follows.*

Step 1 (5 seconds after the beginning)
Only these players who see the hats of one color only come to the basket. There are three possibilities:

- Only one player comes to the basket. Then he guesses he has a hat of the color differing from the one he sees.
- At least two players come to the basket. Then every one of them guesses he has a hat of the color he sees.

- No player comes to the basket. Then we execute Step 2.

Let i be a positive integer.

Step $2i$ ($10i$ seconds after the beginning)
Only these players who see exactly i blue hats come to the basket. There are two possibilities:

- At least one player comes to the basket. Then every one of them guesses he has a blue hat.
- No player comes to the basket. Then we execute Step $2i + 1$.

Step $2i + 1$ (($10i + 5$) seconds after the beginning)
Only these players who see exactly i red hats come to the basket. There are two possibilities:

- At least one player comes to the basket. Then every one of them guesses he has a red hat.
- No player comes to the basket. Then we execute Step $2i + 2$.

We prove that the above strategy always succeeds.

Theorem 62. *Strategy 61 always succeeds for the modified hat problem.*

Let us consider the numbers of blue and red hats on the heads of the players. If there are more blue hats than red hats, then let x mean the number of red hats. Otherwise let it mean the number of blue hats.

Next we show in which step the team wins using Strategy 61.

Fact 63. *If there are less than two blue hats or less than two red hats, then Strategy 61 succeeds in Step* 1. *In the opposite case, for* x *defined above, Strategy 61 succeeds in Step* $2x - 1$ *if there are more blue hats than red hats, and otherwise in Step* $2x - 2$.

1.9 Applications and variations of the hat problem

The hat problem has many applications and connections to different areas of science, for example: information theory [7], linear programming [26, 35], genetic programming [13], economics [1, 36], biology [31], approximating Boolean functions [4], and autoreducibility of random sequences [5, 21–24].

We describe these applications and connections in [43], which is Section 2.10 in this thesis.

There are known many variations of the hat problem. For example, in the papers [1, 14, 36] there was considered a variation in which passing is not allowed, thus everybody has to guess his hat color. The aim is to maximize the number of correct guesses. The authors of [26] investigated several variations of the hat problem in which the aim is to design a strategy guaranteeing a desired number of correct guesses. In [31] there was considered a variation in which the probabilities of getting hats of each color do not have to be equal. For more variations of the hat problem, see for example [3, 9, 12, 15, 18–20, 27, 29, 30, 32, 33, 57–60].

In [43], which is Section 2.10 in this thesis, we give a comprehensive list of variations of the hat problem considered in the literature. We also present what is already known about each variation. For some of them we give a strategy which solves the problem.

Chapter 2

Publications

2.1 Hat problem on a graph

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HAT PROBLEM ON A GRAPH

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Abstract: The topic of our paper is the hat problem. In that problem, each of n people is randomly fitted with a blue or red hat. Then everybody can try to guess simultaneously his own hat color looking at the hat colors of the other people. The team wins if at least one person guesses his hat color correctly and no one guesses his hat color wrong, otherwise the team loses. The aim is to maximize the probability of win. In this version every person can see everybody excluding him. In this paper we consider such problem on a graph, where vertices are people and a person can see these people, to which he is connected by an edge. We prove some general theorems about the hat problem on a graph and solve the problem on trees. We also consider the hat problem on a graph with given degrees of vertices. We give an upper bound that is based only on the degrees of vertices on the chance of success of any strategy for the graph G. We show that this upper bound together with integrality constraints is tight on some toy examples.

1. Introduction

In the hat problem, a team of n people enters a room and a blue or red hat is randomly placed on the head of each person. Each person can see the hats of all of the other people but not his own. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats,

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each person must simultaneously guess the color of his own hat or pass. The team wins if at least one person guesses his hat color correctly and no one guesses his hat color wrong, otherwise the team loses. The aim is to maximize the probability of win.

The hat problem with seven people called "seven prisoners puzzle" was formulated by T. Ebert in his Ph.D. Thesis [10]. The hat problem was also the subject of articles in The New York Times [20], Die Zeit [5], and abcNews [19]. The hat problem with n people and two colors of hat was investigated in [6]. It was solved for $2^k - 1$ people in [12]. The hat problem and Hamming codes were also the subject of an article in Polish math–physics–informatic magazine [9].

There are also known some variants and generalizations of hat problem. The authors of [18] investigate the generalized hat problem with $q \geq$ ≥ 2 colors, they also consider variants in which there are arbitrary input distributions, randomized playing strategies, and symmetric strategies. In the papers [1], [8], and [17] there is considered another variant of hat problem in which passing is not allowed, thus everybody has to try to guess his hat color. The aim is to maximize the number of correct guesses. In [14] the authors investigate several variants of hat problem in which the aim is to design a strategy such that the number of correct guesses is greater than or equal to the given positive integer. In the paper [15] there is considered the hat problem, and also a variant in which the probabilities of getting hats of each colors do not have to be equal. The authors of [2] investigate a problem similar to the hat problem. There are n people which have random bits on foreheads, and they have to vote on the parity of the n bits.

The hat problem and its variants have many applications and connections to other areas of science, for example: information technology [4], linear programming [14, 16], genetic programming [7], economy [1, 17], biology [15], approximating Boolean functions [2], and autoreducibility of random sequences [3, 10–13]. Therefore, it is hoped that the hat problem on a graph considered in this paper, as a natural generalization, is worth exploring, and may also have many applications.

We consider the hat problem on a graph, where vertices are people and a person can see these people, to which he is connected by an edge. We prove some general theorems about the hat problem on a graph and solve the problem on trees. We also consider the hat problem on a graph with given degrees of vertices. We give an upper bound that is based only on the degrees of vertices on the chance of success of any strategy for the graph G. We show that this upper bound together with integrality constraints is tight on some toy examples.

The paper is organized as follows. In Sec. 2 we give the notation and terminology used. In Sec. 3 first we make some general observations about the hat problem on a graph. In Th. 4 we solve that problem on paths, and in Th. 5 we solve the hat problem on trees. Then we consider the hat problem on a graph with given degrees of vertices.

2. Preliminaries

For a graph G, by $V(G)$ and $E(G)$ we denote the set of vertices and the set of edges of this graph, respectively. If H is a subgraph of G, then we write $H \subseteq G$. Let $v \in V(G)$. By $N_G(v)$ we denote the open neighbourhood of v, that is $N_G(v) = \{x \in V(G)\colon vx \in E(G)\}$. By $N_G[v]$ we denote the closed neighbourhood of v, that is $N_G[v] = N_G(v) \cup \{v\}$. By $d_G(v)$ we denote the degree of the vertex v, that is the number of its neighbours, thus $d_G(v) = |N_G(v)|$. By P_n we denote the path with n vertices. By C_n we denote the cycle with n vertices. By K_n we denote the complete graph with n vertices. Let $f\colon X \to Y$ be a function. If $Z \subseteq X$, then by $f_{|Z}$ we denote the restriction of f to Z. If $y \in Y$, then by $f \equiv y$ we denote that for every $x \in X$ we have $f(x) = y$.

Without loss of generality we may assume an ordering of the vertices of a graph G, that is $V(G) = \{v_1, v_2, \ldots, v_n\}$.

Let $\{b, r\}$ be the set of colors (b means blue and r means red). If $v_i \in V(G)$, then $c(v_i)$ is the color of v_i, so $c\colon V(G) \to \{b, r\}$ is a function. By a case for the graph G we mean a sequence $(c(v_1), c(v_2), \ldots, c(v_n))$. The set of all cases for the graph G we denote by $C(G)$, of course $|C(G)| = 2^{|V(G)|}$.

If $v_i \in V(G)$, then by s_i we denote a function $s_i\colon V(G) \to \{b, r, *\}$, where $s_i(v_j) \in \{b, r\}$ is the color of v_j if v_i sees v_j, and mark $*$ otherwise, that is, $s_i(v_j) = c(v_j)$ if $v_j \in N_G(v_i)$, while $s_i(v_j) = *$ if $v_j \in V(G) \setminus N_G(v_i)$. By a situation of the vertex v_i in the graph G we mean the sequence $(s_i(v_1), s_i(v_2), \ldots, s_i(v_n))$. The set of all possible situations of v_i in the graph G we denote by $St_i(G)$. Of course, $|St_i(G)| = 2^{|N_G(v_i)|}$.

Let $v_i \in V(G)$. We say that a case $(c_1, c_2, \ldots, c_n)$ for the graph G corresponds to a situation $(t_1, t_2, \ldots, t_n)$ of the vertex v_i in the graph G if it is created from this situation only by changing every mark $*$ to b or r.

So, a case corresponds to a situation of v_i if every vertex adjacent to v_i, in that case has the same color as in that situation. To every situation of the vertex v_i in the graph G correspond $2^{|V(G)|-|N_G(v_i)|}$ cases, because every situation of v_i has $|V(G)| - |N_G(v_i)|$ marks $*$.

Let G and H be graphs such that $V(H) = \{v_1, v_2, \ldots, v_m\}$, $V(G) = \{v_1, v_2, \ldots, v_m, \ldots, v_n\}$, and $E(H) \subseteq E(G)$. We say that a case $(a_1, a_2, \ldots, a_m, \ldots, a_n)$ for the graph G corresponds to a case $(b_1, b_2, \ldots, b_m)$ for the graph H if $(a_1, a_2, \ldots, a_m) = (b_1, b_2, \ldots, b_m)$, that is, every vertex from the graph H in both cases has the same color. Of course, to every case for the graph H correspond 2^{n-m} cases for the graph G.

Let G and H be graphs such that $V(H) = \{v_1, v_2, \ldots, v_m\}$, $V(G) = \{v_1, v_2, \ldots, v_m, \ldots, v_n\}$, and $E(H) \subseteq E(G)$. Let $i \in \{1, 2, \ldots, m\}$. We say that a situation $(t_1, t_2, \ldots, t_m, \ldots, t_n)$ of the vertex v_i in the graph G corresponds to a situation $(u_1, u_2, \ldots, u_m)$ of the vertex v_i in the graph H if $(t_1, t_2, \ldots, t_m) = (u_1, u_2, \ldots, u_m)$, that is, every vertex adjacent to v_i in the graph H, in both of these situations has the same color.

By a statement of a vertex we mean its declaration about the color it guesses it is. By the result of a case we mean a win or a loss. According to the definition of the hat problem, the result of a case is a win if at least one vertex states its color correctly and no vertex states its color wrong. The result of a case is a loss if no vertex states its color or some vertex states its color wrong.

By a guessing instruction for the vertex $v_i \in V(G)$ (denoted by g_i) we mean a function $g_i \colon St_i(G) \to \{b, r, p\}$ which, for a given situation, gives b or r meaning the color v_i guesses it is, or the letter p if v_i passes. Thus a guessing instruction is a rule which determines the behavior of the vertex v_i in every situation. By a strategy for the graph G we mean a sequence $(g_1, g_2, \ldots, g_n)$. By $\mathcal{F}(G)$ we denote the family of all strategies for the graph G.

Let $v_i \in V(G)$ and $S \in \mathcal{F}(G)$. We say that v_i never states its color in the strategy S if v_i passes in every situation, that is $g_i \equiv p$. We say that v_i always states its color in the strategy S if v_i states its color in every situation, that is, for every $T \in St_i(G)$ we have $g_i(T) \in \{b, r\}$ ($g_i(T) \neq p$, equivalently).

If $S \in \mathcal{F}(G)$, then by $Cw(S)$ and $Cl(S)$ we denote the sets of cases for the graph G in which the team wins or loses, respectively. Of course, $|Cw(S)| + |Cl(S)| = |C(G)|$. Consequently, by the chance of success of the strategy S we mean the number $p(S) = \frac{|Cw(S)|}{|C(G)|}$. By the hat number of the graph G we mean the number $h(G) = \max\{p(S)\colon S \in \mathcal{F}(G)\}$. Certainly $p(S) \le h(G)$. We say that the strategy S is optimal for the graph G if $p(S) = h(G)$. By $\mathcal{F}^0(G)$ we denote the family of all optimal strategies for the graph G.

Let $t, m_1, m_2, \dots, m_t \in \{1, 2, \dots, n\}$ be such that $m_j \neq m_k$ and $c_{m_j} \in \{b, r\}$, for every $j, k \in \{1, 2, \dots, t\}$.

By $C(G, v_{m_1}^{c_{m_1}}, v_{m_2}^{c_{m_2}}, \dots, v_{m_t}^{c_{m_t}})$ we denote the set of cases for the graph G such that the color of v_{m_j} is c_{m_j}.

Let $S \in \mathcal{F}(G)$. By $Cw(S, v_{m_1}^{c_{m_1}}, v_{m_2}^{c_{m_2}}, ..., v_{m_t}^{c_{m_t}})$ ($Cl(S, v_{m_1}^{c_{m_1}}, v_{m_2}^{c_{m_2}}, ..., v_{m_t}^{c_{m_t}})$, respectively) we denote the set of cases for G which belong to the set $C(G, v_{m_1}^{c_{m_1}}, v_{m_2}^{c_{m_2}}, \dots, v_{m_t}^{c_{m_t}})$, and in which the team wins (loses, respectively).

Let $v_i \in V(G)$. If for every $j \in \{1, 2, \dots, t\}$ we have $v_{m_j} \in N_G(v_i)$, then by $St_i(G, v_{m_1}^{c_{m_1}}, v_{m_2}^{c_{m_2}}, \dots, v_{m_t}^{c_{m_t}})$ we denote the set of possible situations of v_i in the graph G such that the color of v_{m_j} is c_{m_j}.

3. Results

First let us observe that indeed we can confine to deterministic strategies (that is strategies such that the decision of each person is determined uniquely by the hat colors of other people). We can do this since for any randomized strategy there exists a not worse deterministic one. It is true, because every randomized strategy is a convex combination of some deterministic strategies. The probability of winning is a linear function on the convex polyhedron corresponding to the set of all randomized strategies which can be achieved combining those deterministic strategies. It is well known that this function achieves its maximum on a vertex of the polyhedron which corresponds to a deterministic strategy.

Let G and H be graphs. Assume that $H \subseteq G$. Since every vertex from the set $V(G) \setminus V(H)$ can always pass, and every vertex $v_i \in V(H)$ can ignore the colors of vertices from the set $N_G(v_i) \setminus N_H(v_i)$, it is easy to see that the hat number of the graph G is greater than or equal to the hat number of the graph H. It is that if $H \subseteq G$, then $h(H) \le h(G)$.

Since K_1 is a subgraph of every graph, we get $h(G) \ge \frac{1}{2}$.

Let S be an optimal strategy for the graph G. By definition we have $p(S) = h(G)$. Since $h(G) \ge \frac{1}{2}$, we get $p(S) \ge \frac{1}{2}$.

Now we prove a fact characterizing the number of cases in which the loss of the team is caused by a statement of a vertex.

Fact 1. *Let G be a graph and let v_i be a vertex of G. Let $S \in \mathcal{F}(G)$. If v_i states its color in a situation, then the team loses in at least half of all cases corresponding to this situation.*

Proof. Assume that v_i states its color in a situation T. Without loss of generality we assume that in this situation v_i states it is blue, that is $g_i(T) = b$. In half of all cases corresponding to T we have $c(v_i) = r$, it means that v_i is red. Thus, the team loses in every one of these cases, because v_i states its color wrong, as $g_i(T) = b \neq r = c(v_i)$. $\Diamond$

Corollary 2. *Let G be a graph and let v be a vertex of G. If $S \in \mathcal{F}^0(G)$ is a strategy such that v always states its color, then $h(G) = \frac{1}{2}$.*

Proof. Assumption indicates that in every case v states its color, so by Fact 1 we have $|Cl(S)| \geq \frac{|C(G)|}{2}$. Consequently,

$$p(S) = \frac{|Cw(S)|}{|C(G)|} = \frac{|C(G)| - |Cl(S)|}{|C(G)|} \leq \frac{|C(G)| - \frac{|C(G)|}{2}}{|C(G)|} = \frac{1}{2}.$$

Since $p(S) \leq \frac{1}{2}$ and $S \in \mathcal{F}^0(G)$, we have $h(G) \leq \frac{1}{2}$ (by definition). On the other hand we have $h(G) \geq \frac{1}{2}$. $\Diamond$

In the following theorem we give a sufficient condition for deleting a vertex of a graph without changing its hat number.

Theorem 3. *Let G be a graph and let v be a vertex of G. If $S \in \mathcal{F}^0(G)$ is a strategy such that v never states its color, then $h(G) = h(G - v)$.*

Proof. Let $S' \in \mathcal{F}(G - v)$ be the strategy as follows: Every vertex not adjacent to v in G behaves in the same way as in S, that is, if $v_i \notin N_G(v)$, then $g'_i = g_i$, where g'_i and g_i are the guessing instructions for the vertex v_i in the strategies S' and S, respectively. First assume that $|Cw(S, v^b)| \geq$ $\geq |Cw(S, v^r)|$. Let every vertex adjacent to v in G behave in the same way as in S when v is blue, that is, if $v_i \in N_G(v)$, then $g'_i = g_{i|St_i(G,v^b)}$. The result of any case C' in the strategy S' is the same as the result of the case C in the strategy S, where C is the corresponding case in which v is blue, because in both strategies S' and S the vertex v never states its color and every vertex in the strategy S' behaves in the same way as in S when v is blue. This implies that $|Cw(S')| = |Cw(S, v^b)|$. Now we get

$$p(S') = \frac{|Cw(S')|}{2^{|V(G-v)|}} = \frac{|Cw(S, v^b)|}{2^{|V(G)|-1}} = \frac{2|Cw(S, v^b)|}{2^{|V(G)|}} \geq$$
$$\geq \frac{|Cw(S, v^b)| + |Cw(S, v^r)|}{2^{|V(G)|}} = p(S).$$

If $|Cw(S, v^b)| < |Cw(S, v^r)|$, then similarly we get a strategy S' such that $p(S') > p(S)$. Since $S \in \mathcal{F}^0(G)$ and $S' \in \mathcal{F}(G - v)$, we have

$$h(G) = p(S) \leq p(S') \leq h(G - v).$$

On the other hand we have $h(G) \geq h(G - v)$. $\Diamond$

Let S be a strategy for the graph G. Let C be a case in which some vertex states its color. Since the rules of the hat problem are such that one correct statement suffices to win, and one wrong statement causes the loss, it is easy to see that a statement of any other vertex cannot improve the result of the case C.

Now we solve the hat problem on paths.

Theorem 4. *For every path P_n we have $h(P_n) = \frac{1}{2}$.*

Proof. Let $E(P_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$. We distinguish six possibilities: $n = 1$, $n = 2$, $n = 3$, $n = 4$, $n = 5$, and $n \geq 6$.

First, we assume that $n = 1$. Since $P_1 = K_1$, we have $h(P_1) = h(K_1) = \frac{1}{2}$.

Now assume that $n = 2$. Let S be an optimal strategy for P_2. If some vertex, say v_i, never states its color, then by Th. 3 we have $h(P_2) = h(P_2 - v_i)$. Since $P_2 - v_i = P_1$, we have $h(P_2) = h(P_1) = \frac{1}{2}$. Now assume that v_1 and v_2 state their colors. If one of them always states its color, then by Cor. 2 we have $h(P_2) = \frac{1}{2}$. If, neither v_1 nor v_2 always states its color, then without loss of generality we assume that v_1 states its color when v_2 is blue, and in this situation it states it is blue. We consider the following four possibilities: $g_2(b, *) = b$ (Table 1); $g_2(b, *) = r$ (Table 2); $g_2(r, *) = b$ (Table 3); $g_2(r, *) = r$ (Table 4). In the next tables b means blue, r means red, + means correct statement (success), − means wrong statement (loss), and blank square means passing.

In Tables 1, 2, and 3 we have $|Cw(S)| = 1$, $|C(P_2)| = 4$, so $p(S) = \frac{1}{4} < \frac{1}{2}$, a contradiction.

Table 1

No	The color of		The statement of		Result
	v_1	v_2	v_1	v_2	
1	b	b	$+$	$+$	$+$
2	b	r		$-$	$-$
3	r	b	$-$		$-$
4	r	r			$-$

Table 2

No	The color of		The statement of		Result
	v_1	v_2	v_1	v_2	
1	b	b	$+$	$-$	$-$
2	b	r		$+$	$+$
3	r	b	$-$		$-$
4	r	r			$-$

Table 3

No	The color of		The statement of		Result
	v_1	v_2	v_1	v_2	
1	b	b	$+$		$+$
2	b	r			$-$
3	r	b	$-$	$+$	$-$
4	r	r		$-$	$-$

Table 4

No	The color of		The statement of		Result
	v_1	v_2	v_1	v_2	
1	b	b	$+$		$+$
2	b	r			$-$
3	r	b	$-$	$-$	$-$
4	r	r		$+$	$+$

In Table 4 we have $|Cw(S)| = 2$, $|C(P_2)| = 4$, so $p(S) = \frac{2}{4} = \frac{1}{2}$. Since $S \in \mathcal{F}^0(P_2)$, we have $h(P_2) = \frac{1}{2}$.

Now assume that $n = 3$. Let S be an optimal strategy for P_3. If v_1 or v_3 never states its color, then without loss of generality we assume that it is v_1. By Th. 3 we have $h(P_3) = h(P_3 - v_1)$. Since $P_3 - v_1 = P_2$, we have $h(P_3) = h(P_2) = \frac{1}{2}$. Now assume that v_1 and v_3 state their colors. If v_1 or v_3 always states its color, then by Cor. 2 we have $h(P_3) = \frac{1}{2}$. If neither v_1 nor v_3 always states its color, then without loss of generality we assume that v_1 states its color when v_2 is blue, and in this situation it states it is blue. We have the following two possibilities: (1) v_3 states its color when v_2 is blue; (2) v_3 does not state its color when v_2 is blue.

(1) Let the strategy S' differ from S only in that v_3 does not state its color when v_2 is blue. Since in every case in which v_2 is blue v_1 states its color, the statement of v_3 cannot improve the result of any of these cases. Therefore, $p(S) \le p(S')$. Since $S \in \mathcal{F}^0(P_3)$, the strategy S' is also optimal for P_3. If v_3 never states its color in the strategy S', then we have the possibility already considered. The other possibility when v_3 states its color we consider in the next paragraph.

(2) Certainly, v_3 states its color when v_2 is red. Since v_1 (v_3, respectively) states its color when v_2 is blue (red, respectively), by Fact 1 we have

$$|Cl(S, v_2^b)| \ge \frac{|C(P_3, v_2^b)|}{2} \quad \left(|Cl(S, v_2^r)| \ge \frac{|C(P_3, v_2^r)|}{2}, \text{ respectively}\right).$$

This implies that

$$|Cl(S)| = |Cl(S, v_2^b)| + |Cl(S, v_2^r)| \ge \frac{|C(P_3, v_2^b)|}{2} + \frac{|C(P_3, v_2^r)|}{2} = \frac{|C(P_3)|}{2}.$$

Consequently,

$$p(S) = \frac{|Cw(S)|}{|C(P_3)|} = \frac{|C(P_3)| - |Cl(S)|}{|C(P_3)|} \le \frac{|C(P_3)| - \frac{|C(P_3)|}{2}}{|C(P_3)|} = \frac{1}{2}.$$

Since $S \in \mathcal{F}^0(P_3)$, we have $h(P_3) \le \frac{1}{2}$. Since $h(P_3) \ge \frac{1}{2}$, we get $h(P_3) = = \frac{1}{2}$.

Now assume that $n = 4$. Let S be an optimal strategy for P_4. If some vertex, say v_i, never states its color, then by Th. 3 we have $h(P_4) = = h(P_4 - v_i)$. If $i \in \{1, 4\}$, then $P_4 - v_i = P_3$, so $h(P_4) = h(P_3) = \frac{1}{2}$. If $i \in \{2, 3\}$, then $P_4 - v_i = P_1 \cup P_2$. Since $P_1 \cup P_2 \subseteq P_3$, we have $h(P_1 \cup P_2) \le h(P_3) = \frac{1}{2}$. Therefore, $h(P_4) = h(P_1 \cup P_2) \le \frac{1}{2}$. Since $h(P_4) \ge \frac{1}{2}$, we get $h(P_4) = \frac{1}{2}$. Now assume that every vertex states its color. If some vertex always states its color, then by Cor. 2 we have $h(P_4) = \frac{1}{2}$. If no vertex always states its color, then without loss of generality we assume that v_1 states its color when v_2 is blue, and in this situation it states it is blue. Similarly, since $N_{P_4}[v_1] \cap N_{P_4}[v_4] = \emptyset$, we may assume that v_4 states its color when v_3 is blue, and in this situation it states it is blue. We consider the following two possibilities: (1) v_2 states its color when v_3 is blue, or v_3 states its color when v_2 is blue; (2) v_2 does not state its color when v_3 is blue, and v_3 does not state its color when v_2 is blue.

(1) Let the strategy S' differ from S only in that v_2 does not state its color when v_3 is blue, and v_3 does not state its color when v_2 is blue. Since in every case in which v_3 (v_2, respectively) is blue v_4 (v_1, respectively) states its color, the statement of v_2 (v_3, respectively) cannot improve the result of any of these cases. Therefore, $p(S) \leq p(S')$. Since $S \in \mathcal{F}^0(P_4)$, the strategy S' is also optimal for P_4. If v_2 or v_3 never states its color in the strategy S', then we have the possibility already considered. The other possibility when v_2 and v_3 state their colors we consider in the next paragraph.

(2) If $c(v_1) = r$ and $c(v_2) = b$, or $c(v_3) = b$ and $c(v_4) = r$, then in each of the 7 cases, the team loses. Certainly, v_2 can state its color only when v_3 is red. Thus there are the following four possibilities: (2.1) $g_2(b, *, r, *) = b$; (2.2) $g_2(b, *, r, *) = r$; (2.3) $g_2(r, *, r, *) = b$; (2.4) $g_2(r, *, r, *) = r$.

(2.1) Since

$$|Cl(S, v_1^b, v_2^r, v_3^r)| = |C(P_4, v_1^b, v_2^r, v_3^r)| = 2$$

and

$$C(P_4, v_1^b, v_2^r, v_3^r) \cap (C(P_4, v_1^r, v_2^b) \cup C(P_4, v_3^b, v_4^r)) = \emptyset,$$

the team loses in at least $7 + 2 = 9$ cases, and wins in at most 7 cases. It means that $p(S) \leq \frac{7}{16} < \frac{1}{2}$, a contradiction.

Possibilities (2.2) and (2.3) are similar to (2.1).

(2.4) Certainly, v_3 can state its color only when v_2 is red. Thus we have the following four possibilities: (2.4.1) $g_3(*, r, *, b) = b$; (2.4.2) $g_3(*, r, *, b) = r$; (2.4.3) $g_3(*, r, *, r) = b$; (2.4.4) $g_3(*, r, *, r) = r$.

In possibilities (2.4.1), (2.4.2), and (2.4.3), without considering the consequences of statements of v_2, we get a similar contradiction as in (2.1), (2.2), and (2.3).

(2.4.4) In this possibility, analyzed in Table 5, we have $|Cw(S)| = 8$, $|C(P_4)| = 16$, so $p(S) = \frac{8}{16} = \frac{1}{2}$. Since $S \in \mathcal{F}^0(P_4)$, we have $h(P_4) = \frac{1}{2}$.

Now assume that $n = 5$. Let S be an optimal strategy for P_5. If for some $i \in \{1, 3, 5\}$ the vertex v_i never states its color, then by Th. 3 we have $h(P_5) = h(P_5 - v_i)$. If $i \in \{1, 5\}$, then $P_5 - v_i = P_4$, so $h(P_5) = h(P_4) = \frac{1}{2}$. If $i = 3$, then $P_5 - v_3 = P_2 \cup P_2$. Since $P_2 \cup P_2 \subseteq P_4$, we have $h(P_2 \cup P_2) \leq h(P_4) = \frac{1}{2}$, so $h(P_5) = h(P_2 \cup P_2) \leq \frac{1}{2}$. Since $h(P_5) \geq \frac{1}{2}$, we get $h(P_5) = \frac{1}{2}$. Now assume that every vertex from the set $\{v_1, v_3, v_5\}$ states its color. If some of these vertices always states its color, then by Cor. 2 we have $h(P_5) = \frac{1}{2}$. If no vertex from the set $\{v_1, v_3, v_5\}$ always states its color, then without loss of generality we assume that v_1 states its color when v_2 is blue, and in this situation it states it is blue. Similarly, since $N_{P_5}[v_1] \cap N_{P_5}[v_5] = \emptyset$, we may assume that v_5 states its color when v_4 is blue, and in this situation it states it is blue. We consider the following two possibilities: (1) v_3 states its color when v_2 or v_4 is blue; (2) v_3 does not state its color when v_2 or v_4 is blue.

(1) Let the strategy S' differ from S only in that v_3 does not state its color when v_2 or v_4 is blue. Since in every case in which v_2 (v_4, respectively) is blue, v_1 (v_5, respectively) states its color, the statement of v_3 cannot improve the result of any of these cases. Therefore, $p(S) \leq p(S')$. Since $S \in \mathcal{F}^0(P_5)$, the strategy S' is also optimal for P_5.

Table 5

No	The color of				The statement of				Result
	v_1	v_2	v_3	v_4	v_1	v_2	v_3	v_4	
1	b	b	b	b	+			+	+
2	b	b	b	r	+			−	−
3	b	b	r	b	+				+
4	b	b	r	r	+				+
5	b	r	b	b				+	+
6	b	r	b	r			−	−	−
7	b	r	r	b					−
8	b	r	r	r			+		+
9	r	b	b	b	−			+	−
10	r	b	b	r	−			−	−
11	r	b	r	b	−	−			−
12	r	b	r	r	−	−			−
13	r	r	b	b				+	+
14	r	r	b	r			−	−	−
15	r	r	r	b		+			+
16	r	r	r	r		+	+		+

If v_3 never states its color in the strategy S', then we have the possibility already considered. The other possibility when v_3 states its color we consider in the next paragraph.

(2) If $c(v_1) = r$ and $c(v_2) = b$, or $c(v_4) = b$ and $c(v_5) = r$, then in each of the $2^3 + 2^3 - 2 = 14$ cases the team loses. Certainly, v_3 states its color only when v_2 and v_4 are red. Without loss of generality we assume that in this situation v_3 states it is blue. If $c(v_2) = c(v_3) = c(v_4) = = r$, then in each of the 4 cases, the team loses. Since $(C(P_5, v_1^r, v_2^b) \cup \cup C(P_5, v_4^b, v_5^r)) \cap C(P_5, v_2^r, v_3^r, v_4^r) = \emptyset$, the team loses in at least $14+4 = 18$ cases, and wins in at most 14 cases. This implies that $p(S) \le \frac{14}{32} < \frac{1}{2}$, a contradiction.

The result for $n \ge 6$ we prove by the induction on the number of vertices of a path. Let us assume that n is an integer such that $n \ge 6$, and $h(P_{n-1}) = \frac{1}{2}$. We will prove that $h(P_n) = \frac{1}{2}$. Let S be an optimal strategy for P_n. If for some $i \in \{1, 3, n\}$ the vertex v_i never states its color, then by Th. 3 we have $h(P_n) = h(P_n - v_i)$. If $i \in \{1, n\}$, then $P_n - v_i = P_{n-1}$, so $h(P_n) = h(P_{n-1}) = \frac{1}{2}$. If $i = 3$, then $P_n - v_3 = P_2 \cup \cup P_{n-3}$. Since $P_2 \cup P_{n-3} \subseteq P_{n-1}$, we have $h(P_2 \cup P_{n-3}) \le h(P_{n-1}) = \frac{1}{2}$, so $h(P_n) = h(P_2 \cup P_{n-3}) \le \frac{1}{2}$. Since $h(P_n) \ge \frac{1}{2}$, we get $h(P_n) = \frac{1}{2}$. Now assume that every vertex from the set $\{v_1, v_3, v_n\}$ states its color. If some from these vertices always states its color, then by Cor. 2 we have $h(P_n) = \frac{1}{2}$. If no vertex from the set $\{v_1, v_3, v_n\}$ always states its color, then without loss of generality we assume that v_1 states its color when v_2 is blue, and in this situation it states it is blue. Similarly, since $N_{P_n}[v_1] \cap N_{P_n}[v_n] = \emptyset$, we may assume that v_n states its color when v_{n-1} is blue, and in this situation it states it is blue. We consider the following two possibilities: (1) v_3 states its color when v_2 is blue; (2) v_3 does not state its color when v_2 is blue.

(1) Let the strategy S' differ from S only in that v_3 does not state its color when v_2 is blue. Since in every case in which v_2 is blue, v_1 states its color, the statement of v_3 cannot improve the result of any of these cases. Therefore, $p(S) \le p(S')$. Since $S \in \mathcal{F}^0(P_n)$, the strategy S' is also optimal for P_n. If v_3 never states its color in the strategy S', then we have the possibility already considered. The other possibility when v_3 states its color we consider in the next paragraph.

(2) If $c(v_1) = r$ and $c(v_2) = b$, or $c(v_{n-1}) = b$ and $c(v_n) = r$, then in each of the $(\frac{1}{4} + \frac{1}{4} - \frac{1}{4^2})|C(P_n)| = \frac{7}{16}|C(P_n)|$ cases the team loses. Certainly, v_3 can state its color only when v_2 is red. Without loss of generality we assume that v_3 states its color when v_2 is red and v_4 is

blue, and in this situation it states it is blue. If $c(v_2) = c(v_3) = r$ and $c(v_4) = b$, then the team loses. All the cases in which $c(v_{n-1}) = b$ and $c(v_n) = r$ have been counted, so it remains to count the such ones that $c(v_2) = c(v_3) = r$, $c(v_4) = b$, and $(c(v_{n-1}) = r$ or $c(v_n) = b)$. There are $\frac{1}{2^3} \cdot \frac{3}{4} \cdot |C(P_n)| = \frac{3}{32}|C(P_n)|$ such cases. This implies that the team loses in at least $(\frac{7}{16} + \frac{3}{32})|C(P_n)| = \frac{17}{32}|C(P_n)|$ cases, and wins in at most $\frac{15}{32}|C(P_n)|$ cases. It means that $p(S) \leq \frac{15}{32} < \frac{1}{2}$, a contradiction. $\Diamond$

Now we solve the hat problem on trees.

Theorem 5. *For every tree T we have $h(T) = \frac{1}{2}$.*

Proof. The result we prove by induction on the number of vertices of a tree. If T has one vertex, that is $T = K_1$, it is obvious that the theorem is true. Let T be any tree with $n \geq 2$ vertices, and let us assume that $h(T') = \frac{1}{2}$ for every tree T' with $n-1$ vertices. Every tree has at least two leafs (that is vertices of a tree having exactly one neighbour). If T has exactly two leafs, then T is a path, and by Th. 4 we have $h(T) = = \frac{1}{2}$. If T has at least three leafs, then let v_1, v_2, and v_3 be any three of them. Let S be an optimal strategy for T. Since v_1, v_2, and v_3 are leafs, there are exactly two possible situations of each of them. If for some $i \in \{1, 2, 3\}$ the vertex v_i never states its color, then by Th. 3 we have $h(T) = h(T - v_i)$. Since $T - v_i$ is a tree with $n-1$ vertices, by the inductive assumption we have $h(T - v_i) = \frac{1}{2}$, and therefore $h(T) = \frac{1}{2}$. Now assume that every vertex from the set $\{v_1, v_2, v_3\}$ states its color. If one of them always states its color, then by Cor. 2 we have $h(T) = \frac{1}{2}$. Now assume that every vertex from the set $\{v_1, v_2, v_3\}$ states its color in exactly one situation. We consider the following two possibilities: (1) at least two leafs from the set $\{v_1, v_2, v_3\}$ have the same neighbour, that is, $N_T(v_i) = N_T(v_j)$ for certain $i, j \in \{1, 2, 3\}$, $i \neq j$; (2) every leaf from the set $\{v_1, v_2, v_3\}$ has another neighbour, that is, $N_T(v_1) \neq N_T(v_2) \neq N_T(v_3)$ and $N_T(v_1) \neq N_T(v_3)$.

(1) Let us denote $\{x\} = N_T(v_i) = N_T(v_j)$. We consider the following two possibilities: (1.1) v_i and v_j state their colors in the same situation; (1.2) v_i and v_j state their colors in different situations.

(1.1) Without loss of generality we assume that v_i and v_j state their colors when x is blue. Let the strategy S' differ from S only in that v_j does not state its color when x is blue, that is, v_j never states its color. Since in every case in which x is blue v_i states its color, the statement of v_j cannot improve the result of any of these cases. Therefore, $p(S) \leq \leq p(S')$. Since $S \in \mathcal{F}^0(T)$, the strategy S' is also optimal for T. Since v_j

never states its color in the strategy S', we have the possibility already considered.

(1.2) Without loss of generality we assume that v_i states its color when x is blue and v_j states its color when x is red. By Fact 1 we have

$$|Cl(S, x^b)| \geq \frac{|C(T, x^b)|}{2} \text{ and } |Cl(S, x^r)| \geq \frac{|C(T, x^r)|}{2}.$$

This implies that

$$|Cl(S)| = |Cl(S, x^b)| + |Cl(S, x^r)| \geq \frac{|C(T, x^b)|}{2} + \frac{|C(T, x^r)|}{2} = \frac{|C(T)|}{2}.$$

Consequently,

$$p(S) = \frac{|Cw(S)|}{|C(T)|} = \frac{|C(T)| - |Cl(S)|}{|C(T)|} \leq \frac{|C(T)| - \frac{|C(T)|}{2}}{|C(T)|} = \frac{1}{2}.$$

Since $S \in \mathcal{F}^0(T)$, we have $h(T) \leq \frac{1}{2}$. Since $h(T) \geq \frac{1}{2}$, we get $h(T) = \frac{1}{2}$.

(2) If $i \in \{1, 2, 3\}$, then let us denote $N_T(v_i) = \{v_i'\}$. Without loss of generality we assume that v_1 states its color when v_1' is blue, and in this situation it states it is blue. Similarly, since $v_1' \neq v_2' \neq v_3'$ and $v_1' \neq$ $\neq v_3'$, we may assume that v_2 states its color when v_2' is blue and in this situation it states it is blue, and v_3 states its color when v_3' is blue and in this situation it states it is blue. No vertex from the set $\{v_1, v_2, v_3\}$ states its color if and only if $c(v_1') = c(v_2') = c(v_3') = r$. If $(c(v_1) = r$ and $c(v_1') = b)$ or $(c(v_2) = r$ and $c(v_2') = b)$, or $(c(v_3) = r$ and $c(v_3') = b)$, then in each of the $(1 - (1 - \frac{1}{4})^3)|C(T)| = \frac{37}{64}|C(T)|$ cases the team loses. This implies that the team wins in at most $\frac{27}{64}|C(T)|$ cases. Consequently,

$$p(S) = \frac{|Cw(S)|}{|C(T)|} \leq \frac{\frac{27}{64}|C(T)|}{|C(T)|} = \frac{27}{64} < \frac{1}{2},$$

a contradiction. $\Diamond$

Now we consider the hat problem on a graph such that the only information we know about are the degrees of vertices. In the following theorem we give an upper bound on the chance of success of any strategy for the hat problem on a graph with given degrees of vertices.

Theorem 6. *Let G be a graph and let S be any strategy for this graph. Then*

$$|Cw(S)| \leq \sum_{v \in V(G)} \left\lfloor 2^{d_G(v)+1} - \frac{|Cw(S)|}{2^{|V(G)|-d_G(v)-1}} \right\rfloor \cdot 2^{|V(G)|-d_G(v)-1}.$$

Proof. Let v_i be a vertex of G. Every statement of the color in any situation done by v_i is wrong in exactly $2^{|V(G)|-d_G(v_i)-1}$ cases, because to every situation of v_i correspond $2^{|V(G)|-d_G(v_i)}$ cases, and in the half of them v_i has another color than it states it has. The vertex v_i cannot state its color in at least

$$\left\lfloor 2^{d_G(v_i)+1} - \frac{|Cw(S)|}{2^{|V(G)|-d_G(v_i)-1}} \right\rfloor + 1$$

situations, otherwise its statements are wrong in at least

$$2^{|V(G)|-d_G(v_i)-1}\left(\left\lfloor 2^{d_G(v_i)+1} - \frac{|Cw(S)|}{2^{|V(G)|-d_G(v_i)-1}} \right\rfloor + 1\right) >$$
$$> 2^{|V(G)|-d_G(v_i)-1}\left(2^{d_G(v_i)+1} - \frac{|Cw(S)|}{2^{|V(G)|-d_G(v_i)-1}}\right) = 2^{|V(G)|} - |Cw(S)|$$

cases. This implies that the team loses in more than $2^{|V(G)|} - |Cw(S)|$ cases, and wins in less than

$$|C(G)| - (2^{|V(G)|} - |Cw(S)|) = 2^{|V(G)|} - 2^{|V(G)|} + |Cw(S)| = |Cw(S)|$$

cases, but $|Cw(S)|$ is the number of cases in which the team wins, a contradiction. Since the vertex v_i does not state its color in at least

$$\left\lfloor 2^{d_G(v_i)+1} - \frac{|Cw(S)|}{2^{|V(G)|-d_G(v_i)-1}} \right\rfloor + 1$$

situations, it states its color in at most

$$\left\lfloor 2^{d_G(v_i)+1} - \frac{|Cw(S)|}{2^{|V(G)|-d_G(v_i)-1}} \right\rfloor$$

situations. Every statement of the color in any situation done by v_i is correct in exactly $2^{|V(G)|-d_G(v_i)-1}$ cases, because to every situation of v_i correspond $2^{|V(G)|-d_G(v_i)-1}$ cases, and in the half of them v_i has the color it states it has. Therefore, the statements of v_i are correct in at most

$$\left\lfloor 2^{d_G(v_i)+1} - \frac{|Cw(S)|}{2^{|V(G)|-d_G(v_i)-1}} \right\rfloor \cdot 2^{|V(G)|-d_G(v_i)-1}$$

cases. This implies that the team wins in at most

$$\sum_{v \in V(G)} \left\lfloor 2^{d_G(v)+1} - \frac{|Cw(S)|}{2^{|V(G)|-d_G(v)-1}} \right\rfloor \cdot 2^{|V(G)|-d_G(v)-1}$$

cases. $\Diamond$

In the following three facts we show that the upper bound from the previous theorem together with integrality constraints is tight on complete graphs with two, three, and four vertices, respectively.

Fact 7. $h(K_2) = \frac{1}{2}$.

Proof. Let S be any strategy for K_2. By Th. 6 we have

$$|Cw(S)| \leq \sum_{v \in V(K_2)} \left\lfloor 2^{d_{K_2}(v)+1} - \frac{|Cw(S)|}{2^{|V(K_2)|-d_{K_2}(v)-1}} \right\rfloor \cdot 2^{|V(K_2)|-d_{K_2}(v)-1}.$$

Since $|V(K_2)| = 2$ and every vertex in K_2 has exactly one neighbour, we get

$$|Cw(S)| \leq 2 \cdot \lfloor 2^2 - |Cw(S)| \rfloor \Leftrightarrow |Cw(S)| \leq 8 - 2|Cw(S)| \Leftrightarrow |Cw(S)| \leq 2\frac{2}{3}.$$

This implies that $|Cw(S)| \leq 2$, as $n \in N$. Consequently,

$$p(S) = \frac{|Cw(S)|}{|C(K_2)|} \leq \frac{2}{2^2} = \frac{1}{2}.$$

Since S is any strategy for K_2, we have $h(K_2) \leq \frac{1}{2}$. Since $h(K_2) \geq \frac{1}{2}$, we get $h(K_2) = \frac{1}{2}$. $\Diamond$

Fact 8. $h(K_3) = \frac{3}{4}$.

Proof. Let S be any strategy for K_3. By Th. 6 we have

$$|Cw(S)| \leq \sum_{v \in V(K_3)} \left\lfloor 2^{d_{K_3}(v)+1} - \frac{|Cw(S)|}{2^{|V(K_3)|-d_{K_3}(v)-1}} \right\rfloor \cdot 2^{|V(K_3)|-d_{K_3}(v)-1}.$$

Since $|V(K_3)| = 3$ and every vertex in K_3 has exactly two neighbours, we get

$$|Cw(S)| \leq 3 \cdot \lfloor 2^3 - |Cw(S)| \rfloor \Leftrightarrow |Cw(S)| \leq 24 - 3|Cw(S)| \Leftrightarrow |Cw(S)| \leq 6.$$

Consequently,

$$p(S) = \frac{|Cw(S)|}{|C(K_3)|} \leq \frac{6}{2^3} = \frac{3}{4}.$$

Since S is any strategy for K_3, we have $h(K_3) \leq \frac{3}{4}$. Let $S_1 \in \mathcal{F}(K_3)$ be the strategy such that every vertex considers colors of its two neighbours, and if they are the same, it states it has the opposite color. If they are different, it passes. It is easy to verify that $|Cw(S_1)| = 6$. Since $|C(K_3)| = 2^3 = 8$, we have $p(S_1) = \frac{|Cw(S)|}{|C(K_3)|} = \frac{6}{8} = \frac{3}{4}$. Since $p(S_1) \leq$ $\leq h(K_3)$, we have $h(K_3) \geq \frac{3}{4}$. Since $h(K_3) \geq \frac{3}{4}$ and $h(K_3) \leq \frac{3}{4}$, we get $h(K_3) = \frac{3}{4}$. $\Diamond$

Fact 9. $h(K_4) = \frac{3}{4}$.

Proof. Let S be any strategy for K_4. By Th. 6 we have

$$|Cw(S)| \leq \sum_{v \in V(K_4)} \left\lfloor 2^{d_{K_4}(v)+1} - \frac{|Cw(S)|}{2^{|V(K_4)|-d_{K_4}(v)-1}} \right\rfloor \cdot 2^{|V(K_4)|-d_{K_4}(v)-1}.$$

Since $|V(K_4)| = 4$ and every vertex in K_4 has three neighbours, we get

$$|Cw(S)| \leq 4 \cdot \lfloor 2^4 - |Cw(S)| \rfloor \Leftrightarrow |Cw(S)| \leq 64 - 4|Cw(S)| \Leftrightarrow |Cw(S)| \leq 12\frac{4}{5}.$$

This implies that $|Cw(S)| \leq 12$, as $|Cw(S)| \in N$. Consequently,

$$p(S) = \frac{|Cw(S)|}{|C(K_4)|} \leq \frac{12}{2^4} = \frac{3}{4}.$$

Since S is any strategy for K_4, we have $h(K_4) \leq \frac{3}{4}$. Since $K_3 \subseteq K_4$ and $h(K_3) = \frac{3}{4}$, we get $h(K_3) \leq h(K_4)$. Since $h(K_3) = \frac{3}{4}$, we have $h(K_4) \geq \frac{3}{4}$. This implies that $h(K_4) = \frac{3}{4}$. $\Diamond$

In the next fact we solve the hat problem on the graph $K_3 \cup K_2$.

Fact 10. $h(K_3 \cup K_2) = \frac{3}{4}$.

Proof. Let $E(K_3 \cup K_2) = \{v_1v_2, v_2v_3, v_3v_1, v_4v_5\}$. Let S be any strategy for the graph $K_3 \cup K_2$. By Th. 6 we have

$$|Cw(S)| \leq$$

$$\leq \sum_{v \in V(K_3 \cup K_2)} \left\lfloor 2^{d_{K_3 \cup K_2}(v)+1} - \frac{|Cw(S)|}{2^{|V(K_3 \cup K_2)| - d_{K_3 \cup K_2}(v) - 1}} \right\rfloor \cdot 2^{|V(K_3 \cup K_2)| - d_{K_3 \cup K_2}(v) - 1}.$$

Since $d_{K_3 \cup K_2}(v_1) = d_{K_3 \cup K_2}(v_2) = d_{K_3 \cup K_2}(v_3) = 2$ and $d_{K_3 \cup K_2}(v_4) = d_{K_3 \cup K_2}(v_5) = 1$, we get

$$|Cw(S)| \leq 3 \cdot 2^2 \cdot \left\lfloor 2^3 - \frac{|Cw(S)|}{2^2} \right\rfloor + 2 \cdot 2^3 \cdot \left\lfloor 2^2 - \frac{|Cw(S)|}{2^3} \right\rfloor =$$

$$= 12 \left\lfloor 8 - \frac{|Cw(S)|}{4} \right\rfloor + 16 \left\lfloor 4 - \frac{|Cw(S)|}{8} \right\rfloor.$$

This implies that

$$|Cw(S)| \leq 12 \left(8 - \frac{|Cw(S)|}{4} \right) + 16 \left(4 - \frac{|Cw(S)|}{8} \right) =$$

$$= 96 - 3|Cw(S)| + 64 - 2|Cw(S)| = 160 - 5|Cw(S)|.$$

Now we easily get $|Cw(S)| \leq \frac{160}{6} = 26\frac{2}{3}$. Since $|Cw(S)|$ is an integer, we have $|Cw(S)| \leq 26$. Assume that $|Cw(S)| = 26$. We have

$$26 \leq 12 \left\lfloor 8 - \frac{26}{4} \right\rfloor + 16 \left\lfloor 4 - \frac{26}{8} \right\rfloor = 12 \cdot 1 + 16 \cdot 0 = 12,$$

a contradiction. Now assume that $|Cw(S)| = 25$. We have

$$25 \leq 12 \left\lfloor 8 - \frac{25}{4} \right\rfloor + 16 \left\lfloor 4 - \frac{25}{8} \right\rfloor = 12 \cdot 1 + 16 \cdot 0 = 12,$$

a contradiction. This implies that $|Cw(S)| \leq 24$, and consequently, $p(S) = \frac{|Cw(S)|}{|C(K_3 \cup K_2)|} \leq \frac{24}{32}$. Since S is any strategy for $K_3 \cup K_2$, we have $h(K_3 \cup K_2) \leq \frac{3}{4}$. Since $K_3 \subseteq K_3 \cup K_2$ and $h(K_3) = \frac{3}{4}$, we get $h(K_3 \cup K_2) \geq h(K_3) = \frac{3}{4}$. This implies that $h(K_3 \cup K_2) = \frac{3}{4}$. $\Diamond$

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2.2 Hat problem on the cycle C_4

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Hat Problem on the Cycle C_4

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Abstract

The topic of our paper is the hat problem. In that problem, each of n people is randomly fitted with a blue or red hat. Then everybody can try to guess simultanously his own hat color looking at the hat colors of the other people. The team wins if at least one person guesses his hat color correctly and no one guesses his hat color wrong, otherwise the team loses. The aim is to maximize the probability of win. In this version every person can see everybody excluding him. We consider such problem on a graph, where vertices are people, and a person can see these people to which he is connected by an edge. The solution of the hat problem is known for trees. In this paper we solve the problem on the cycle C_4.

Mathematics Subject Classification: 05C38, 05C57, 91A43

Keywords: hat problem, graph, cycle

1 Introduction

In the hat problem, a team of n people enters a room and a blue or red hat is randomly placed on the head of each person. Each person can see the hats of all of the other people but not his own. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats, each person must simultanously guess the color of his own hat or pass. The team wins if at least one person guesses his hat color correctly and no one guesses his hat color wrong, otherwise the team loses. The aim is to maximize the probability of win.

The hat problem with seven people called ”seven prisoners puzzle” was formulated by T. Ebert in his Ph.D. Thesis [1]. The hat problem with three people was the subject of an article in The New York Times [3].

The hat problem has many applications and connections to other areas of science, for example: information technology, linear programming, genetic programming, economy, biology, approximating Boolean functions, and autoreducibility of random sequences. Therefore, it is hoped that the hat problem on a graph considered in this paper, as a natural generalization, is worth exploring, and may also have many applications.

In the hat problem on a graph, vertices are people and a person can see these people, to which he is connected by an edge. This variant of the hat problem was first considered in [2] where there are proved some general theorems about the hat problem on a graph, and the problem is solved on trees.

In this paper we solve the hat problem on the cycle with four vertices.

2 Preliminaries

For a graph G, by $V(G)$ and $E(G)$ we denote the set of vertices and the set of edges of this graph, respectively. If H is a subgraph of G, then we write $H \subseteq G$. Let $v \in V(G)$. By $N_G(v)$ we denote the neighbourhood of v, that is $N_G(v) = \{x \in V(G)\colon vx \in E(G)\}$. By P_n (C_n, K_n, respectively) we denote the path (cycle, complete graph, respectively) with n vertices.

Without loss of generality we may assume an ordering of the vertices of a graph G, that is $V(G) = \{v_1, v_2, \ldots, v_n\}$.

If $v_i \in V(G)$, then $c(v_i)$ is the first letter of the color of v_i, so $c\colon V(G) \to \{b, r\}$ is a function. By a case for the graph G we mean a sequence $(c(v_1), c(v_2), \ldots, c(v_n))$. The set of all cases for the graph G we denote by $C(G)$, of course $|C(G)| = 2^{|V(G)|}$.

If $v_i \in V(G)$, then by s_i we denote a function $s_i\colon V(G) \to \{b, r, *\}$, where $s_i(v_j)$ is the first letter of the color of v_j if v_i sees v_j, and mark $*$ otherwise, that is, $s_i(v_j) = c(v_j)$ if $v_j \in N_G(v_i)$, while $s_i(v_j) = *$ if $v_j \in V(G) \setminus N_G(v_i)$. By a situation of the vertex v_i in the graph G we mean the sequence $(s_i(v_1), s_i(v_2), \ldots, s_i(v_n))$. The set of all possible situations of v_i in the graph G we denote $St_i(G)$. Of course, $|St_i(G)| = 2^{|N_G(v_i)|}$.

Let $v_i \in V(G)$. We say that a case $(c_1, c_2, \ldots, c_n)$ for the graph G corresponds to a situation $(t_1, t_2, \ldots, t_n)$ of the vertex v_i in the graph G if it is created from this situation only by changing every mark $*$ to the letter b or r. So, a case corresponds to a situation of v_i if every vertex adjacent to v_i, in that case case has the same color as in that situation. To every situation of the vertex v_i in the graph G correspond $2^{|V(G)|-d_G(v_i)}$ cases, because every situation of v_i has $|V(G)| - d_G(v_i)$ marks $*$.

By a statement of a vertex we mean its declaration about the color it guesses it is. By the effect of a case we mean a win or a loss. According to the definition of the hat problem, the effect of a case is a win if at least one vertex states its color correctly and no vertex states its color wrong. The effect of a case is a loss if no vertex states its color or somebody states its color wrong.

By a guessing instruction for the vertex $v_i \in V(G)$ (denoted by g_i) we mean a function $g_i \colon St_i(G) \to \{b, r, p\}$ which, for a given situation, gives the first letter of the color v_i guesses it is or a letter p if v_i passes. Thus a guessing instruction is a rule which determines the conduct of the vertex v_i in every situation. By a strategy for the graph G we mean a sequence $(g_1, g_2, \ldots, g_n)$. By $\mathcal{F}(G)$ we denote the family of all strategies for the graph G.

Let $v_i \in V(G)$ and $S \in \mathcal{F}(G)$. We say that v_i never states its color in the strategy S if v_i passes in every situation, that is $g_i \equiv p$. We say that v_i always states its color in the strategy S if v_i states its color in every situation, that is, for every $T \in St_i(G)$ we have $g_i(T) \in \{b, r\}$ ($g_i(T) \neq p$, equivalently).

If $S \in \mathcal{F}(G)$, then by $Cw(S)$ and $Cl(S)$ we denote the sets of cases for the graph G in which the team wins or loses, respectively. Of course, $|Cw(S)| + |Cl(S)| = |C(G)|$. Consequently, by the chance of success of the strategy S we mean the number $p(S) = \frac{|Cw(S)|}{|C(G)|}$. By the hat number of the graph G we mean the number $h(G) = \max\{p(S) \colon S \in \mathcal{F}(G)\}$. Certainly $p(S) \leq h(G)$. We say that the strategy S is optimal for the graph G if $p(S) = h(G)$. By $\mathcal{F}^0(G)$ we denote the family of all optimal strategies for the graph G.

Let $t, m_1, m_2, \ldots, m_t \in \{1, 2, \ldots, n\}$ be such that $m_j \neq m_k$ and $c_{m_j} \in \{b, r\}$, for every $j, k \in \{1, 2, \ldots, t\}$. By $C(G, v_{m_1}^{c_{m_1}}, v_{m_2}^{c_{m_2}}, \ldots, v_{m_t}^{c_{m_t}})$ we denote the set of cases for the graph G such that the first letter of the color of v_{m_j} is c_{m_j}.

The following theorems are from [2]. The first of them presents a relation between the hat number of a graph and the hat number of its any subgraph.

Theorem 1 *If H is a subgraph of G, then $h(H) \leq h(G)$.*

Since the graph K_1 is a subgraph of every graph, we get the following Corollary.

Corollary 2 *For every graph G we have $h(G) \geq \frac{1}{2}$.*

In the next two theorems there are considered optimal strategies such that some vertex always (never, respectively) states its color.

Theorem 3 *Let G be a graph and let v be a vertex of G. If $S \in \mathcal{F}^0(G)$ is a strategy such that v always states its color, then $h(G) = \frac{1}{2}$.*

Theorem 4 *Let G be a graph and let v be a vertex of G. If $S \in \mathcal{F}^0(G)$ is a strategy such that v never states its color, then $h(G) = h(G - v)$.*

The following theorem is the solution of the hat problem on paths.

Theorem 5 *For every path P_n we have $h(P_n) = \frac{1}{2}$.*

The next fact is about the unnecessity of statements of any further vertices in a case in which some vertex already states its color.

Fact 6 *Let G be a graph and let S be a strategy for this graph. Let C be a case in which some vertex states its color. Then a statement of any other vertex cannot improve the effect of the case C.*

Now we characterize the number of cases in which the loss of the team is caused by a statement of a vertex.

Fact 7 *Let G be a graph and let v_i be a vertex of G. Let $S \in \mathcal{F}(G)$. If v_i states its color in a situation, then the team loses in at least half of all cases corresponding to this situation.*

3 Results

In the following theorem we solve the hat problem on the cycle with four vertices.

Theorem 8 $h(C_4) = \frac{1}{2}$.

Proof. Let S be an optimal strategy for C_4 such that there is no situation in which both v_1 and v_3 state its colors, and there is no situation in which both v_2 and v_4 state its colors. Now we prove that such strategy exists. Let S' be an optimal strategy for C_4. Assume in S' there is a situation in which both v_1 and v_3 state its colors, or there is a situation in which both v_2 and v_4 state its colors. Let the strategy S differ from S' only by that v_3 does not state its color when v_1 states its color, and v_4 does not state its color when v_2 states its color. By Fact 6 the statements of v_3 and v_4 cannot improve the effect of any from that cases. Therefore, $p(S) \geq p(S')$. Since $S' \in \mathcal{F}^0(C_4)$, the strategy S is also optimal. In the strategy S there is no such situation in which both v_1 and v_3 state its colors, and there is no such situation in which both v_2 and v_4 state its colors. If some vertex in C_4 never states its color, then let $i \in \{1,2,3,4\}$ be such that v_i never states its color. By Theorem 4 we have $h(C_4) = h(C_4 - v_i)$. Since $C_4 - v_i = P_3$, and by Theorem 5 we have $h(P_3) = \frac{1}{2}$, we get $h(C_4) = \frac{1}{2}$. Now assume every vertex in C_4 states its color. If some vertex in C_4 always states its color, then by Theorem 3 we have $h(C_4) = \frac{1}{2}$. Now assume there is no vertex in C_4 such that always states its color. Every vertex states its color in one, two, or three situations. We consider the following two possibilities: (1) every vertex states its color in exactly one situation; (2) there is a vertex which states its color in at least two situations.

(1) Any statement of any vertex in any situation is correct in exactly two cases, because to every situation of any vertex correspond four cases, and in

the half of them this vertex has the color it states it has. If every vertex states its color in exactly one situation, then there are exactly 8 correct statements, and even if every of them is in another case, then the team can win in at most 8 cases. This implies that $p(S) \le \frac{8}{16} = \frac{1}{2}$. Since $S \in \mathcal{F}^0(C_4)$, we have $h(C_4) \le \frac{1}{2}$. Since by Corollary 2 we have $h(C_4) \ge \frac{1}{2}$, we get $h(C_4) = \frac{1}{2}$.

(2) We consider the following two possibilities: (2.1) there is a vertex which states its color in exactly three situations; (2.2) every vertex states its color in at most two situations.

(2.1) Without loss of generality we assume v_1 states its color in exactly three situations. Since there is no such situation in which both v_1 and v_3 state its colors, and v_3 states its color in at least one situation, v_3 states its color in exactly one situation. Since in every from the situations $(*, b, *, b)$, $(*, b, *, r)$, $(*, r, *, b)$, and $(*, r, *, r)$ the vertex v_1 or v_3 states his hat color, by Fact 7 we have

$$|Cl(S, v_2^b, v_2^b)| \ge \frac{|C(C_4, v_2^b, v_2^b)|}{2}, \quad |Cl(S, v_2^b, v_2^r)| \ge \frac{|C(C_4, v_2^b, v_2^r)|}{2},$$

$$|Cl(S, v_2^r, v_2^b)| \ge \frac{|C(C_4, v_2^r, v_2^b)|}{2}, \quad \text{and} \quad |Cl(S, v_2^r, v_2^r)| \ge \frac{|C(C_4, v_2^r, v_2^r)|}{2}.$$

Consequently

$$|Cl(S)| = |Cl(S, v_2^b, v_2^b)| + |Cl(S, v_2^b, v_2^r)| + |Cl(S, v_2^r, v_2^b)| + |Cl(S, v_2^r, v_2^r)|$$

$$\ge \frac{|C(C_4, v_2^b, v_2^b)|}{2} + \frac{|C(C_4, v_2^b, v_2^r)|}{2} + \frac{|C(C_4, v_2^r, v_2^b)|}{2} + \frac{|C(C_4, v_2^r, v_2^r)|}{2} = \frac{|C(C_4)|}{2}.$$

Now we get

$$p(S) = \frac{|Cw(S)|}{|C(C_4)|} = \frac{|C(C_4)| - |Cl(S)|}{|C(C_4)|} \le \frac{|C(C_4)| - \frac{|C(C_4)|}{2}}{|C(C_4)|} = \frac{1}{2}.$$

Since $S \in \mathcal{F}^0(C_4)$, we have $h(C_4) \le \frac{1}{2}$. Since by Corollary 2 we have $h(C_4) \ge \frac{1}{2}$, we get $h(C_4) = \frac{1}{2}$.

(2.2) Since there is a vertex which states its color in exactly two situations, without loss of generality we assume v_1 states its color in exactly two situations. We consider the following two possibilities: (2.2.1) v_3 states its color in exactly two situations; (2.2.2) v_3 states its color in exactly one situation.

(2.2.1) Since in every from the situations $(*, b, *, b)$, $(*, b, *, r)$, $(*, r, *, b)$, $(*, b, *, b)$ v_1 or v_3 states his hat color, by the same arguments as in (2.1), we get $h(C_4) = \frac{1}{2}$.

(2.2.2) We consider the following two possibilities: (a1) in both situations in which v_1 states its color, v_2 has the same color or v_4 has the same color; (a2) in both situations in which v_1 states its color, v_2 has different colors, and v_4

has different colors. We consider the following two possibilities: (b1) in both situations v_1 states it has the same color; (b2) in both situations v_1 states it has different colors.

Let $v_i \in \{v_1, v_2\}$. Now we consider the following four possibilities: (2.2.2.1) (a1),(b1); (2.2.2.2) (a1),(b2); (2.2.2.3) (a2),(b1); (2.2.2.4) (a2),(b2).

(2.2.2.1) Without loss of generality we assume v_1 states its color in the situations $(*, b, *, b)$ and $(*, b, *, r)$, and in these situations it states it is blue. Also without loss of generality we assume v_3 states its color in the situation $(*, r, *, b)$, and in this situation it states it is blue. These statements are correct in the cases: (b, b, b, b), (b, b, r, b), (b, b, b, r), (b, b, r, r), (b, r, b, b), and (r, r, b, b), and are wrong in the cases: (r, b, b, b), (r, b, r, b), (r, b, b, r), (r, b, r, r), (b, r, r, b), and (r, r, r, b),. To the situation $(b, *, b, *)$ correspond three cases in which v_1 or v_3 states its color correctly, and the case (b, r, b, r) in which neither v_1 nor v_3 states its color. By Fact 6, among cases corresponding to the situation $(b, *, b, *)$, the effect only of (b, r, b, r) can be improved. In two cases corresponding to the situation $(b, *, b, *)$ the statement of v_i is wrong. This implies that in at least one case corresponding to the situation $(b, *, b, *)$ in which v_1 or v_3 states its color correctly, v_i states its color wrong. Therefore, the statement of v_i in the situation $(b, *, b, *)$ cannot improve the chance of success. Thus we assume v_i does not states its color in the situation $(b, *, b, *)$. Now let us consider the cases corresponding to the situation $(b, *, r, *)$. To the situation $(b, *, r, *)$ correspond two cases in which v_1 or v_3 states its color correctly, one case in which v_1 or v_3 states its color wrong, and one in which v_i does not state its color. By Fact 6, among cases corresponding to the situation $(b, *, r, *)$, the effect only of (b, r, r, r) can be improved. To improve the effect of this case, the statement of v_i has to be correct in this case. Among four cases corresponding to the situation $(b, *, r, *)$ in two of them the statement of v_i is wrong. This implies that in some case corresponding to the situation $(b, *, b, *)$ in which v_1 or v_3 states its color correctly, v_i states its color wrong, making worse the evaluation of this case. Therefore, the statement of v_i in the situation $(b, *, r, *)$ cannot improve the chance of success. Thus we may assume v_i does not state its color in the situation $(b, *, r, *)$. There is only one case corresponding to the situation $(r, *, b, *)$ in which neither v_1 nor v_3 states its color. There in also only one case corresponding to the situation $(r, *, r, *)$ in which neither v_1 nor v_3 states its color. Therefore, there are only two cases which effects can be improved. This implies that the team wins in at most eight cases, so $p(S) = \frac{|Cw(S)|}{|C(C_n)|} \leq \frac{8}{16} = \frac{1}{2}$. Since S is an optimal strategy for G, we have $h(C_4) \leq \frac{1}{2}$. Since by Theorem 1 we have $h(C_4) \geq \frac{1}{2}$, we get $h(C_4) = \frac{1}{2}$.

(2.2.2.2) Without loss of generality we assume in the situation $(*, b, *, b)$ v_1 states it is blue, and in the situation $(*, b, *, r)$ it states it is red. Also without loss of generality we assume v_3 states its color in the situation $(*, r, *, b)$, and in this situation it states it is blue. These statements are correct in the cases: (b,b,b,b), (b,b,r,b), (r,b,b,r), (r,b,r,r), (b,r,b,b), and (r,r,b,b), and are wrong in the cases: (r,b,b,b), (r,b,r,b), (b,b,b,r), (b,b,r,r), (b,r,r,b), and (r,r,r,b). To the situation $(b,*,b,*)$ correspond two correspond two cases in which v_1 or v_3 states its color correctly, one case in which v_1 or v_3 states its color wrong, and one in which neither v_1 nor v_3 states its color. To the situation $(r,*,b,*)$ also correspond two cases in which v_1 or v_3 states its color correctly, one case in which v_1 or v_3 states its color wrong, and one in which neither v_1 nor v_3 states its color. By reasons similar as when considering the situation $(b,*,r,*)$ in (2.2.2.1), we may assume v_i does not states its color in any of the situations $(b,*,b,*)$ and $(r,*,b,*)$. To the situation $(b,*,r,*)$ correspond three cases in which v_1 or v_3 states its color, and one in which neither v_1 nor v_3 states its color. To the situation $(r,*,r,*)$ also correspond three cases in which v_1 or v_3 states its color, and one in which neither v_1 nor v_3 states its color. Therefore, by Fact 6, there are two cases which effects can be improved. This implies that the team wins in at most eight cases, so $p(S)=\frac{|Cw(S)|}{|C(C_4)|}\le\frac{8}{16}=\frac{1}{2}$.

(2.2.2.3) Without loss of generality we assume v_1 states its color in the situations $(*,b,*,b)$ and $(*,r,*,r)$, and in these situations it states it is blue. Also without loss of generality we assume v_3 states its color in the situation $(*,r,*,b)$, and in this situation it states it is blue. These statements are correct in the cases: (b,b,b,b), (b,b,r,b), (b,r,b,r), (b,r,r,r), (b,r,b,b), and (r,r,b,b), and are wrong in the cases: (r,b,b,b), (r,b,r,b), (r,r,b,r), (r,r,r,r), (b,r,r,b), and (r,r,r,b). To the situation $(b,*,b,*)$ correspond three cases in which v_1 or v_3 states its color correctly, and one in which neither v_1 nor v_3 states its color. By reasons similar as in the situation $(b,*,b,*)$ in (2.2.2.1), we may assume v_i does not state its color in the situation $(b,*,b,*)$. To the situation $(b,*,r,*)$ correspond two cases in which v_1 or v_3 states its color correctly, one case in which v_1 or v_3 states its color wrong, and one in which neither v_1 nor v_3 states its color. By reasons similar in the situation $(b,*,r,*)$ in (2.2.2.1), we may assume v_i does not states its color in the situation $(b,*,r,*)$. To the situation $(r,*,b,*)$ corresponds only one case in which neither v_1 nor v_3 states its color, and therefore statements of v_2 or v_4 can improve the effects of only two cases. To the situation $(r,*,r,*)$ also corresponds only one case in which neither v_1 nor v_3 states its color. Therefore, by Fact 6, there are two cases which effects can be improved. This implies that the team wins in at most eight cases, so $p(S)=\frac{|Cw(S)|}{|C(C_4)|}\le\frac{8}{16}=\frac{1}{2}$.

(2.2.2.4) Without loss of generality we assume in the situation $(*,b,*,b)$ v_1 states it is blue, and in the situation $(*,r,*,r)$ it states it is red. Also without loss of generality we assume v_3 states its color in the situation $(*,r,*,b)$, and in this situation it states it is blue. These statements are correct in the cases: (b,b,b,b), (b,b,r,b), (r,r,b,r), (r,r,r,r), (b,r,b,b), and (r,r,b,b), and are wrong in the cases: (r,b,b,b), (r,b,r,b), (b,r,b,r), (b,r,r,r), (b,r,r,b), and (r,r,r,b). To the situation $(b,*,b,*)$ correspond two cases in which v_1 or v_3

states its color correctly, one case in which v_1 or v_3 states its color wrong, and one in which neither v_1 or v_3 states its color. To the situation $(r, *, b, *)$ correspond two cases in which v_1 or v_3 states its color correctly, one case in which v_1 or v_3 states its color wrong, and one in which neither v_1 or v_3 states its color. By reasons similar as when considering the situation $(b, *, r, *)$ in (2.2.2.1), we may assume v_i does not states its color in any of the situations $(b, *, b, *)$ and $(r, *, b, *)$. To the situation $(b, *, r, *)$ corresponds only one case in which neither v_1 nor v_3 states its color, and therefore the statements of v_2 or v_4 can improve the effects of only two cases. To the situation $(r, *, r, *)$ corresponds only one case in which neither v_1 nor v_3 states its color. Therefore, by Fact 6, there are two cases which effects can be improved. This implies that the team wins in at most eight cases, so $p(S) = \frac{|Cw(S)|}{|C(C_4)|} \leq \frac{8}{16} = \frac{1}{2}$, and consequenty $h(C_4) = \frac{1}{2}$. ■

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2.3 The hat problem on cycles on at least nine vertices

The hat problem on cycles on at least nine vertices

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Abstract

The topic is the hat problem in which each of n players is randomly fitted with a blue or red hat. Then everybody can try to guess simultaneously his own hat color by looking at the hat colors of the other players. The team wins if at least one player guesses his hat color correctly, and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of winning. In this version every player can see everybody excluding himself. We consider such a problem on a graph, where vertices correspond to players, and a player can see each player to whom he is connected by an edge. The solution of the hat problem on a graph is known for trees and for the cycle C_4. We solve the problem on cycles on at least nine vertices.
Keywords: hat problem, graph, cycle.
AMS Subject Classification: 05C38, 05C99, 91A12.

1 Introduction

In the hat problem, a team of n players enters a room and a blue or red hat is randomly placed on the head of each player. Each player can see the hats of all of the other players but not his own. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats, each player must simultaneously guess the color of his own hat or pass. The team wins if at least one player guesses his hat color correctly and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of winning.

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The hat problem with seven players, called the "seven prisoners puzzle", was formulated by T. Ebert in his Ph.D. Thesis [12]. The hat problem was also the subject of articles in The New York Times [24], Die Zeit [6], and abcNews [23]. It is also a one of subjects of the webpage [4].

The hat problem with $2^k - 1$ players was solved in [14], and for 2^k players in [11]. The problem with n players was investigated in [7]. The hat problem and Hamming codes were the subject of [8]. The generalized hat problem with n people and q colors was investigated in [22].

There are many known variations of the hat problem (for a comprehensive list, see [21]). For example in the papers [1, 10, 18] there was considered a variation in which passing is not allowed, thus everybody has to guess his hat color. The aim is to maximize the number of correct guesses. The authors of [16] investigated several variations of the hat problem in which the aim is to design a strategy guaranteeing a desired number of correct guesses. In [17] there was considered a variation in which the probabilities of getting hats of each colors do not have to be equal. The authors of [2] investigated a problem similar to the hat problem. There are n players which have random bits on foreheads, and they have to vote on the parity of the n bits.

The hat problem and its variations have many applications and connections to different areas of science (for a survey on this topic, see [21]), for example: information technology [5], linear programming [16], genetic programming [9], economics [1, 18], biology [17], approximating Boolean functions [2], and autoreducibility of random sequences [3, 12–15]. Therefore, it is hoped that the hat problem on a graph is worth exploring as a natural generalization, and may also have many applications.

We consider the hat problem on a graph, where vertices correspond to players and a player can see each player to whom he is connected by an edge. This variation of the hat problem was first considered in [19]. There were proven some general theorems about the hat problem on a graph, and the problem was solved on trees. Additionally, there was considered the hat problem on a graph such that the only known information are degrees of vertices. In [20] the problem was solved on the cycle C_4. It has been proven that for both trees and the cycle C_4 the maximum chance of success is one by two. Thus in such graph an optimal strategy is for example such in which one vertex always guesses it is blue, while the remaining vertices always pass. It means that the structure of such graph does not improve the maximum chance of success in the hat problem on a graph comparing to the one-vertex graph.

We solve the hat problem on cycles on at least nine vertices.

2 Preliminaries

For a graph G, the set of vertices and the set of edges we denote by $V(G)$ and $E(G)$, respectively. Let $v \in V(G)$. The degree of vertex v, that is, the number of its neighbors, we denote by $d_G(v)$. The path (cycle, respectively) on n vertices we denote by P_n (C_n, respectively).

Let $f\colon X \to Y$ be a function, and let $y \in Y$. If for every $x \in X$ we have $f(x) = y$, then we write $f \equiv y$.

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. By $Sc = \{1, 2\}$ we denote the set of colors, where 1 corresponds to blue, and 2 corresponds to red.

By a case for a graph G we mean a function $c\colon V(G) \to \{1, 2\}$, where $c(v_i)$ means color of vertex v_i. The set of all cases for the graph G we denote by $C(G)$; of course $|C(G)| = 2^{|V(G)|}$.

By a situation of a vertex v_i we mean a function $s_i\colon V(G) \to Sc \cup \{0\} = \{0, 1, 2\}$, where $s_i(v_j) \in Sc$ if v_i and v_j are adjacent, and 0 otherwise. The set of all possible situations of v_i in the graph G we denote by $St_i(G)$; of course $|St_i(G)| = 2^{d_G(v_i)}$.

We say that a case c for the graph G corresponds to a situation s_i of vertex v_i if $c(v_j) = s_i(v_j)$, for every v_j adjacent to v_i. This implies that a case corresponds to a situation of v_i if every vertex adjacent to v_i in that case has the same color as in that situation. Of course, to every situation of the vertex v_i correspond exactly $2^{|V(G)|-d_G(v_i)}$ cases.

By a guessing instruction of a vertex $v_i \in V(G)$ we mean a function $g_i\colon St_i(G) \to Sc \cup \{0\} = \{0, 1, 2\}$, which for a given situation gives the color v_i guesses it is, or 0 if v_i passes. Thus, a guessing instruction is a rule determining the behavior of a vertex in every situation. We say that v_i never guesses its color if v_i passes in every situation, that is, $g_i \equiv 0$.

Let c be a case, and let s_i be the situation (of vertex v_i) corresponding to that case. The guess of v_i in the case c is correct (wrong, respectively) if $g_i(s_i) = c(v_i)$ ($0 \neq g_i(s_i) \neq c(v_i)$, respectively). Let $S \in \mathcal{F}(G)$ and let $v_i \in V(G)$. By $L(S, v_i)$ we denote the set of cases for the graph G such that in the strategy S the vertex v_i guesses its color wrong. By result of the case c we mean a win if at least one vertex guesses its color correctly, and no vertex guesses its color wrong, that is, $g_i(s_i) = c(v_i)$ (for some i) and there is no j such that $0 \neq g_j(s_j) \neq c(v_j)$. Otherwise the result of the case c is a loss.

By a strategy for the graph G we mean a sequence $(g_1, g_2, \ldots, g_n)$, where g_i is the guessing instruction of vertex v_i. The family of all strategies for a graph G we denote by $\mathcal{F}(G)$.

If $S \in \mathcal{F}(G)$, then the set of cases for the graph G for which the team wins (loses, respectively) using the strategy S we denote by $W(S)$ ($L(S)$, respectively). By the chance of success of the strategy S we mean the number $p(S) = |W(S)|/|C(G)|$. By the hat number of the graph G we mean the number $h(G) = \max\{p(S)\colon S \in \mathcal{F}(G)\}$. We say that a strategy S is optimal for the graph G if $p(S) = h(G)$. The family of all optimal strategies for the graph G we denote by $\mathcal{F}^0(G)$.

Let $t \in \{1, 2, \ldots, n\}$, and let $m_1, m_2, \ldots, m_t \in \{1, 2, \ldots, n\}$ be such that $m_j \neq m_k$ for every $j \neq k$. Let $c_{m_1}, c_{m_2}, \ldots, c_{m_t} \in \{1, 2\}$. The set of cases c for the graph G such that $c(v_{m_j}) = c_{m_j}$ we denote by $C(G, v_{m_1}^{c_{m_1}}, v_{m_2}^{c_{m_2}}, \ldots, v_{m_t}^{c_{m_t}})$.

By solving the hat problem on a graph G we mean finding the number $h(G)$.

Now we give an example of notation for the hat problem on the graph K_3. Of course, there are $2^3 = 8$ possible cases. The vertices we denote by v_1, v_2, and v_3. Assume for example that in a case c the vertices v_1 and v_3 have the first color, and the vertex v_2 has the second color. Thus $c(v_1) = c(v_3) = 1$ and $c(v_2) = 2$. Now let us consider situations of some vertex, say v_1. The vertex v_1 can see that v_2 has the second color and v_3 has the first color. Of course, the vertex v_1 cannot see its own color. Thus $s_1(v_1) = 0$, $s_1(v_2) = 2$, and $s_1(v_3) = 1$. We say that a case corresponds to that situation if each one of the neighbors of v_1 has the same color as in that situation. It is easy to see that the case in which v_1 and v_2 have the second color and v_3 has the first color corresponds to that situation. These are the only two cases corresponding to that situation as $2^{|V(K_3)| - d_{K_3}(v_1)} = 2^{3-2} = 2$. Now let us consider a guessing instruction of some vertex, say v_2. Assume for example that the vertex v_2 guesses it has the first color when v_1 and v_3 have the second color; it guesses it has the second color when v_1 and v_3 have the first color; otherwise it passes. We have $g_2(202) = 1$, $g_2(101) = 2$, and $g_2(102) = g_2(201) = 0$. If a case c is such that $c(v_1) = c(v_3) = 1$ and $c(v_2) = 2$, then the guess of v_2 is correct as $g_2(101) = 2 = c(v_2)$.

The following theorems are from [19]. The first of them is a lower bound on the chance of success of an optimal strategy.

Theorem 1 *Let G be a graph. If S is an optimal strategy for G, then $p(S) \geq 1/2$.*

Now we give a sufficient condition for deleting a vertex of a graph without changing its hat number.

Theorem 2 *Let G be a graph and let v be a vertex of G. If there exists a strategy $S \in \mathcal{F}^0(G)$ such that v never guesses its color, then $h(G) = h(G - v)$.*

The following theorem is the solution of the hat problem on paths.

Theorem 3 *For every path P_n we have $h(P_n) = 1/2$.*

3 Results

In the next few pages we solve the hat problem on cycles on at least nine vertices.

We assume that $E(C_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$. Let S be a strategy for C_n such that every vertex guesses its color (rather than passing) in exactly one situation. Let $\alpha_i(S), \beta_i(S), \gamma_i(S) \in \{1, 2\}$ (we write $\alpha_i, \beta_i, \gamma_i$) be such that the guess of v_i is wrong when $c(v_{i-1}) = \alpha_i$, $c(v_i) = \beta_i$, and $c(v_{i+1}) = \gamma_i$ ($i \in \{2, 3, \ldots, n-1\}$), the guess of v_1 is wrong when $c(v_n) = \alpha_1$, $c(v_1) = \beta_1$, and $c(v_2) = \gamma_1$, and the guess of v_n is wrong when $c(v_{n-1}) = \alpha_n$, $c(v_n) = \beta_n$, and $c(v_1) = \gamma_n$. For example, if the vertex v_2 guesses it has the second color when v_1 has the first color and v_3 has the second color, then it follows that the vertex v_2 guesses its color wrong when $c(v_1) = c(v_2) = 1$ and $c(v_3) = 2$. Therefore $\alpha(v_2) = \beta(v_2) = 1$ and $\gamma(v_2) = 2$.

Let us consider strategies such that every vertex guesses its color (rather than passing) in exactly one situation. In the following lemma we give such strategy for which the number of cases in which some vertex guesses its color wrong is as small as possible.

Lemma 4 *Let us consider the family of all strategies for C_n such that every vertex guesses its color (rather than passing) in exactly one situation. The number of cases in which some vertex guesses its color wrong is minimal for a strategy S such that $\gamma_{i-1} = \beta_i = \alpha_{i+1}$ ($i \in \{2, 3, \ldots, n-1\}$), $\gamma_{n-1} = \beta_n = \alpha_1$, and $\gamma_n = \beta_1 = \alpha_2$.*

Proof. First, we prove that we may assume that $\alpha_n = \gamma_{n-2}$. Consider the possibility $\alpha_n \neq \gamma_{n-2}$. Thus $\beta_{n-1} = \alpha_n$ or $\beta_{n-1} = \gamma_{n-2}$, otherwise $\alpha_n = \gamma_{n-2}$, a contradiction. Without loss of generality we assume that $\beta_{n-1} = \gamma_{n-2}$. Since $\alpha_n \neq \gamma_{n-2}$, we have $\gamma_{n-2} = \beta_{n-1} \neq \alpha_n$. Let a strategy S' differ from S only in that $\alpha_n(S') \neq \alpha_n(S) = \alpha_n$. Thus $\alpha_n(S') = \beta_{n-1} = \gamma_{n-2}$. Let B (B', respectively) denote the set of cases in which in the strategy S (S', respectively) the vertex v_n guesses its color wrong, and at the same time another vertex also guesses its color wrong. Thus $B = L(S, v_n) \cap \bigcup_{i=1}^{n-1} L(S, v_i)$ and $B' = L(S', v_n) \cap \bigcup_{i=1}^{n-1} L(S', v_i)$. We want to minimize the number of cases in which some vertex guesses its color wrong. Therefore we want the number of cases in which v_n guesses its color wrong, and at the same time another vertex also guesses its color wrong to be as great as possible. Since the strategies S and S' differ only in the behavior of the vertex v_n, and for each set A_i ($i \in \{1, 2, \ldots, n-3\}$) the color of the vertex v_{n-1} is not determined, we have $|L(S, v_n)| \cap \bigcup_{i=1}^{n-3} L(S, v_i)| = |L(S', v_n)| \cap \bigcup_{i=1}^{n-3} L(S', v_i)|$. We also get $|L(S, v_n) \cap L(S, v_{n-2})| = |C(C_n, v_{n-1}^{\alpha_n}, v_n^{\beta_n}, v_1^{\gamma_n}) \cap C(C_n, v_{n-3}^{\alpha_{n-2}}, v_{n-2}^{\beta_{n-2}}, v_{n-1}^{\gamma_{n-2}})| = 0$ as

$\alpha_n \neq \gamma_{n-2}$. Since $\alpha_n \neq \beta_{n-1}$, we have $|L(S, v_n) \cap L(S, v_{n-1})| = |C(C_n, v_{n-1}^{\alpha_n}, v_n^{\beta_n}, v_1^{\gamma_n}) \cap C(C_n, v_{n-2}^{\alpha_{n-1}}, v_{n-1}^{\beta_{n-1}}, v_n^{\gamma_{n-1}})| = 0$. This implies that $|B'| \geq |B|$, and therefore we may assume that $\alpha_n = \gamma_{n-2}$. Let us make this assumption.

Now we prove that we may assume that $\beta_{n-1} = \gamma_{n-2}$. Consider the possibility $\beta_{n-1} \neq \gamma_{n-2}$. Let a strategy S'' differ from S only in that $\beta_{n-1}(S'') \neq \beta_{n-1}(S) = \beta_{n-1}$, thus $\beta_{n-1}(S'') = \gamma_{n-2} = \alpha_n$. Let us define sets D and D'' analogically as the sets B and B'. Similarly we get $|D''| \geq |D|$. Therefore we may assume that $\beta_{n-1} = \gamma_{n-2}$.

Because of the possibility of cyclic renumbering of vertices of the cycle, we may assume that $\gamma_{i-1} = \beta_i = \alpha_{i+1}$ ($i \in \{2, 3, \ldots, n-1\}$), $\gamma_{n-1} = \beta_n = \alpha_1$, and $\gamma_n = \beta_1 = \alpha_2$. ∎

If $n \geq 3$ is an integer, then let

$$A_n = \{c \in C(C_n)\colon c(v_{i-1}) = c(v_i) = c(v_{i+1}) = 1, \text{ for an } i \in \{2, 3, \ldots, n-1\}\},$$

that is, A_n is the set of cases for C_n such that there are three vertices of the first color the indices of which are consecutive integers. Let the sequence $\{a_n\}_{n=1}^{\infty}$ be such that $a_n = |A_n|$ ($n \geq 3$), and also $a_1 = a_2 = 0$.

In the following lemma we give a recursive formula for a_n (with $n \geq 4$).

Lemma 5 *For every integer $n \geq 4$ we have $a_n = 2^{n-3} + a_{n-3} + a_{n-2} + a_{n-1}$.*

Proof. To find the number a_n, we have to count the cases for C_n such that $c(v_{i-1}) = c(v_i) = c(v_{i+1}) = 1$, for some $i \in \{2, 3, \ldots, n-1\}$. Let c be any case for C_n. We consider the following four possibilities: (1) $\min\{i\colon c(v_i) = 2\} = 1$; (2) $\min\{i\colon c(v_i) = 2\} = 2$; (3) $\min\{i\colon c(v_i) = 2\} = 3$; (4) $c(v_1) = c(v_2) = c(v_3) = 1$.

(1) There are a_{n-1} such cases, because there are $n-1$ vertices which can form a triple of vertices of the first color the indices of which are consecutive integers.

(2) There are a_{n-2} such cases, because there are $n-2$ vertices which can form a triple of vertices of the first color the indices of which are consecutive integers, as v_2 has the second color, and v_1 cannot belong to any triple of vertices of the first color the indices of which are consecutive integers because of the interruption of v_2.

(3) There are a_{n-3} such cases, due to reasons similar to those in (2).

(4) There are 2^{n-3} such cases, because v_1, v_2, and v_3 form a triple of vertices of the first color the indices of which are consecutive integers, and there are 2^{n-3} possibilities of coloring the remaining $n-3$ vertices.

From (1)–(4) it follows that $a_n = 2^{n-3} + a_{n-3} + a_{n-2} + a_{n-1}$. ∎

If n is an integer such that $n \geq 3$, then let

$$\begin{aligned} B_n = \{c \in C(C_n)\colon c(v_{i-1}) = c(v_i) = c(v_{i+1}) = 1 \text{ (for an } i \in \{2, 3, \dots, n-1\}) \\ \text{or } c(v_{n-1}) = c(v_n) = c(v_1) = 1 \text{ or } c(v_n) = c(v_1) = c(v_2) = 1\}, \end{aligned}$$

that is, B_n is the set of cases for C_n such that there are three consecutive vertices of the first color. Let the sequence $\{b_n\}_{n=3}^{\infty}$ be such that $b_n = |B_n|$.

Now we give a relation between the number b_n (with $n \geq 6$), and the elements of the sequence $\{a_n\}_{n=1}^{\infty}$.

Lemma 6 *If $n \geq 6$ is an integer, then $b_n = 5 \cdot 2^{n-6} + a_n - 2a_{n-5} - a_{n-6}$.*

Proof. Let us consider the partition of the set B_n (the set of cases for C_n such that there are three consecutive vertices of the first color) into the following two sets. In the first set there are the cases for C_n such that there are three vertices of the first color the indices of which are consecutive integers. In the second set there are the cases for C_n such that there are three consecutive vertices of the first color, but there are not any three vertices of the first color the indices of which are consecutive integers. Thus

$$\begin{aligned} B_n &= \{c \in C(C_n)\colon c(v_{i-1}) = c(v_i) = c(v_{i+1}) = 1 \text{ (for an } i \in \{2, 3, \dots, n-1\}) \\ &\qquad \text{or } c(v_{n-1}) = c(v_n) = c(v_1) = 1 \text{ or } c(v_n) = c(v_1) = c(v_2) = 1\} \\ &= \{c \in C(C_n)\colon c(v_{i-1}) = c(v_i) = c(v_{i+1}) = 1, \text{ for an } i \in \{2, 3, \dots, n-1\}\} \\ &\quad \cup \{c \in C(C_n)\colon c(v_{n-1}) = c(v_n) = c(v_1) = 1 \text{ or } c(v_n) = c(v_1) = c(v_2) = 1, \\ &\qquad \text{and at the same time there is no } i \in \{2, 3, \dots, n-1\} \text{ such that} \\ &\qquad c(v_{i-1}) = c(v_i) = c(v_{i+1}) = 1\} \\ &= A_n \cup (B_n \setminus A_n). \end{aligned}$$

We have

$$b_n = |B_n| = |A_n \cup (B_n \setminus A_n)| = |A_n| + |B_n \setminus A_n| = a_n + |B_n \setminus A_n|.$$

Now let us find a formula for $|B_n \setminus A_n|$. Let c be any case for C_n belonging to the set $B_n \setminus A_n$. We consider the following three possibilities: (1) $c(v_{n-1}) = c(v_n) = c(v_1) = c(v_2) = 1$ (so also $c(v_{n-2}) = c(v_3) = 2$, as this case does not belong to the set A_n); (2) $c(v_{n-1}) = 2$ and $c(v_n) = c(v_1) = c(v_2) = 1$ (so also $c(v_3) = 2$, as this case does not belong to the set A_n); (3) $c(v_{n-1}) = c(v_n) = c(v_1) = 1$ and $c(v_2) = 2$ (so also $c(v_{n-2}) = 2$, as this case does not belong to the set A_n), see Figure 1.

(1) There are $2^{n-6} - a_{n-6}$ such cases, because there are 2^{n-6} possibilities of coloring the remaining $n-6$ vertices, and we do not count the a_{n-6} cases such that there are three vertices of the first color the indices of which are consecutive integers.

(2) There are $2^{n-5} - a_{n-5}$ such cases, due to reasons analogical to that in (1).

(3) There are $2^{n-5} - a_{n-5}$ such cases, also due to reasons analogical to that in (1).

It follows from (1), (2), and (3) that

$$|B_n \setminus A_n| = 2^{n-6} - a_{n-6} + 2(2^{n-5} - a_{n-5}) = 5 \cdot 2^{n-6} - 2a_{n-5} - a_{n-6}.$$

Since $b_n = a_n + |B_n \setminus A_n|$, we get $b_n = 5 \cdot 2^{n-6} + a_n - 2a_{n-5} - a_{n-6}$. ■

- ◉ a vertex of the first color
- ⋇ a vertex of the second color
- • a vertex of unknown color

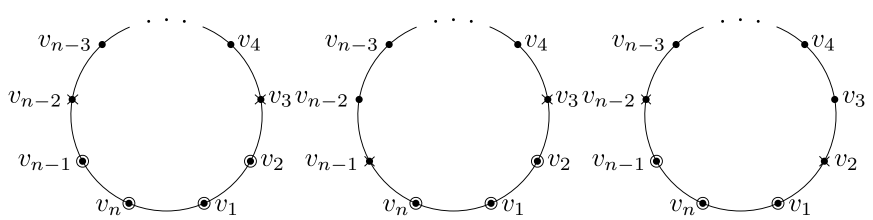

Figure 1: Illustrations to the proof of Lemma 6:
possibilities (1), (2), and (3), respectively

Now we give a lower bound on the number b_n (with $n \geq 9$).

Lemma 7 *For every integer $n \geq 9$ we have $b_n > 2^{n-1}$.*

Proof. First, we find the eleven initial elements of the sequence $\{a_n\}_{n=1}^{\infty}$. We calculate them recursively. If we try to solve the recurrence which determines the elements of the sequence $\{a_n\}_{n=1}^{\infty}$, then in the generating function we get the expression x^3+x^2+x+1 corresponding to the so-called tribonacci sequence for which the iterative formula is not known. Solving the recurrence of the sequence $\{a_n\}_{n=1}^{\infty}$ using tribonacci numbers, we can only get a formula which is also recursive.

Using Lemma 5, the definition of the sequence $\{a_n\}_{n=1}^{\infty}$, and the fact that $a_3 = 1$ (as the case in which every vertex has the first color is the only one such case), we get

$$\begin{array}{lllllllllllllll}
a_1 & = & 0, \\
a_2 & = & 0, \\
a_3 & = & 1, \\
a_4 & = & 2 & + & a_1 & + & a_2 & + & a_3 & = & 2 & + 0 & + 0 & + 1 & = 3, \\
a_5 & = & 2^2 & + & a_2 & + & a_3 & + & a_4 & = & 4 & + 0 & + 1 & + 3 & = 8, \\
a_6 & = & 2^3 & + & a_3 & + & a_4 & + & a_5 & = & 8 & + 1 & + 3 & + 8 & = 20, \\
a_7 & = & 2^4 & + & a_4 & + & a_5 & + & a_6 & = & 16 & + 3 & + 8 & + 20 & = 47,
\end{array}$$

$$
\begin{array}{lllllllllllllllll}
a_8 & = & 2^5 & + & a_5 & + & a_6 & + & a_7 & = & 32 & + & 8 & + & 20 & + & 47 & = & 107, \\
a_9 & = & 2^6 & + & a_6 & + & a_7 & + & a_8 & = & 64 & + & 20 & + & 47 & + & 107 & = & 238, \\
a_{10} & = & 2^7 & + & a_7 & + & a_8 & + & a_9 & = & 128 & + & 47 & + & 107 & + & 238 & = & 520, \\
a_{11} & = & 2^8 & + & a_8 & + & a_9 & + & a_{10} & = & 256 & + & 107 & + & 238 & + & 520 & = & 1121.
\end{array}
$$

By Lemma 6 we get

$$
\begin{array}{lllllllll}
b_9 & = & 5 \cdot 2^3 & + & a_9 & - & 2a_4 & - & a_3 \\
 & = & 40 & + & 238 & - & 2 \cdot 3 & - & 1 \\
 & = & 271 & & & & & & \\
 & > & 256 & = & 2^8. & & & &
\end{array}
$$

Now assume that $n \geq 10$. Since $a_n = |A_n|$, $b_n = |B_n|$, and $A_n \subseteq B_n$ (see the definition of the set B_n), we get $a_n \leq b_n$. This implies that it suffices to prove that $a_n > 2^{n-1}$. We prove this by induction. We have $a_{10} = 520 > 512 = 2^9$ and $a_{11} = 1121 > 1024 = 2^{10}$. Assume that $n \geq 10$ is an integer, and we have $a_n > 2^{n-1}$ and $a_{n+1} > 2^n$. We prove that $a_{n+2} > 2^{n+1}$. By Lemma 5 and the inductive hypothesis we get

$$
\begin{array}{lllllllll}
a_{n+2} & = & 2^{n-1} & + & a_{n-1} & + & a_n & + & a_{n+1} \\
 & > & 2^{n-1} & + & 0 & + & 2^{n-1} & + & 2^n \\
 & = & 2^{n+1}. & & & & & &
\end{array}
$$

■

Now we solve the hat problem on cycles on at least nine vertices.

Theorem 8 *For every integer $n \geq 9$ we have $h(C_n) = 1/2$.*

Proof. Let S be an optimal strategy for C_n. If some vertex, say v_i, never guesses its color, then by Theorem 2 we have $h(C_n) = h(C_n - v_i)$. Since $C_n - v_i = P_{n-1}$ and $h(P_{n-1}) = 1/2$ (by Theorem 3), we get $h(C_n) = 1/2$. Now assume that every vertex guesses its color (rather than passing) in at least one situation. We are interested in the possibility when the number of cases for which the team loses is as small as possible. We assume that every vertex guesses its color (rather than passing) in exactly one situation, and we prove that these guesses suffice to cause the loss of the team in more than half of all cases. Let us consider the strategy $S' \in \mathcal{F}(C_n)$ such that $\gamma_{i-1} = \beta_i = \alpha_{i+1}$ ($i \in \{2, 3, \ldots, n-1\}$), $\gamma_{n-1} = \beta_n = \alpha_1$, and $\gamma_n = \beta_1 = \alpha_2$. Without loss of generality we assume that $\gamma_{i-1} = \beta_i = \alpha_{i+1} = 1$ ($i \in \{2, 3, \ldots, n-1\}$), $\gamma_{n-1} = \beta_n = \alpha_1 = 1$, and $\gamma_n = \beta_1 = \alpha_2 = 1$. Some vertex guesses its color wrong in the cases such that there are three consecutive vertices of the first color. Using the definition of the sequence $\{b_n\}_{n=3}^{\infty}$, there are b_n such cases. From Lemma 4 we know that the number of cases in which some vertex guesses its color wrong in the strategy S' is minimal among all strategies for C_n such that every vertex guesses its color (rather than passing) in exactly one situation. This implies that in the strategy S in at least b_n cases some vertex guesses its color wrong.

Therefore the team loses for at least b_n cases, that is, $|L(S)| \geq b_n$. Since $b_n > 2^{n-1}$ (by Lemma 7), we have $|L(S)| > 2^{n-1}$. Now we get

$$p(S) = \frac{|W(S)|}{|C(C_n)|} = \frac{|C(C_n)| - |L(S)|}{|C(C_n)|} < \frac{2^n - 2^{n-1}}{2^n} = \frac{1}{2},$$

a contradiction to Corollary 1. ■

Of course, $h(C_3) = 3/4$. A natural issue is to determine the hat numbers of cycles of length between four and eight. This will make the hat problem on cycles solved. One can also investigate the problem on another classes of graphs. This may be helpful for solving generally the hat problem on an arbitrary graph.

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2.4 On the hat problem on a graph

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ON THE HAT PROBLEM ON A GRAPH

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Abstract. The topic of this paper is the hat problem in which each of n players is uniformly and independently fitted with a blue or red hat. Then everybody can try to guess simultaneously his own hat color by looking at the hat colors of the other players. The team wins if at least one player guesses his hat color correctly, and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of winning. In this version every player can see everybody excluding himself. We consider such a problem on a graph, where vertices correspond to players, and a player can see each player to whom he is connected by an edge. The solution of the hat problem on a graph is known for trees and for cycles on four or at least nine vertices. In this paper first we give an upper bound on the maximum chance of success for graphs with neighborhood-dominated vertices. Next we solve the problem on unicyclic graphs containing a cycle on at least nine vertices. We prove that the maximum chance of success is one by two. Then we consider the hat problem on a graph with a universal vertex. We prove that there always exists an optimal strategy such that in every case some vertex guesses its color. Moreover, we prove that there exists a graph with a universal vertex for which there exists an optimal strategy such that in some case no vertex guesses its color. We also give some Nordhaus-Gaddum type inequalities.

Keywords: hat problem, graph, degree, neighborhood, neighborhood-dominated, unicyclic, universal vertex, Nordhaus-Gaddum.

Mathematics Subject Classification: 05C07, 05C38, 05C99, 91A12.

1. INTRODUCTION

In the hat problem, a team of n players enters a room and a blue or red hat is uniformly and independently placed on the head of each player. Each player can see the hats of all of the other players but not his own. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats, each player must simultaneously guess the color of his own hat or pass. The team wins if at least one player guesses his hat color correctly and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of winning.

The hat problem with seven players, called the "seven prisoners puzzle", was formulated by T. Ebert in his Ph.D. Thesis [12]. The hat problem was also the subject of articles in The New York Times [26], Die Zeit [6], and abcNews [25]. It is also one of the Berkeley Riddles [4].

The hat problem with $2^k - 1$ players was solved in [14], and for 2^k players in [11]. The problem with n players was investigated in [7]. The hat problem and Hamming codes were the subject of [8]. The generalized hat problem with n people and q colors was investigated in [24].

There are many known variations of the hat problem (for a comprehensive list, see [22]). For example in [19] there was considered a variation in which players do not have to guess their hat colors simultaneously. In the papers [1, 10, 18] there was considered a variation in which passing is not allowed, thus everybody has to guess his hat color. The aim is to maximize the number of correct guesses. The authors of [16] investigated several variations of the hat problem in which the aim is to design a strategy guaranteeing a desired number of correct guesses. In [17] there was considered a variation in which the probabilities of getting hats of each colors do not have to be equal. The authors of [2] investigated a problem similar to the hat problem. There are n players which have random bits on foreheads, and they have to vote on the parity of the n bits.

The hat problem and its variations have many applications and connections to different areas of science (for a survey on this topic, see [22]), for example: information technology [5], linear programming [16], genetic programming [9], economics [1, 18], biology [17], approximating Boolean functions [2], and autoreducibility of random sequences [3, 12–15]. Therefore, it is hoped that the hat problem on a graph considered in this paper is worth exploring as a natural generalization, and may also have many applications.

We consider the hat problem on a graph, where vertices correspond to players and a player can see each player to whom he is connected by an edge. This variation of the hat problem was first considered in [20]. There were proven some general theorems about the hat problem on a graph, and the problem was solved on trees. Additionally, there was considered the hat problem on a graph such that the only known information are degrees of vertices. In [21] the problem was solved on the cycle C_4. The problem on cycles on at least nine vertices was solved in [23].

In this paper first we give an upper bound on the maximum chance of success for graphs with neighborhood-dominated vertices. We use this bound to solve the hat problem on the graph obtained from K_4 by the subdivision of one edge. We also prove that there exists a graph having two vertices with the same open neighborhood for which there exists an optimal strategy such that in some situation both those vertices guess their colors. Next we solve the problem on unicyclic graphs containing a cycle on at least nine vertices. We prove that the maximum chance of success is one by two. Then we consider the hat problem on a graph with a universal vertex. We prove that there always exists an optimal strategy such that in every case some vertex guesses its color. Moreover, we prove that there exists a graph with a universal vertex for which there exists an optimal strategy such that in some case no vertex guesses its color. We also give some Nordhaus-Gaddum type inequalities.

2. PRELIMINARIES

For a graph G, the set of vertices and the set of edges we denote by $V(G)$ and $E(G)$, respectively. By complement of G, denoted by $\overline{G}$, we mean a graph which has the same vertices as G, and two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. If H is a subgraph of G, then we write $H \subseteq G$. Let $v \in V(G)$. The open neighborhood of v, that is $\{x \in V(G)\colon vx \in E(G)\}$, we denote by $N_G(v)$. We say that a vertex of G is universal if it is adjacent to every one of the remaining vertices. By a leaf we mean a vertex having exactly one neighbor. We say that a vertex v of a graph G is neighborhood-dominated in G if there is some other vertex $w \in V(G)$ such that $N_G(v) \subseteq N_G(w)$. We say that a graph is unicyclic if it contains exactly one cycle as a subgraph.

The degree of vertex v, that is, the number of its neighbors, we denote by $d_G(v)$. Thus $d_G(v) = |N_G(v)|$. The path (cycle, complete graph, respectively) on n vertices we denote by P_n (C_n, K_n, respectively).

Let $f\colon X \to Y$ be a function. If for every $x \in X$ we have $f(x) = y$, then we write $f \equiv y$.

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. By $Sc = \{1, 2\}$ we denote the set of colors, where 1 corresponds to the blue color, and 2 corresponds to the red color.

By a case for a graph G we mean a function $c\colon V(G) \to \{1, 2\}$, where $c(v_i)$ means color of vertex v_i. The set of all cases for the graph G we denote by $C(G)$, of course $|C(G)| = 2^{|V(G)|}$.

By a situation of a vertex v_i we mean a function $s_i\colon V(G) \to Sc \cup \{0\} = \{0, 1, 2\}$, where $s_i(v_j) \in Sc$ if v_i and v_j are adjacent, and 0 otherwise. The set of all possible situations of v_i in the graph G we denote by $St_i(G)$, of course $|St_i(G)| = 2^{d_G(v_i)}$.

We say that a case c for the graph G corresponds to a situation s_i of vertex v_i if $c(v_j) = s_i(v_j)$, for every v_j adjacent to v_i. This implies that a case corresponds to a situation of v_i if every vertex adjacent to v_i in the case has the same color as in the situation. Of course, to every situation of the vertex v_i correspond exactly $2^{|V(G)|-d_G(v_i)}$ cases.

By a guessing instruction of a vertex $v_i \in V(G)$ we mean a function $g_i\colon St_i(G) \to Sc \cup \{*\} = \{1, 2, *\}$, which for a given situation gives the color v_i guesses if $g_i(s_i) \neq *$, otherwise v_i passes. Thus a guessing instruction is a rule determining the behavior of a vertex in every situation. We say that v_i never guesses its color if v_i passes in every situation, that is, $g_i \equiv *$.

Let c be a case, and let s_i be the situation (of vertex v_i) corresponding to that case. The guess of v_i in the case c is correct (wrong, respectively) if $g_i(s_i) = c(v_i)$ ($* \neq g_i(s_i) \neq c(v_i)$, respectively). By result of the case c we mean a win if at least one vertex guesses its color correctly, and no vertex guesses its color wrong, that is, $g_i(s_i) = c(v_i)$ (for some i) and there is no j such that $* \neq g_j(s_j) \neq c(v_j)$. Otherwise the result of the case c is a loss.

By a strategy for the graph G we mean a sequence $(g_1, g_2, \ldots, g_n)$, where g_i is the guessing instruction of vertex v_i. The family of all strategies for a graph G we denote by $\mathcal{F}(G)$.

If $S \in \mathcal{F}(G)$, then the set of cases for the graph G for which the team wins (loses, respectively) using the strategy S we denote by $W(S)$ ($L(S)$, respectively). The set of cases for which the team loses, and no vertex guesses its color we denote by $Ln(S)$. By the chance of success of the strategy S we mean the number $p(S) = |W(S)|/|C(G)|$. By the hat number of the graph G we mean the number $h(G) = \max\{p(S)\colon S \in \mathcal{F}(G)\}$. We say that a strategy S is optimal for the graph G if $p(S) = h(G)$. The family of all optimal strategies for the graph G we denote by $\mathcal{F}^0(G)$.

By solving the hat problem on a graph G we mean finding the number $h(G)$.

Let G and H be graphs. Assume that $H \subseteq G$. Since every vertex from the set $V(G) \setminus V(H)$ can always pass, and every vertex $v_i \in V(H)$ can ignore the colors of vertices from the set $N_G(v_i) \setminus N_H(v_i)$, we get the following relation between numbers $h(H)$ and $h(G)$.

Fact 2.1. *If H is a subgraph of G, then $h(H) \leq h(G)$.*

Since the one-vertex graph is a subgraph of every graph, we get the following corollary.

Corollary 2.2. *For every graph G we have $h(G) \geq 1/2$.*

Using the definition of an optimal strategy, we immediately get the following corollary.

Corollary 2.3. *Let G be a graph. If $S \in \mathcal{F}^0(G)$, then $p(S) \geq 1/2$.*

The following four results are from [20]. The first of them states that there does not exist any graph such that the team can always win.

Fact 2.4. *For every graph G we have $h(G) < 1$.*

Now we state that a guess of any other vertex is unnecessary in a case in which some vertex already guesses its color.

Fact 2.5. *Let G be a graph, and let S be a strategy for this graph. Let c be a case in which some vertex guesses its color. Then a guess of any other vertex cannot improve the result of the case c.*

Now there is a sufficient condition for deleting a vertex of a graph without changing its hat number.

Theorem 2.6. *Let G be a graph, and let v be a vertex of G. If there exists a strategy $S \in \mathcal{F}^0(G)$ such that v never guesses its color, then $h(G) = h(G - v)$.*

The next theorem is the solution of the hat problem on trees.

Theorem 2.7. *For every tree T we have $h(T) = 1/2$.*

The following solution of the hat problem on cycles on at least nine vertices is a result from [23]. It was obtained by proving that even if every vertex guesses its color in exactly one situation, then in at least half of all cases some vertex guesses its color wrong, causing the loss of the team.

Theorem 2.8. *For every integer $n \geq 9$ we have $h(C_n) = 1/2$.*

3. HAT PROBLEM ON A GRAPH WITH NEIGHBORHOOD – DOMINATED VERTICES

In this section we consider the hat problem on graphs with neighborhood-dominated vertices.

First, we investigate optimal strategies for such graphs.

Theorem 3.1. *Let G be a graph, and let v_1 and v_2 be vertices of G. If $N_G(v_1) \subseteq N_G(v_2)$, then there exists an optimal strategy for the graph G such that there is no case in which both vertices v_1 and v_2 guess their colors.*

Proof. Suppose that for every optimal strategy for the graph G there exists a case in which both v_1 and v_2 guess their colors. Let S be any optimal strategy for G. Let $c_1, c_2, \ldots, c_k$ be the cases in which both vertices v_1 and v_2 guess their colors. These cases correspond to the situations $s_2^1, s_2^2, \ldots, s_2^l$ of v_2 ($s_2^i \neq s_2^j$ for $i \neq j$). Let the strategy S' for the graph G differ from S only in that v_2 does not guess its color in the situations $s_2^1, s_2^2, \ldots, s_2^l$. Since in every one of the cases corresponding to these situations v_1 guesses its color, by Fact 2.5 the guess of v_2 cannot improve the result of any one of these cases. Therefore $p(S) \leq p(S')$. Since $S \in \mathcal{F}^0(G)$, the strategy S' is also optimal. In this strategy there is no case in which both v_1 and v_2 guess their colors. □

Corollary 3.2. *Let G be a graph, and let $v_1, v_2, \ldots, v_k$ be vertices of G such that $N_G(v_1) = N_G(v_2) = \ldots = N_G(v_k)$. Then there exists an optimal strategy for the graph G such that in each situation at most one of the vertices $v_1, v_2, \ldots, v_k$ guesses its color.*

In the next fact we state that there exists a graph having two vertices with the same open neighborhood for which there exists an optimal strategy such that in some situation both those vertices guess their colors.

Fact 3.3. *There exists an optimal strategy for the path P_3 such that in some situation the two vertices having the same open neighborhood guess their colors.*

Proof. Let $E(P_3) = \{v_1v_2, v_2v_3\}$, and let $S = (g_1, g_2, g_3) \in \mathcal{F}(P_3)$ be the strategy as follows.

$$g_1(s_1) = \begin{cases} 1 & \text{if } s_1(v_2) = 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$g_2(s_2) \equiv 1,$$

$$g_3(s_3) = \begin{cases} 1 & \text{if } s_3(v_2) = 2, \\ 0 & \text{otherwise.} \end{cases}$$

It means that the vertices proceed as follows.

— The vertex v_1. If v_2 has the second color, then v_1 guesses it has the first color, otherwise it passes.

— The vertex v_2 always guesses it has the first color.

— The vertex v_3. If v_2 has the second color, then v_3 guesses it has the first color, otherwise it passes.

It is not difficult to verify that $|W(S)| = 4$. Since $|C(P_3)| = 8$, we get $p(S) = 4/8 = 1/2$. By Theorem 2.7 we have $h(P_3) = 1/2$, therefore the strategy S is optimal. We have $N_{P_3}(v_1) = N_{P_3}(v_3)$, and in the strategy S both vertices v_1 and v_3 guess their colors in the situation when v_2 has the second color. $\square$

Let G be a graph, and let $A_1, A_2, \ldots, A_k$ be a partition of the set of vertices of G such that the open neighborhoods of the vertices of each set A_i can be linearly ordered by inclusion.

Now we give an upper bound on the chance of success for any strategy for the hat problem on a graph with neighborhood-dominated vertices.

Theorem 3.4. *Let G be a graph, and let k mean the minimum number of sets to which $V(G)$ can be partitioned in a way described above. Then $h(G) \le k/(k+1)$.*

Proof. Theorem 3.1 implies that there exists a strategy $S \in \mathcal{F}^0(G)$ such that in every case at most one vertex from each set A_i guesses its color. The number of cases in which the vertices of A_i guess their colors in the strategy S is at most $2(2^{|V(G)|} - |W(S)|)$, otherwise the number of cases in which some of these vertices guesses its color wrong is greater than $2^{|V(G)|} - |W(S)|$. This implies that the team loses for more than $2^{|V(G)|} - |W(S)|$ cases, and therefore the number of cases for which the team wins is less than $|W(S)|$. This is a contradiction as $|W(S)|$ is the number of cases for which the team wins. In half of all cases the guesses of the vertices of A_i are correct, thus their guesses are correct in at most $2^{|V(G)|} - |W(S)|$ cases. Therefore the number of cases for which the team wins using the strategy S is less than or equal to $k(2^{|V(G)|} - |W(S)|)$. This implies that $p(S) = |W(S)|/2^{|V(G)|} \le k/(k+1)$. $\square$

Now we use the previous theorem to solve the hat problem on the graph H (given in Figure 1). This graph is obtained from K_4 by the subdivision of one edge.

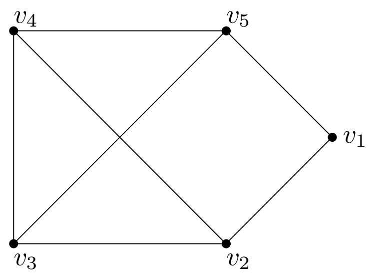

Fig. 1. The graph H

Fact 3.5. $h(H) = 3/4$.

Proof. It is easy to observe that $N_H(v_1) \subseteq N_H(v_3)$ and $N_H(v_2) = N_H(v_5)$. This implies that we can partition the set of vertices of H into three sets the open neighborhoods of which can be linearly ordered. By Theorem 3.4 we have $h(G) \le 3/4$. On the other hand, by Fact 2.1 we get $3/4 = h(K_3) \le h(H)$ as $K_3 \subseteq H$. $\square$

4. HAT PROBLEM ON A UNICYCLIC GRAPH

In this section we solve the hat problem on unicyclic graphs containing a cycle on at least nine vertices.

Theorem 4.1. *If G is a unicyclic graph containing the cycle C_k for some $k \geq 9$, then $h(G) = 1/2$.*

Proof. The result we prove by induction on the number n of vertices of G. For $n = k$ the Theorem holds by Theorem 2.8. Now assume that $n > k$. Assume that for every unicyclic graph G' with $n-1$ vertices containing C_k we have $h(G') = 1/2$. Let S be an optimal strategy for G. If some vertex, say v_i, never guesses its color, then by Theorem 2.6 we have $h(G) = h(G - v_i)$. If v_i is a vertex of the cycle, then the graph $G - v_i$ is a subgraph of a tree. Using Theorem 2.7 we get $h(G - v_i) \leq 1/2$, and therefore $h(G) \leq 1/2$. On the other hand, by Fact 2.2 we have $h(G) \geq 1/2$. If v_i is a leaf (obviously, G has at least one leaf), then the graph $G - v_i$ is a unicyclic graph with $n-1$ vertices containing C_k. By the inductive hypothesis we have $h(G - v_i) = 1/2$, and therefore $h(G) = 1/2$. Now assume that every vertex of the cycle, and every leaf guesses its color, that is, every one of these vertices guesses its color in at least one situation. We are interested in the possibility when the number of cases for which the team loses is as small as possible. We assume that every one of those vertices guesses its color in exactly one situation, and we prove that these guesses suffice to cause the loss of the team in more than a half of all cases. The vertices of the cycle we denote by $v_1, v_2, \ldots, v_k$. Let v_i and v_{i+1} be adjacent.

First assume that at least three vertices of the cycle have degree at least three. This implies that G has at least three leaves having different neighbors. Observe that each one of the leaves guesses its color wrong in a quarter of all cases. Since the closed neighborhoods of the leaves are pairwise disjoint, the team wins for at most $(3/4)^3 = 27/64 < 1/2$ of all cases. This is a contradiction to Corollary 2.3.

Now assume that exactly two vertices of the cycle have degree at least three. Thus G has at least two leaves, say x and y, which have different neighbors. The neighbor of x (y, respectively) we denote by x' (y', respectively). Let v_i mean a vertex of the cycle such that $x, y, x', y' \notin N_G[v_i]$. Let us observe that the vertex v_i guesses its color wrong in $1/8$ of all cases as it has two neighbors. Each one of the leaves x and y guesses its color wrong in a quarter of all cases. Since the closed neighborhoods of the vertices x, y, and v_i are pairwise disjoint, the team wins for at most $(3/4)^2 \cdot 7/8 = 63/128 < 1/2$ of all cases. This is a contradiction to Corollary 2.3.

Now assume that exactly one vertex of the cycle, say v_1, has degree at least three. Let x mean a leaf of T_k which is joined with v_1 by a path which does not go through any other vertex of the cycle. The vertex x guesses its color wrong in a quarter of all cases. Each one of the vertices v_3 and v_4 guesses its color wrong in $1/8$ of all cases. Let us observe that both these vertices at the same time guess their colors wrong in at most $1/16$ of all cases. Thus they guess their colors wrong in at least $1/8 + 1/8 - 1/16 = 3/16$ of all cases. Similarly we conclude that the vertices v_7 and v_8 guess their colors wrong in at least $3/16$ of all cases. Disjointness of proper neighborhoods implies that the team wins for at most $(13/16)^2 \cdot 3/4 = 507/1024 < 1/2$ of all cases. This is a contradiction to Corollary 2.3. □

5. HAT PROBLEM ON A GRAPH WITH A UNIVERSAL VERTEX

Now we consider the hat problem on graphs with a universal vertex.

We have the following property of optimal strategies for such graphs.

Fact 5.1. *Let G be a graph, and let v be a universal vertex of G. If S is an optimal strategy for the graph G, then for every situation of v, in at least one of two cases corresponding to this situation some vertex guesses its color.*

Proof. Let s be a situation of v. Suppose that the strategy S for the graph G is optimal, and in the cases c and d corresponding to the situation s no vertex guesses its color. Of course, for both these cases the team loses. Let the strategy S' for the graph G differ from S only in that in the situation s the vertex v guesses it has the color which it has in the case c. In the strategy S' the result of the case c is a win, and d is a loss. This implies that $|W(S')| = |W(S)| + 1$, and consequently,

$$p(S') = \frac{|W(S')|}{|C(G)|} = \frac{|W(S)| + 1}{|C(G)|} > \frac{|W(S)|}{|C(G)|} = p(S),$$

a contradiction to the optimality of S. □

Now, let us consider a strategy for a graph with a universal vertex such that there are two cases corresponding to the same situation of a universal vertex, and in one of them no vertex guesses its color, while in the second some vertex guesses its color. In the following lemma we give a method of creating a strategy which is not worse than that.

Lemma 5.2. *Let G be a graph and let v be a universal vertex of G. Let c and d be any cases corresponding to the same situation of v. Let S be a strategy for the graph G such that in the case c no vertex guesses its color, and in the case d some vertex guesses its color. Let the strategy S' for the graph G differ from S only in that v, in the situation to which correspond cases c and d, guesses it has the color which it has in the case c. Then $p(S') \geq p(S)$.*

Proof. The result of the case c in the strategy S' is a win, and in the strategy S is a loss. The result of the case d in the strategy S' is a loss. If the result of the case d in the strategy S is also a loss, then $|W(S')| = |W(S)| + 1$. If the result of the case d in the strategy S is a win, then $|W(S')| = |W(S)|$. This implies that $|W(S')| \geq |W(S)|$. Therefore $p(S') \geq p(S)$. □

It is possible to prove that if a graph has a universal vertex, then there exists an optimal strategy such that in every case some vertex guesses its color. This implies that to solve the hat problem on a graph with a universal vertex it suffices to examine only strategies such that in every case some vertex guesses its color. Thus if in some case of a strategy no vertex guesses its color, then we can cease further examining this strategy.

Theorem 5.3. *If G is a graph with a universal vertex, then there exists a strategy $S \in \mathcal{F}^0(G)$ such that $|Ln(S)| = 0$.*

Proof. Suppose that for every optimal strategy S for the graph G we have $|Ln(S)| > 0$. Let S' be an optimal strategy for G, and let $c_1, c_2, \ldots, c_n$ be the cases in which no vertex guesses its color. By Fact 5.1, any two of them do not correspond to the same situation of v. Let the strategy S_1 for the graph G differ from S' only in that v, in the situation to which corresponds the case c_1, guesses it has the color which it has in the case c_1. By Lemma 5.2 we have $p(S_1) \geq p(S')$. Let the strategy S_2 for the graph G differ from S_1 only in that v, in the situation to which corresponds the case c_2, guesses it has the color which it has in the case c_2. By Lemma 5.2 we have $p(S_2) \geq p(S_1)$. After $n-2$ further analogical steps we get the strategy $S = S_n$ for the graph G such that $p(S) \geq p(S_{n-1}) \geq \ldots \geq p(S_2) \geq p(S_1) \geq p(S')$, and there is no case in which no vertex guesses its color. Since the strategy S' for the graph G is optimal, and $p(S) \geq p(S')$, the strategy S is also optimal. In every case in the strategy S_n some vertex guesses its color, thus $|Ln(S)| = 0$. □

In the next fact we state that there exists a graph with a universal vertex for which there exists an optimal strategy such that in some case no vertex guesses its color.

Fact 5.4. *There exists a strategy $S \in \mathcal{F}^0(K_2)$ such that $|Ln(S)| > 0$.*

Proof. Let $S = (g_1, g_2) \in \mathcal{F}(K_2)$ be the strategy as follows.

$$g_1(s_1) = \begin{cases} 1 & \text{if } s_1(v_2) = 1, \\ * & \text{otherwise,} \end{cases}$$

$$g_2(s_2) = \begin{cases} 2 & \text{if } s_2(v_1) = 2, \\ * & \text{otherwise.} \end{cases}$$

All cases we present in Table 1.

Table 1.

No	The color of		The guess of		Result
	v_1	v_2	v_1	v_2	
1	1	1	+		+
2	1	2			−
3	2	1	−	−	−
4	2	2		+	+

From Table 1 we know that $|W(S)| = 2$ and $|Ln(S)| = 1$. We have $|C(K_2)| = 4$, thus $p(S) = 2/4 = 1/2$. The graph K_2 is a tree, therefore by Theorem 2.7 we have $h(K_2) = 1/2$. Since $h(K_2) = 1/2$, the strategy S is optimal for K_2. Both vertices v_1 and v_2 are universal, and $|Ln(S)| = 1$ as in the case in which v_1 has the first color, and v_2 has the second color no vertex guesses its color. □

6. A NORDHAUS-GADDUM TYPE INEQUALITIES

In this section we give some Nordhaus-Gaddum type inequalities.

In the following two theorems we give a lower and upper bounds on the product (sum, respectively) of the hat number of a graph and the hat number of its complement.

Theorem 6.1. *For every graph G we have $1/4 \leq h(G)h(\overline{G}) < 1$.*

Proof. By Corollary 2.2 we have $h(G) \geq 1/2$ and $h(\overline{G}) \geq 1/2$, so $h(G)h(\overline{G}) \geq 1/4$. Since by Fact 2.4 we have $h(G)<1$ and $h(\overline{G})<1$, we get $h(G)h(\overline{G})<1$. □

Theorem 6.2. *For every graph G we have $1 \leq h(G) + h(\overline{G}) < 2$.*

The proof is similar to that of Theorem 6.1.

Now we prove that for every number greater than or equal to quarter, and smaller than one, there exists a graph for which the product of its hat number and the hat number of its complement is greater than that number.

Theorem 6.3. *For every $\alpha \in [1/4; 1)$ there is a graph G such that $h(G)h(\overline{G}) > \alpha$.*

Proof. Let G be a graph with $2n$ vertices such that $V(G) = \{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n\}$ and $E(G) = \{v_iv_j : i, j \in \{1, 2, \ldots, n\}, i \neq j\}$. It is easy to see that $E(\overline{G}) = \{v_iv'_j : i, j \in \{1, 2, \ldots, n\}\} \cup \{v'_iv'_j : i, j \in \{1, 2, \ldots, n\}, i \neq j\}$. Since K_n is a subgraph of both graphs G and $\overline{G}$, by Fact 2.1 we have $h(G) \geq h(K_n)$ and $h(\overline{G}) \geq h(K_n)$. To prove that $h(G)h(\overline{G}) > \alpha$, it suffices to prove that $(h(K_n))^2 > \alpha$, that is $h(K_n) > \sqrt{\alpha}$. The authors of [14] have proven that for the hat problem with $n = 2^k - 1$ players there exists a strategy giving the chance of success $(2^k-1)/2^k$. Since $\lim_{k\to\infty}(2^k-1)/2^k = 1$, for every $\alpha \in [1/4; 1)$ there exists a positive integer k such that for the hat problem with $n = 2^k - 1$ players there exists a strategy S such that $p(S) \geq 1 - 1/2^k = 1 - 1/(n+1) > \sqrt{\alpha}$. By definition we have $h(K_n) \geq p(S)$, thus $h(K_n) > \sqrt{\alpha}$. □

The following theorem says that for every number greater than or equal to one, and smaller than two, there exists a graph for which the sum of its hat number and the hat number of its complement is greater than that number.

Theorem 6.4. *For every $\alpha \in [1; 2)$ there is a graph G such that $h(G) + h(\overline{G}) > \alpha$.*

The proof is similar to that of Theorem 6.3.

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2.5 The hat problem on a union of disjoint graphs

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The hat problem on a union of disjoint graphs

Abstract. The topic is the hat problem in which each of n players is randomly fitted with a blue or red hat. Then everybody can try to guess simultaneously his own hat color by looking at the hat colors of the other players. The team wins if at least one player guesses his hat color correctly, and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of winning. In this version every player can see everybody excluding himself. We consider such a problem on a graph, where vertices correspond to players, and a player can see each player to whom he is connected by an edge. The solution of the hat problem is known for cycles and bipartite graphs. We investigate the problem on a union of disjoint graphs.

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1. Introduction. In the hat problem, a team of n players enters a room and a blue or red hat is randomly placed on the head of each player. Each player can see the hats of all of the other players but not his own. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats, each player must simultaneously guess the color of his own hat or pass. The team wins if at least one player guesses his hat color correctly and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of winning.

The hat problem with seven players, called the "seven prisoners puzzle", was formulated by T. Ebert in his Ph.D. Thesis [12]. The hat problem was also the subject of articles in The New York Times [28], Die Zeit [6], and abcNews [27]. It was also one of the Berkeley Riddles [4].

The hat problem with $2^k - 1$ players was solved in [14], and for 2^k players in [11]. The problem with n players was investigated in [7]. The hat problem and Hamming codes were the subject of [8]. The generalized hat problem with n people and q colors was investigated in [26].

There are many known variations of the hat problem (for a comprehensive list, see [24]). For example in [20] there was considered a variation in which players do not have to guess their hat colors simultaneously. In the papers [1, 10, 19] there was considered a variation in which passing is not allowed, thus everybody has to guess his hat color. The aim is to maximize the number of correct guesses. The authors of [17] investigated several variations of the hat problem in which the aim is to design a strategy guaranteeing a desired number of correct guesses. In [18] there was considered a variation in which the probabilities of getting hats of each colors do not have to be equal. The authors of [2] investigated a problem similar to the hat problem. There are n players which have random bits on foreheads, and they have to vote on the parity of the n bits.

The hat problem and its variations have many applications and connections to different areas of science (for a survey on this topic, see [24]), for example: information technology [5], linear programming [17], genetic programming [9], economics [1, 19], biology [18], approximating Boolean functions [2], and autoreducibility of random sequences [3, 12–15]. Therefore, it is hoped that the hat problem on a graph considered in this paper is worth exploring as a natural generalization, and may also have many applications.

We consider the hat problem on a graph, where vertices correspond to players and a player can see each player to whom he is connected by an edge. This variation of the hat problem was first considered in [21]. There were proven some general theorems about the hat problem on a graph, and the problem was solved on trees. Additionally, there was considered the hat problem on a graph such that the only known information are degrees of vertices. In [23] the hat problem was solved on the cycle C_4. In [25] the problem was solved on cycles on at least nine vertices. Then the problem was solved on all odd cycles [22]. Uriel Feige [16] conjectured that for any graph the maximum chance of success in the hat problem is equal to the maximum chance of success for the hat problem on the maximum clique in the graph. He provided several results that support this conjecture, and solved the hat problem for bipartite graphs and planar graphs containing a triangle. He also proved that the hat number of a union of disjoint graphs is the maximum hat number among that graphs.

In this paper we consider the hat problem on a union of disjoint graphs. By the union of two strategies (each for another graph) we mean the strategy for the union of that graphs such that every vertex behaves in the same way as in the proper strategy which is an element of the union. First, we give a sufficient condition for that the union of strategies gives worse chance of success than some component of the union. Next, we characterize when the union of strategies gives at least the same (better, the same, respectively) chance of success as each component of the union. Finally, we prove that there exists a disconnected graph for which there exists an optimal strategy such that every vertex guesses its color.

2. Preliminaries. For a graph G, the set of vertices and the set of edges we denote by $V(G)$ and $E(G)$, respectively. If H is a subgraph of G, then we write $H \subseteq G$. Let $v \in V(G)$. The degree of vertex v, that is, the number of its neighbors, we denote by $d_G(v)$. The path (complete graph, respectively) on n vertices we denote by P_n (K_n, respectively).

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. By $Sc = \{1, 2\}$ we denote the set of colors, where 1 corresponds to blue, and 2 corresponds to red.

By a case for a graph G we mean a function $c\colon V(G) \to \{1, 2\}$, where $c(v_i)$ means color of vertex v_i. The set of all cases for the graph G we denote by $C(G)$, of course $|C(G)| = 2^{|V(G)|}$.

By a situation of a vertex v_i we mean a function $s_i\colon V(G) \to Sc \cup \{0\} = \{0, 1, 2\}$, where $s_i(v_j) \in Sc$ if v_i and v_j are adjacent, and 0 otherwise. The set of all possible situations of v_i in the graph G we denote by $St_i(G)$, of course $|St_i(G)| = 2^{d_G(v_i)}$.

We say that a case c for the graph G corresponds to a situation s_i of vertex v_i if $c(v_j) = s_i(v_j)$, for every v_j adjacent to v_i. This implies that a case corresponds to a situation of v_i if every vertex adjacent to v_i in that case has the same color as in that situation. Of course, to every situation of the vertex v_i correspond exactly $2^{|V(G)|-d_G(v_i)}$ cases.

By a guessing instruction of a vertex $v_i \in V(G)$ we mean a function $g_i\colon St_i(G) \to Sc \cup \{0\} = \{0, 1, 2\}$, which for a given situation gives the color v_i guesses it is, or 0 if v_i passes. Thus, a guessing instruction is a rule determining the behavior of a vertex in every situation.

Let c be a case, and let s_i be the situation (of vertex v_i) corresponding to that case. The guess of v_i in the case c is correct (wrong, respectively) if $g_i(s_i) = c(v_i)$ ($0 \neq g_i(s_i) \neq c(v_i)$, respectively). By result of the case c we mean a win if at least one vertex guesses its color correctly, and no vertex guesses its color wrong, that is, $g_i(s_i) = c(v_i)$ (for some i) and there is no j such that $0 \neq g_j(s_j) \neq c(v_j)$. Otherwise the result of the case c is a loss.

By a strategy for the graph G we mean a sequence $(g_1, g_2, \ldots, g_n)$, where g_i is the guessing instruction of vertex v_i. The family of all strategies for a graph G we denote by $\mathcal{F}(G)$.

If $S \in \mathcal{F}(G)$, then the set of cases for the graph G for which the team wins (loses, respectively) using the strategy S we denote by $W(S)$ ($L(S)$, respectively). The set of cases for which the team loses, and some vertex guesses its color (no vertex guesses its color, respectively) we denote by $Ls(S)$ ($Ln(S)$, respectively). By the chance of success of the strategy S we mean the number $p(S) = |W(S)|/|C(G)|$. By the hat number of the graph G we mean the number $h(G) = \max\{p(S)\colon S \in \mathcal{F}(G)\}$. We say that a strategy S is optimal for the graph G if $p(S) = h(G)$. The family of all optimal strategies for the graph G we denote by $\mathcal{F}^0(G)$.

By solving the hat problem on a graph G we mean finding the number $h(G)$.

Since for every graph we can apply a strategy in which one vertex always guesses it has, let us say, the first color, and the other vertices always pass, we immediately get the following lower bound on the hat number of a graph.

FACT 2.1 *For every graph G we have $h(G) \geqslant 1/2$.*

The next solution of the hat problem on paths is a result from [21].

THEOREM 2.2 *For every path P_n we have $h(P_n) = 1/2$.*

3. Results. Let G and H be vertex-disjoint graphs, and let $S_1 \in \mathcal{F}(G)$ and $S_2 \in \mathcal{F}(H)$. By the union of the strategies S_1 and S_2 we mean the strategy $S \in \mathcal{F}(G \cup H)$ such that every vertex of G behaves in the same way as in S_1, and every vertex of H behaves in the same way as in S_2. If S is the union of S_1 and S_2, then we write $S = S_1 \cup S_2$.

From now writing that G and H are graphs, we assume that they are vertex-disjoint.

In the following theorem we give a sufficient condition for that the union of strategies gives worse chance of success than some component of the union.

THEOREM 3.1 *Let G and H be graphs, and let $S = S_1 \cup S_2$, where $S_1 \in \mathcal{F}(G)$ and $S_2 \in \mathcal{F}(H)$. Assume that $p(S_1) > 0$ and $p(S_2) > 0$. If $|Ln(S_1)| \cdot |Ln(S_2)| < |Ls(S_1)| \cdot |Ls(S_2)|$, then $p(S) < \max\{p(S_1), p(S_2)\}$.*

PROOF First, let us observe that $|Ls(S_1)| > 0$, otherwise no vertex guesses its color, and therefore $|W(S_1)| = 0$. Consequently, $p(S_1) = 0$, a contradiction. Similarly we get $|Ls(S_2)| > 0$. Now let us consider the strategy $S = S_1 \cup S_2$ for the graph $G \cup H$. The team wins if at least one vertex guesses its color correctly, and no vertex guesses its color wrong, thus the team wins if:

(i) some vertex of G guesses its color correctly and no vertex of G guesses its color wrong, and some vertex of H guesses its color correctly and no vertex of H guesses its color wrong, or

(ii) some vertex of G guesses its color correctly and no vertex of G guesses its color wrong, and no vertex of H guesses its color, or

(iii) no vertex of G guesses its color, and some vertex of H guesses its color correctly and no vertex of H guesses its color wrong.

This implies that

$$|W(S)| = |W(S_1)||W(S_2)| + |W(S_1)||Ln(S_2)| + |Ln(S_1)||W(S_2)|.$$

Since $|C(G \cup H)| = |C(G)||C(H)|$, we get

$$p(S) = \frac{|W(S)|}{|C(G \cup H)|} = \frac{|W(S_1)||W(S_2)| + |W(S_1)||Ln(S_2)| + |Ln(S_1)||W(S_2)|}{|C(G)||C(H)|}.$$

We have

$$p(S) \geqslant \max\{p(S_1), p(S_2)\} \Leftrightarrow (p(S) \geqslant p(S_1) \text{ and } p(S) \geqslant p(S_2)).$$

Now we get the following chain of equivalences

$$\begin{aligned}
p(S_1) \leqslant p(S) \quad &\Leftrightarrow \quad \frac{|W(S_1)|}{|C(G)|} \leqslant \frac{|W(S_1)||W(S_2)| + |W(S_1)||Ln(S_2)| + |Ln(S_1)||W(S_2)|}{|C(G)||C(H)|} \\
&\Leftrightarrow \quad |W(S_1)||C(H)| \leqslant |W(S_1)||W(S_2)| + |W(S_1)||Ln(S_2)| \\
&\qquad\qquad + |Ln(S_1)||W(S_2)| \\
&\Leftrightarrow \quad |C(H)| \leqslant |W(S_2)| + |Ln(S_2)| + \frac{|Ln(S_1)||W(S_2)|}{|W(S_1)|}
\end{aligned}$$

$$\Leftrightarrow |Ls(S_2)| \leqslant \frac{|Ln(S_1)||W(S_2)|}{|W(S_1)|}.$$

Similarly we get

$$p(S) \geqslant p(S_2) \Leftrightarrow \frac{|Ln(S_2)||W(S_1)|}{|W(S_2)|} \geqslant |Ls(S_1)|.$$

Therefore we have

$$p(S) \geqslant \max\{p(S_1), p(S_2)\}$$

if and only if

$$\frac{|Ln(S_1)||W(S_2)|}{|W(S_1)|} \geqslant |Ls(S_2)| \quad \text{and} \quad \frac{|Ln(S_2)||W(S_1)|}{|W(S_2)|} \geqslant |Ls(S_1)|.$$

Consequently,

$$p(S) \geqslant \max\{p(S_1), p(S_2)\} \Rightarrow |Ln(S_1)| \cdot |Ln(S_2)| \geqslant |Ls(S_1)| \cdot |Ls(S_2)|.$$

Equivalently,

$$|Ln(S_1)| \cdot |Ln(S_2)| < |Ls(S_1)| \cdot |Ls(S_2)| \Rightarrow p(S) < \max\{p(S_1), p(S_2)\}.$$

Corollary 3.2 *Let G and H be graphs, and let $S = S_1 \cup S_2$, where $S_1 \in \mathcal{F}(G)$ and $S_2 \in \mathcal{F}(H)$. Assume that $p(S_1) > 0$ and $p(S_2) > 0$. If $|Ln(S_1)| = 0$ or $|Ln(S_2)| = 0$, then $p(S) < \max\{p(S_1), p(S_2)\}$.*

Proof As we have observed in the proof of Theorem 3.1, we have $|Ls(S_1)| > 0$ and $|Ls(S_2)| > 0$. Therefore $|Ln(S_1)| \cdot |Ln(S_2)| = 0 < |Ls(S_1)| \cdot |Ls(S_2)|$. Now, by Theorem 3.1 we have $p(S) < \max\{p(S_1), p(S_2)\}$. ■

From now writing $S_1 \in \mathcal{F}(G)$ and $S_2 \in \mathcal{F}(H)$, we assume that $p(S_1) > 0$, $p(S_2) > 0$, and $|Ln(S_1)| \cdot |Ln(S_2)| \geqslant |Ls(S_1)| \cdot |Ls(S_2)|$.

The following theorem determines when the union of strategies gives at least the same chance of success as each component of the union.

Theorem 3.3 *If G and H are graphs and $S = S_1 \cup S_2$, where $S_1 \in \mathcal{F}(G)$ and $S_2 \in \mathcal{F}(H)$, then*

$$p(S) \geqslant \max\{p(S_1), p(S_2)\} \Leftrightarrow \frac{|W(S_1)|}{|W(S_2)|} \in \left[\frac{|Ls(S_1)|}{|Ln(S_2)|}; \frac{|Ln(S_1)|}{|Ls(S_2)|}\right].$$

Proof From the proof of Theorem 3.1 we know that

$$p(S) \geqslant \max\{p(S_1), p(S_2)\}$$

if and only if

$$\frac{|Ln(S_1)||W(S_2)|}{|W(S_1)|} \geqslant |Ls(S_2)| \quad \text{and} \quad \frac{|Ln(S_2)||W(S_1)|}{|W(S_2)|} \geqslant |Ls(S_1)|.$$

Since $|Ln(S_1)| > 0$, $|Ln(S_2)| > 0$, and $|Ls(S_2)| > 0$ (see the proof of Theorem 3.1), the condition above is equivalent to that

$$\frac{|W(S_1)|}{|W(S_2)|} \leqslant \frac{|Ln(S_1)|}{|Ls(S_2)|} \quad \text{and} \quad \frac{|W(S_1)|}{|W(S_2)|} \geqslant \frac{|Ls(S_1)|}{|Ln(S_2)|},$$

that is

$$\frac{|Ls(S_1)|}{|Ln(S_2)|} \leqslant \frac{|W(S_1)|}{|W(S_2)|} \leqslant \frac{|Ln(S_1)|}{|Ls(S_2)|}.$$

The interval

$$\left[\frac{|Ls(S_1)|}{|Ln(S_2)|}; \frac{|Ln(S_1)|}{|Ls(S_2)|}\right]$$

is nonempty, because $|Ln(S_1)| \cdot |Ln(S_2)| \geqslant |Ls(S_1)| \cdot |Ls(S_2)|$. Concluding,

$$p(S) \geqslant \max\{p(S_1), p(S_2)\} \Leftrightarrow \frac{|W(S_1)|}{|W(S_2)|} \in \left[\frac{|Ls(S_1)|}{|Ln(S_2)|}; \frac{|Ln(S_1)|}{|Ls(S_2)|}\right].$$

Corollary 3.4 *If G and H are graphs and $S = S_1 \cup S_2$, where $S_1 \in \mathcal{F}(G)$ and $S_2 \in \mathcal{F}(H)$, then*

$$p(S) < \max\{p(S_1), p(S_2)\} \Leftrightarrow \frac{|W(S_1)|}{|W(S_2)|} \notin \left[\frac{|Ls(S_1)|}{|Ln(S_2)|}; \frac{|Ln(S_1)|}{|Ls(S_2)|}\right].$$

The following two theorems determine when the union of strategies gives better (or the same) chance of success than each component of the union. The proof of each one of these two theorems is similar to proofs of Theorems 3.1 and 3.3. Therefore we do not prove them.

Theorem 3.5 *If G and H are graphs and $S = S_1 \cup S_2$, where $S_1 \in \mathcal{F}(G)$ and $S_2 \in \mathcal{F}(H)$, then*

$$p(S) > \max\{p(S_1), p(S_2)\} \Leftrightarrow \frac{|W(S_1)|}{|W(S_2)|} \in \left(\frac{|Ls(S_1)|}{|Ln(S_2)|}; \frac{|Ln(S_1)|}{|Ls(S_2)|}\right).$$

Theorem 3.6 *If G and H are graphs and $S = S_1 \cup S_2$, where $S_1 \in \mathcal{F}(G)$ and $S_2 \in \mathcal{F}(H)$, then*

$$p(S) = p(S_1) \Leftrightarrow \frac{|W(S_1)|}{|W(S_2)|} = \frac{|Ln(S_1)|}{|Ls(S_2)|}$$

and

$$p(S) = p(S_2) \Leftrightarrow \frac{|W(S_1)|}{|W(S_2)|} = \frac{|Ls(S_1)|}{|Ln(S_2)|}.$$

COROLLARY 3.7 *Assume that G and H are graphs, and $S = S_1 \cup S_2$, where $S_1 \in \mathcal{F}(G)$ and $S_2 \in \mathcal{F}(H)$. Let $i \in \{1,2\}$ be such that $p(S_i) = \max\{p(S_1), p(S_2)\}$, and let $j \in \{1,2\}$, $j \neq i$. Then*

$$p(S) = \max\{p(S_1), p(S_2)\} \Leftrightarrow \frac{|W(S_i)|}{|W(S_j)|} = \frac{|Ln(S_i)|}{|Ls(S_j)|}.$$

It is possible to prove that there exists a disconnected graph for which there exists an optimal strategy such that every vertex guesses its color. First, we solve the hat problem on such graph, $K_2 \cup K_2$.

FACT 3.8 $h(K_2 \cup K_2) = 1/2$.

PROOF We have $K_2 \cup K_2 \subseteq P_4$, thus $h(K_2 \cup K_2) \leqslant h(P_4)$. By Theorem 2.2 we have $h(P_4) = 1/2$. Therefore $h(K_2 \cup K_2) \leqslant 1/2$. On the other hand, by Fact 2.1 we have $h(K_2 \cup K_2) \geqslant 1/2$. ■

FACT 3.9 *There exists an optimal strategy for the graph $K_2 \cup K_2$ such that every vertex guesses its color.*

PROOF Let $S' = (g_1, g_2) \in \mathcal{F}(K_2)$ be the strategy as follows.

$$g_1(s_1) = \begin{cases} 1 & \text{if } s_1(v_2) = 1, \\ 0 & \text{otherwise;} \end{cases}$$

$$g_2(s_2) = \begin{cases} 2 & \text{if } s_2(v_1) = 2, \\ 0 & \text{otherwise.} \end{cases}$$

It means that the vertices proceed as follows.

- **The vertex** v_1. If v_2 has the first color, then it guesses it has the first color, otherwise it passes.
- **The vertex** v_2. If v_1 has the second color, then it guesses it has the second color, otherwise it passes.

All cases we present in Table 1, where the symbol + means correct guess (success), − means wrong guess (loss), and blank square means passing.

No	The color of		The guess of		Result
	v_1	v_2	v_1	v_2	
1	1	1	+		+
2	1	2			−
3	2	1	−	−	−
4	2	2		+	+

Table 1

From Table 1 we know that $|W(S')| = 2$, $|Ln(S')| = 1$, and $|Ls(S')| = 1$. Since $|C(K_2)| = 4$, we get $p(S') = |W(S')|/|C(K_2)| = 2/4 = 1/2$. Let $S = S_1 \cup S_2 \in \mathcal{F}(K_2 \cup K_2)$, where $S_1 = S_2 = S'$. We have $|W(S_1)|/|W(S_2)| = |W(S')|/|W(S')| = 1$ and $|Ln(S_1)|/|Ls(S_2)| = |Ln(S')|/|Ls(S')| = 1/1 = 1$. Since $|W(S_1)|/|W(S_2)| = |Ln(S_1)|/|Ls(S_2)|$, by Theorem 3.6 we get $p(S) = p(S_1)$. We have $S_1 = S'$ and $p(S') = 1/2$, thus $p(S) = 1/2$. By Fact 3.8 we have $h(K_2 \cup K_2) = 1/2$. This implies that the strategy S is optimal. In this strategy every vertex guesses its color. ∎

Now we prove this fact elementary.

Proof Let $E(K_2 \cup K_2) = \{v_1v_2, v_3v_4\}$, and let $S = (g_1, g_2, g_3, g_4)$ be the strategy for $K_2 \cup K_2$ as follows.

$$g_1(s_1) = \begin{cases} 1 & \text{if } s_1(v_2) = 1, \\ 0 & \text{otherwise;} \end{cases}$$

$$g_2(s_2) = \begin{cases} 2 & \text{if } s_2(v_1) = 2, \\ 0 & \text{otherwise;} \end{cases}$$

$$g_3(s_3) = \begin{cases} 1 & \text{if } s_3(v_4) = 1, \\ 0 & \text{otherwise;} \end{cases}$$

$$g_4(s_4) = \begin{cases} 2 & \text{if } s_4(v_3) = 2, \\ 0 & \text{otherwise.} \end{cases}$$

It means that the vertices proceed as follows.

- **The vertex** v_1. If v_2 has the first color, then it guesses it has the first color, otherwise it passes.
- **The vertex** v_2. If v_1 has the second color, then it guesses it has the second color, otherwise it passes.
- **The vertex** v_3. If v_4 has the first color, then it guesses it has the first color, otherwise it passes.
- **The vertex** v_4. If v_3 has the second color, then it guesses it has the second color, otherwise it passes.

All cases we present in Table 2. From this table we get $|W(S)| = 8$. We have $|C(K_2 \cup K_2)| = 16$, thus $p(S) = 8/16 = 1/2$. Similarly as in the previous proof we conclude that the strategy S is optimal. In this strategy every vertex guesses its color. ∎

No	The color of				The guess of				Result
	v_1	v_2	v_3	v_4	v_1	v_2	v_3	v_4	
1	1	1	1	1	+		+		+
2	1	1	1	2	+				+
3	1	1	2	1	+		−	−	−
4	1	1	2	2	+			+	+
5	1	2	1	1			+		+
6	1	2	1	2					−
7	1	2	2	1			−	−	−
8	1	2	2	2				+	+
9	2	1	1	1	−	−	+		−
10	2	1	1	2	−	−			−
11	2	1	2	1	−	−	−	−	−
12	2	1	2	2	−	−		+	−
13	2	2	1	1		+	+		+
14	2	2	1	2		+			+
15	2	2	2	1		+	−	−	−
16	2	2	2	2		+		+	+

Table 2

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2.6 Hat problem on odd cycles

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HAT PROBLEM ON ODD CYCLES

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Abstract. The topic is the hat problem in which each of n players is randomly fitted with a blue or red hat. Then everybody can try to guess simultaneously his own hat color by looking at the hat colors of the other players. The team wins if at least one player guesses his hat color correctly, and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of a win. In this version every player can see everybody excluding himself. We consider such a problem on a graph, where vertices correspond to players, and a player can see each player to whom he is connected by an edge. The hat problem on a graph was solved for trees and for the cycle on four vertices. Then Uriel Feige conjectured that for any graph the maximum chance of success in the hat problem is equal to the maximum chance of success for the hat problem on the maximum clique in the graph. He provided several results that support this conjecture, and solved the hat problem for bipartite graphs and planar graphs containing a triangle. We make a step towards proving the conjecture of Feige. We solve the hat problem on all cycles of odd length. Of course, the maximum chance of success for the hat problem on the cycle on three vertices is three fourths. We prove that the hat number of every odd cycle of length at least five is one half, which is consistent with the conjecture of Feige.

1. Introduction

In the hat problem, a team of n players enters a room and a blue or red hat is randomly placed on the head of each player. Each player can see the hats of all of the other players but not his own. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats, each player must simultaneously guess the

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color of his own hat or pass. The team wins if at least one player guesses his hat color correctly and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of a win.

The hat problem with seven people called "seven prisoners puzzle" was formulated by T. Ebert in his Ph.D. Thesis [4]. There are known many variations of the hat problem (for a comprehensive list, see [9]). For example in [6] there was considered a variation in which players do not have to guess their hat colors simultaneously. In [2] there was considered a variation in which passing is not allowed, thus everybody has to guess his hat color. The aim is to maximize the number of correct guesses. N. Alon [1] has proved a lower bound on the chance of success for the generalized hat problem with n people and q colors. This problem was also studied in [10]. The hat problem with three people was the subject of an article in The New York Times [11].

We consider the hat problem on a graph, where vertices correspond to players and a player can see each player to whom he is connected by an edge. This variation of the hat problem was first considered in [7]. There were proven some general theorems about the hat problem on a graph, and the problem was solved on trees. Additionally, there was considered the hat problem on a graph such that the only known information are degrees of vertices. In [8] the problem was solved on the cycle C_4. Uriel Feige [5] conjectured that for any graph the maximum chance of success in the hat problem is equal to the maximum chance of success for the hat problem on the maximum clique in the graph. He provided several results that support this conjecture, and solved the hat problem for bipartite graphs and planar graphs containing a triangle. Feige proved that if a graph is such that the chromatic number equals the number of vertices of the maximum clique, then the conjecture is true. A well known class of graphs for which the chromatic number equals the number of vertices of the maximum clique is that of perfect graphs (where that equality holds not only for the graph, but also for all its subgraphs). Thus Feige solved the hat problem for all perfect graphs. By the strong perfect graph theorem [3], every graph for which neither it nor its complement contains an induced odd cycle of length at least five is perfect. We solve the hat problem on all cycles of odd length. Of course, the maximum chance of success for the hat problem on the cycle on three vertices is three fourths. We prove that the hat number of every odd cycle of length at least five is one half, which is consistent with the conjecture of Feige.

2. Preliminaries

For a graph G, the set of vertices and the set of edges we denote by $V(G)$ and $E(G)$, respectively. If H is a subgraph of G, then we write $H \subseteq G$. The path (cycle, respectively) on n vertices we denote by P_n (C_n, respectively). The neighborhood of a vertex v of G, that is $\{x \in V(G)\colon vx \in E(G)\}$, we denote by $N_G(v)$. We say that a vertex v is neighborhood-dominated if there is some other vertex u such that $N_G(v) \subseteq N_G(u)$.

Let $f\colon X \to Y$ be a function. If for every $x \in X$ we have $f(x) = y$, then we write $f \equiv y$.

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. By $Sc = \{1, 2\}$ we denote the set of colors, where 1 corresponds to the blue color, and 2 corresponds to the red color.

By a case for a graph G we mean a function $c\colon V(G) \to \{1, 2\}$, where $c(v_i)$ means color of vertex v_i. The set of all cases for the graph G we denote by $C(G)$, of course $|C(G)| = 2^{|V(G)|}$. If $c \in C(G)$, then to simplify notation, we write $c = c(v_1)c(v_2)\ldots c(v_n)$ instead of $c = \{(v_1, c(v_1)), (v_2, c(v_2)), \ldots, (v_n, c(v_n))\}$. For example, if a case $c \in C(C_5)$ is such that $c(v_1) = 2$, $c(v_2) = 1$, $c(v_3) = 1$, $c(v_4) = 2$, and $c(v_5) = 1$, then we write $c = 21121$.

By a situation of a vertex v_i we mean a function $s_i\colon V(G) \to Sc \cup \{0\} = \{0, 1, 2\}$, where $s_i(v_j) \in Sc$ if v_i and v_j are adjacent, and 0 otherwise. The set of all possible situations of v_i in the graph G we denote by $St_i(G)$, of course $|St_i(G)| = 2^{d_G(v_i)}$. If $s_i \in St_i(G)$, then for simplicity of notation, we write $s_i = s_i(v_1)s_i(v_2)\ldots s_i(v_n)$ instead of $s_i = \{(v_1, s_i(v_1)), (v_2, s_i(v_2)), \ldots, (v_n, s_i(v_n))\}$. For example, if $s_3 \in St_3(C_5)$ is such that $s_3(v_2) = 2$ and $s_3(v_4) = 1$, then we write $s_3 = 02010$.

By a guessing instruction of a vertex $v_i \in V(G)$ we mean a function $g_i\colon St_i(G) \to Sc \cup \{0\} = \{0, 1, 2\}$, which for a given situation gives the color v_i guesses it is, or 0 if v_i passes. Thus guessing instruction is a rule determining behavior of a vertex in every situation. We say that v_i never guesses its color if v_i passes in every situation, that is, $g_i \equiv 0$. We say that v_i always guesses its color if v_i guesses its color in every situation, that is, for every $s_i \in St_i(G)$ we have $g_i(s_i) \in \{1, 2\}$ ($g_i(s_i) \neq 0$, equivalently).

Let c be a case, and let s_i be the situation (of vertex v_i) corresponding to that case. The guess of v_i in the case c is correct (wrong, respectively) if $g_i(s_i) = c(v_i)$ ($0 \neq g_i(s_i) \neq c(v_i)$, respectively). By result of the case c we mean a win if at least one vertex guesses its color correctly, and no vertex guesses its color wrong, that is, $g_i(s_i) = c(v_i)$ (for some i) and there is no j such that $0 \neq g_j(s_j) \neq c(v_j)$. Otherwise the result of the case c is a loss.

By a strategy for the graph G we mean a sequence $(g_1, g_2, \ldots, g_n)$, where g_i is the guessing instruction of vertex v_i. The family of all strategies for a graph G we denote by $\mathcal{F}(G)$.

If $S \in \mathcal{F}(G)$, then the set of cases for the graph G for which the team wins (loses, respectively) using the strategy S we denote by $W(S)$ ($L(S)$, respectively). By the chance of success of the strategy S we mean the number $p(S) = |W(S)|/|C(G)|$. By the hat number of the graph G we mean the number $h(G) = \max\{p(S) \colon S \in \mathcal{F}(G)\}$. We say that a strategy S is optimal for the graph G if $p(S) = h(G)$. The family of all optimal strategies for the graph G we denote by $\mathcal{F}^0(G)$.

By solving the hat problem on a graph G we mean finding the number $h(G)$.

The following four results are from [7].

Theorem 2.1. *If H is a subgraph of G, then $h(H) \le h(G)$.*

Corollary 2.2. *For every graph G we have $h(G) \ge 1/2$.*

The following theorem is the solution of the hat problem on paths.

Theorem 2.3. *For every path P_n we have $h(P_n) = 1/2$.*

Now there is a sufficient condition for the removal of a vertex of a graph without changing its hat number.

Theorem 2.4. *Let G be a graph, and let v be a vertex of G. If there exists a strategy $S \in \mathcal{F}^0(G)$ such that v never guesses its color, then $h(G) = h(G - v)$.*

Uriel Feige [5] proved the following result.

Lemma 2.5. *Let G be a graph. If v is a neighborhood-dominated vertex of G, then $h(G) = h(G - v)$.*

3. Results

To solve the hat problem on odd cycles of length at least five, we need the fact that $h(C_5) = 1/2$, see Lemma 3.2. Now we prove our main result.

Theorem 3.1. *If $n \ge 5$ is an odd integer, then $h(C_n) = 1/2$.*

Proof. We obtain the result by induction on the length of the cycle. For $n = 5$ the theorem is true by Lemma 3.2. Now assume that $n \ge 7$ is an odd integer, and $h(C_{n-2}) = 1/2$. Let $H_n = C_n \cup v_1v_4$. By Theorem 2.1 we have $h(H_n) \ge h(C_n)$. Observe that $N_{H_n}(v_3) \subset N_{H_n}(v_1)$. Let $H'_n = H_n - v_3$. By Lemma 2.5 we get $h(H_n) = h(H'_n)$. Moreover, since $N_{H'_n}(v_2) \subset N_{H'_n}(v_n)$, again by Lemma 2.5 we get $h(H'_n) = h(H'_n - v_2)$. Let us observe that the graph $H'_n - v_2$ is isomorphic to the cycle C_{n-2}. By the inductive hypothesis we have $h(C_{n-2}) = 1/2$. Now we get $h(C_n) \leq h(H_n) = h(H'_n) = h(C_{n-2}) = 1/2$. On the other hand, by Corollary 2.2 we have $h(C_n) \geq 1/2$. $\square$

Now we solve the hat problem on the cycle on five vertices. We use a Mathematica code to solve the problem. The code is available on the web edition.

Lemma 3.2. $h(C_5) = 1/2$.

Proof. Let S be an optimal strategy for C_5. If some vertex, say v_i, never guesses its color, then by Theorem 2.4 we have $h(C_5) = h(C_5 - v_i)$. Since $C_5 - v_i = P_4$ and $h(P_4) = 1/2$ (by Theorem 2.3), we get $h(C_5) = 1/2$. Now assume that every vertex guesses its color.

Let us consider a guessing instruction of a vertex. If in every case in which this instruction gives a correct guess some other vertex also guesses its color, then we say that the guessing instruction is dominated. Let us observe that we do not have to consider strategies with a dominated guessing instruction because such instruction cannot improve the chance of success. Even if it is the only one guess of a vertex, then by Theorem 2.4 we get $p(S) \leq h(C_5 - v_i) = h(P_4) = 1/2$ implying that $h(C_5) = 1/2$.

Now we explain a way in which the result can be easily verified using computer. We consider only guessing instructions which are not passing. First consider strategies S with exactly one instruction for every vertex. There are exactly 8 possible instructions for each vertex (because of the colors of two neighbors and the guess it is going to make). Thus the total number of possibilities for S is $8^5 = 2^{15}$. Let us observe that from a strategy we can obtain a group of 320 (not necessarily distinguishable) symmetrical strategies. We can perform each one of the following operations: rotating the vertices (gives 5 possibilities), reflecting the vertices (gives 2 possibilities), and relabeling the colors of the vertices (gives $2^5 = 32$ possibilities). Reducing modulo this symmetry group gives only 120 possibilities for S. Now for every one of these possibilities we check the number of cases in which some vertex guesses its color wrong. If in at least 16 cases some vertex guesses its color wrong, then the team loses for at least 16 cases implying that $p(S) \leq 1/2$. For 61 of those 120 strategies in at least 16 cases some vertex guesses its color wrong. Therefore it suffices to consider only the remaining 59 strategies. Now we reduce the set of possibilities by using the idea of dominance from the previous paragraph. In this way we exclude 37 strategies, having only 22 strategies left. Now for every one of them we check the number of cases in which the team so far wins, that is, cases in which some vertex guesses its color correctly while at the same time no vertex guesses its color wrong. The best score among those 22 strategies is 12 successes. Thus we now examine adding an additional instruction to each one of the 22 strategies. Again we exclude strategies for which the team loses for at least 16 cases, or some guessing instruction is dominated. As a result there are only 23 strategies (each one consisting of six guessing instructions) left. Among them, the best score of cases for which the team wins is 12. Now we try to add an additional instruction to each one of the 23 strategies. We verify that for every one of them the team loses for at least 16 cases, or some guessing instruction is dominated. This implies that for every strategy $S \in \mathcal{F}(C_5)$ we have $p(S) \leq 1/2$. Now, by definition we get $h(C_5) = 1/2$. $\square$

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2.7 A construction for the hat problem on a directed graph

A construction for the hat problem on a directed graph

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Abstract

A team of n players plays the following game. After a strategy session, each player is randomly fitted with a blue or red hat. Then, without further communication, everybody can try to guess simultaneously his own hat color by looking at the hat colors of the other players. Visibility is defined by a directed graph; that is, vertices correspond to players, and a player can see each player to whom he is connected by an arc. The team wins if at least one player guesses his hat color correctly, and no one guesses his hat color wrong; otherwise the team loses. The team aims to maximize the probability of a win, and this maximum is called the hat number of the graph.

Previous works focused on the hat problem on complete graphs and on undirected graphs. Some cases were solved, e.g., complete graphs of certain orders, trees, cycles, and bipartite graphs. These led Uriel Feige to conjecture that the hat number of any graph is equal to the hat number of its maximum clique.

We show that the conjecture does not hold for directed graphs. Moreover, for every value of the maximum clique size, we provide a tight characterization of the range of possible values of the hat number. We construct families of directed graphs with a fixed clique number the hat number of which is asymptotically optimal. We also determine the hat number of tournaments to be one half.

Keywords: hat problem, directed graph, digraph, skeleton, clique number.

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1 Introduction

In the hat problem, a team of n players enters a room and a blue or red hat is randomly and independently placed on the head of each player. Each player can see the hats of all of the other players but not his own. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats, each player must simultaneously guess the color of his own hat or pass. The team wins if at least one player guesses his hat color correctly and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of a win.

The hat problem with seven players, called the "seven prisoners puzzle", was formulated by Todd Ebert in his Ph.D. Thesis [6]. It is often posed as a puzzle (e.g., in the Berkeley Riddles [2]) and was also the subject of articles in popular media [3, 20, 21].

The hat problem with $q \geq 2$ possible colors was investigated in [19]. Noga Alon [1] proved that the q-ary hat number of the complete graph tends to one as the graph grows.

Many other variations of the problem exist (for a comprehensive list, see [15]), among them a random but non-uniform hat color distribution [10], an adversarial allocation of hat from a pool known by the players [9], a variation in which passing is not allowed [4], a variation in which players do not have to guess their hat colors simultaneously [11], and many more.

The hat problem can be considered on a graph, where vertices correspond to players, and a player can see each player to whom he is connected by an edge. We seek to determine the hat number of the graph, that is, the maximum chance of success for the hat problem on it. This variation of the hat problem was first considered in [12], and further studied for example in [8, 13, 14, 16–18].

Note that the hat problem on the complete graph is equivalent to the original hat problem. This case was solved for $2^k - 1$ players in [7] and for 2^k players in [5]. In [19] it was shown that a strategy for n players in the complete graph is equivalent to a covering code of radius 1 in the Hamming cube.

The hat problem was solved for trees [12], cycles [8, 13, 14, 18], bipartite graphs [8], perfect graphs [8], and planar graphs containing a triangle [8]. Feige [8] conjectured that for any graph the hat number is equal to the hat number of its maximum clique. He proved this for graphs with clique number $2^k - 1$. Thus triangle-free graphs are the simplest remaining open case.

We consider the hat problem on directed graphs. Under an appropriate definition of the clique number for directed graphs, we provide a tight characterization of the range of possible values of the hat number, for every size of the maximum clique. We construct families of directed graphs with a fixed clique number the hat number of which is asymptotically optimal. We also determine the hat number of tournaments to be one half.

2 Preliminaries

For a graph G, the set of vertices and the set of edges we denote by $V(G)$ and $E(G)$, respectively. If H is a subgraph of G, then we write $H \subseteq G$. The degree of vertex v, that is, the number of its neighbors, we denote by $d_G(v)$.

Let $f\colon X \to Y$ be a function. If for every $x \in X$ we have $f(x) = y$, then we write $f \equiv y$.

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. By $Sc = \{1, 2\}$ we denote the set of colors, where 1 corresponds to blue and 2 corresponds to red.

By a case for a graph G we mean a function $c\colon V(G) \to \{1, 2\}$, where $c(v_i)$ means color of vertex v_i. The set of all cases for the graph G we denote by $C(G)$, of course $|C(G)| = 2^{|V(G)|}$.

By a situation of a vertex v_i we mean a function $s_i\colon V(G) \to Sc \cup \{0\} = \{0, 1, 2\}$, where $s_i(v_j) \in Sc$ if v_i and v_j are adjacent, and 0 otherwise. The set of all possible situations of v_i in the graph G we denote by $St_i(G)$, of course $|St_i(G)| = 2^{d_G(v_i)}$.

By a guessing instruction of a vertex $v_i \in V(G)$ we mean a function $g_i\colon St_i(G) \to Sc \cup \{0\} = \{0, 1, 2\}$, which for a given situation gives the color v_i guesses it is, or 0 if v_i passes. Thus a guessing instruction is a rule determining the behavior of a vertex in every situation. We say that v_i never guesses its color if v_i passes in every situation, that is, $g_i \equiv 0$. We say that v_i always guesses its color if v_i guesses its color in every situation, that is, for every $s_i \in St_i(G)$ we have $g_i(s_i) \in \{1, 2\}$ ($g_i(s_i) \neq 0$, equivalently).

Let c be a case, and let s_i be the situation (of vertex v_i) corresponding to that case. The guess of v_i in the case c is correct (wrong, respectively) if $g_i(s_i) = c(v_i)$ ($0 \neq g_i(s_i) \neq c(v_i)$, respectively). By result of the case c we mean a win if at least one vertex guesses its color correctly, and no vertex guesses its color wrong, that is, $g_i(s_i) = c(v_i)$ (for some i) and there is no j such that $0 \neq g_j(s_j) \neq c(v_j)$. Otherwise the result of the case c is a loss.

By a strategy for the graph G we mean a sequence $(g_1, g_2, \ldots, g_n)$, where g_i is the guessing instruction of vertex v_i. The family of all strategies for a graph G we denote by $\mathcal{F}(G)$.

If $S \in \mathcal{F}(G)$, then the set of cases for the graph G for which the team wins (loses, respectively) using the strategy S we denote by $W(S)$ ($L(S)$, respectively). The set of cases for which the team loses, and some vertex guesses its color we denote by $Ls(S)$. By the chance of success of the strategy S we mean the number $p(S) = |W(S)|/|C(G)|$. By the hat number of the graph G we mean the number $h(G) = \max\{p(S)\colon S \in \mathcal{F}(G)\}$. We say that a strategy S is optimal for the graph G if $p(S) = h(G)$. The family of all optimal strategies for the graph G we denote by $\mathcal{F}^0(G)$.

By solving the hat problem on a graph G we mean finding the number $h(G)$.

For a directed graph (digraph) D, the set of vertices and the set of arcs we denote by $V(D)$ and $A(D)$, respectively.

By the skeleton of a digraph D, denoted by skel(D), we mean the undirected graph on the vertex set V in which x and y are adjacent if both arcs between them belong to the set A, that is, if they form a directed 2-cycle in D.

By the clique number of a digraph D we mean the clique number of its skeleton;

that is, $\omega(D) = \omega(\mathrm{skel}(D))$.

The transpose of a digraph $D = (V, A)$ is the digraph $D^t = (V, A^t)$, where $A^t = \{(x, y) \colon (y, x) \in A\}$.

Slightly abusing notation, we identify a digraph D with its (undirected) skeleton in the case that $D = D^t$, that is, if all arcs of D have anti-parallel counterparts.

We can also consider the hat problem on directed graphs. If there is an arc from u to v, then the vertex u can see the vertex v. All concepts we define similarly as when considering the hat problem on undirected graphs treated for example in [8, 12]. We now cite four propositions that generalize to digraphs with little or no change.

Proposition 1 *For every two digraphs D and E such that $E \subseteq D$ we have $h(E) \leq h(D)$.*

Proposition 2 *For every digraph D we have $h(D) \geq 1/2$.*

Proposition 3 *Let D be a digraph and let v be a vertex of D. If $S \in \mathcal{F}^0(D)$ is a strategy such that v always guesses its color, then $h(D) = 1/2$.*

Proposition 4 *Let D be a digraph and let v be a vertex of D. If $S \in \mathcal{F}^0(D)$ is a strategy such that v never guesses its color, then $h(D) = h(D - v)$.*

We have the following corollary from Propositions 2, 3, and 4.

Proposition 5 *Let D be a digraph and let v be a vertex of D. If v has no outgoing arcs, then $h(D) = h(D - v)$.*

3 Constructions

For an undirected graph G, it is known that if G contains a triangle, then $h(G) \geq 3/4$, and in [8] it is conjectured that if G is triangle-free, then $h(G) = 1/2$. Do directed graphs introduce anything in between? The answer is yes.

Let us consider the hat problem on the digraph D_1 given in Figure 1.

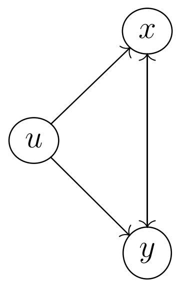


Figure 1: The digraph D_1

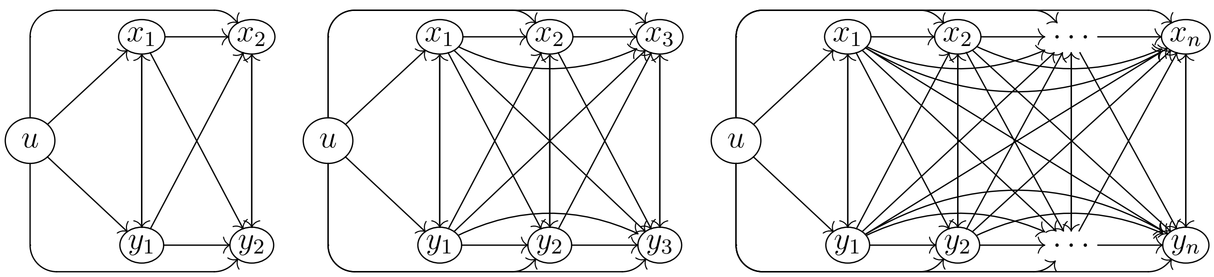
Fact 6 $h(D_1) = 5/8$.

We omit the proof of this fact in favor of extending D_1 to a construction of a family $\{D_n\}_{n=0}^{\infty}$ of semi-complete digraphs that asymptotically achieve the hat number $2/3$, with the property that $\omega(D_n) = 2$. The skeleton of D_n is a matching of size n plus an isolated vertex. For short, we write $\mathrm{skel}(D_n) = nK_2 \cup K_1$.

Definition 7 *Given two disjoint digraphs C and D, we define the directed union of C and D, denoted by $C \to D$, to be the union of these two digraphs with the additional arcs from all vertices of C to all vertices of D. Note that this operator is associative, that is, $C \to (D \to E) = (C \to D) \to E$, for any three digraphs C, D and E. Thus the notation $C \to D \to E$ is unambiguous. The directed union of n disjoint copies of a digraph D, that is $\underbrace{D \to D \to \ldots \to D}_{n}$, we denote by $D^{\to n}$.*

Expressed in the terms of the directed union, $D_1 = K_1 \to K_2$. We extend this to a family of digraphs by defining $D_n = K_1 \to K_2^{\to n}$. Note that the family $\{D_n\}_{n=0}^{\infty}$ satisfies the recurrence relation $D_{n+1} = D_n \to K_2$.

In Figure 2 we give examples of D_n for $n = 2$, $n = 3$, and a general n.



(a) $n = 2$ (b) $n = 3$ (c) General n

Figure 2: The directed, semi-complete graphs D_2, D_3, and D_n. All vertical arcs have anti-parallel counterparts. The remaining arcs are rightwards

We proceed to compute the hat number of the digraphs of the family $\{D_n\}_{n=0}^{\infty}$. First we prove an upper bound.

Lemma 8 *For every digraph D we have $h(D \to K_2) \leq \max\{h(D), 1/2 + h(D)/4\}$.*

Proof. Let S be an optimal strategy for $D \to K_2$. The vertices of the K_2 we denote by x and y. If one of them, say x, never guesses its color, then using Propositions 4 and 5 we get $h(D \to K_2) = h(D \to K_2 - x) = h(D \to K_1) = h(D)$. Now assume that each one of the vertices x and y guesses its color. If x or y always guesses its color, then by Proposition 3 we have $h(D \to K_2) = 1/2$. Now assume that neither x nor y always guesses its color. This implies that each one of them guesses its color in one of the two situations as every one of them has just one outgoing arc. Hence, with probability at least $1/4$ at least one of the vertices x and y is wrong. The chance of the success of the strategy S benefits from the behavior of the vertices of D only when both x and y pass, and this happens exactly with probability 1/4 since they see different vertices (that is, each other). This implies that $p(S) = 1/2 + h(D)/4$. Now we get $h(D \to K_2) = p(S) \leq \max\{h(D), 1/2 + h(D)/4\}$. ■

Now we prove a lower bound.

Lemma 9 *For every digraph D we have $h(D \to K_2) \geq 1/2 + h(D)/4$.*

Proof. Let S be an optimal strategy for the digraph D. The vertices of the K_2 we denote by x and y. Let S' be a strategy for $D \to K_2$ as follows. If y is blue, then x guesses it is also blue; otherwise it passes. If x is red, then y guesses it is also red; otherwise it passes. If x is blue and y is red, then the vertices of D behave as in the strategy S, otherwise they pass. Let us observe that if x and y have the same color, then the team wins. If x is red and y is blue, then the team loses. If x is blue and y is red, then the team wins with probability $p(S)$. Therefore $p(S') = 1/2 + p(S)/4 = 1/2 + h(D)/4$. Consequently, $h(D \to K_2) \geq p(S') = 1/2 + h(D)/4$. ■

Now we prove a lower bound for a more general setting.

Lemma 10 *For every positive integer m there exists $c \geq 1/2$ such that for any digraph D we have $h(D \to K_m) \geq cm/(m+1) + (1-c) \cdot h(D)$. Moreover, if $m = 2$, then $c = 3/4$ satisfies the inequality.*

Proof. Let S be an optimal strategy for the digraph D. The vertices of K_m we denote by $x_1, x_2, \ldots, x_m$. Let $C \subset \{\text{blue}, \text{red}\}^m$ be a code of distance 3, and consider the packing of stars $K_{1,m}$ in the hypercube graph H_m formed by selecting balls of radius one around each codeword. Let $\mathcal{A}$ mean the event that the case of $x_1, x_2, \ldots, x_m$ is covered by the packing. Now let S' be a strategy for $D \to K_m$ as follows. All vertices of D pass if $\mathcal{A}$ occurred, otherwise they behave according to S. The vertices of K_m behave as follows. If x_i is in a situation consistent with some codeword, then it guesses the color that disagrees with it; otherwise it passes. When $\mathcal{A}$ occurs, either m vertices guess their colors wrong, or exactly one vertex guesses its color and the guess is correct; then the team wins with probability $m/(m+1)$. Let $c = p(\mathcal{A})$. We get $p(S') = p(\mathcal{A}) \cdot m/(m+1) + (1 - p(\mathcal{A})) \cdot p(S)$ $= cm/(m+1) + (1-c) \cdot h(D)$. Now, the existence of codes of distance 3, length m, and size $\lceil 2^{m-1}/(m+1) \rceil$ implies that $c \geq 1/2$. ■

We use Lemmas 8 and 9 to calculate the hat number of D_n.

Proposition 11 *For every non-negative integer n we have*

$$h(D_n) = \frac{2}{3} - \frac{1}{6} \cdot \frac{1}{4^n}.$$

Proof. The result we prove by induction on the number n. For $n = 0$ the result is obviously true as D_0 is a single vertex and $h(D_0) = 1/2 = 2/3 - 1/6$. Let n be a positive integer, and assume that $h(D_{n-1}) = 2/3 - 4^{1-n}/6$. Since $h(D_{n-1}) < 2/3$, using Lemma 8 we get $h(D_n) \leq \max\{h(D_{n-1}), 1/2 + h(D_{n-1})/4\} = 1/2 + h(D_{n-1})/4$. The lower bound is matched by Lemma 9. ∎

Corollary 12 *For every $\varepsilon > 0$ there exists a digraph D satisfying $\omega(D) = 2$ such that $h(D) > 2/3 - \varepsilon$.*

The previous result can be generalized to an arbitrary clique number m.

Theorem 13 *For every $\varepsilon > 0$ there exists a digraph D satisfying $\omega(D) = m$ such that $h(D) > m/(m+1) - \varepsilon$.*

Proof. Let us consider $D = K_m^{\to n}$, where $n = \lceil \log_{1-c}(\varepsilon) \rceil$ and c is the appropriate constant from Lemma 10. By repeatedly applying the lemma we get $h(D) \geq (1-(1-c)^n) \cdot m/(m+1) \geq (1-\varepsilon) \cdot m/(m+1) > m/(m+1) - \varepsilon$. ∎

A natural question is whether $m/(m+1)$ is the best possible hat number of such digraphs. In the following section we show that indeed this is the best possible, i.e., the chance of success $m/(m+1)$ is asymptotically optimal for digraphs with the clique number m.

4 The upper bound

Feige [8] proved that for every undirected graph G we have $h(G) \leq \omega(G)/(\omega(G)+1)$. We repeat his proof, refining it a bit to show that the same holds for digraphs.

Proposition 14 *For every digraph D we have $h(D) \leq \omega(D)/(\omega(D)+1)$.*

Proof. Let S be an optimal strategy for D. We define a bipartite graph B whose left-hand side is $Ls(S)$, and the right-hand side is $W(S)$. A losing case $l \in Ls(S)$ is adjacent to a winning case $w \in W(S)$ if they differ only by one coordinate, which is the color of a vertex $v \in V(D)$ that guesses its color in these cases. Since v cannot see its own hat color, it acts the same in both hat cases l and w. Now let us examine the right and left degrees in B.

Right degree. Let $w \in W(S)$, and let $v \in V(D)$ be a vertex that guesses its color correctly in w. Let l be a case which differs from w only in the color of the vertex v. Since v does not distinguish between the cases w and l, it makes the same guess in l, but now it is incorrect. Therefore $l \in Ls(S)$ is a neighbor of w in B, and $d(w) \geq 1$.

Left degree. Let $l \in Ls(S)$, and let $w_1, w_2, \ldots, w_{d_G(l)} \in W(S)$ mean the neighbors of l in B. Let $v_i \in V(D)$ be the vertex whose color differs in the cases l and w_i, for every $i \in \{1, 2, \ldots, d\}$. Suppose that some arc $v_i \to v_j$ is not present in D. By the definition of v_i, it makes a correct guess at the case w_i. It cannot tell w_i apart from l, and thus it makes the same, now wrong, guess at the case l. But then it must make the same incorrect guess at the case w_j, which only differs from l by the color of v_j, unseen by v_i. This contradicts the fact that w_j is a winning case. Therefore $\{v_i\}_{i=1}^d$ is a clique in $\mathrm{skel}(D)$ and $d = d(l) \leq \omega(\mathrm{skel}(D)) = \omega(D)$.

We have shown that the right degree in B is at least one and the left degree in B is at most $\omega(D)$. This implies that $|W(S)| \leq |E(B)| \leq \omega(D)|Ls(S)|$, and consequently, $h(D) = p(S) = |W(S)|/2^{|V(D)|} \leq |W(S)|/(|W(S)| + |Ls(S)|) \leq \omega(D)/(\omega(D) + 1)$. ■

Observe that for a digraph D, the hat number $h(D)$ is always a rational number whose denominator is a power of two. Therefore $h(D) < \omega(D)/(\omega(D) + 1)$ unless $\omega(D) + 1$ is a power of two. When $\omega(D) + 1 = 2^k$ is a power of two, the upper bound is met by a complete graph K_{2^k-1} as $h(K_{2^k-1}) = (2^k - 1)/2^k$.

Corollary 15 *For every tournament T we have $h(T) = 1/2$.*

Proof. Apply Proposition 14 with $\omega(T) = 1$. The lower bound is by Proposition 2. ■

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2.8 A more colorful hat problem

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A more colorful hat problem

Abstract.

The topic is the hat problem in which each of n players is randomly fitted with a blue or red hat. Then everybody can try to guess simultaneously his own hat color by looking at the hat colors of the other players. The team wins if at least one player guesses his hat color correctly, and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of winning. We consider a generalized hat problem with $q \geq 2$ colors. We solve the problem with three players and three colors. Next we prove some upper bounds on the chance of success of any strategy for the generalized hat problem with n players and q colors. We also consider the numbers of strategies that suffice to be examined to solve the hat problem, or the generalized hat problem.

1. Introduction

In the hat problem, a team of n players enters a room and a blue or red hat is randomly placed on the head of each player. Each player can see the hats of all of the other players but not his own. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats, each player must simultaneously guess the color of his own hat or pass. The team wins if at least one player guesses his hat color correctly and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of winning.

The hat problem with seven players, called the “seven prisoners puzzle”, was formulated by T. Ebert in his Ph.D. Thesis [13]. The hat problem was also the subject of articles in The New York Times [22], Die Zeit [7], and abcNews [21]. It is also one of the Berkeley Riddles [5].

The hat problem with $2^k - 1$ players was solved in [15], and for 2^k players in [12]. The problem with n players was investigated in [8]. The hat problem and Hamming codes were the subject of [9]. The generalized hat problem with n people and q colors was investigated in [20].

There are known many variations of the hat problem. For example in the papers [1, 11, 19] there was considered a variation in which passing is not allowed, thus

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everybody has to guess his hat color. The aim is to maximize the number of correct guesses. The authors of [17] investigated several variations of the hat problem in which the aim is to design a strategy guaranteeing a desired number of correct guesses. In [18] there was considered a variation in which the probabilities of getting hats of each colors do not have to be equal. The authors of [3] investigated a problem similar to the hat problem, in that paper there are n players which have random bits on foreheads, and they have to vote on the parity of the n bits.

The hat problem and its variations have many applications and connections to different areas of science, for example: information technology [6], linear programming [17], genetic programming [10], economics [1, 19], biology [18], approximating Boolean functions [3], and autoreducibility of random sequences [4, 13–16].

In this paper we consider a generalized hat problem with $q \geq 2$ colors which was first investigated in [20]. Every player has got a hat of one from q possible colors, and the probabilities of getting hats of all colors are equal. We solve the problem with three players and three colors. Next we prove some upper bounds on the chance of success of any strategy for the generalized hat problem with n players and q colors. We also consider the numbers of strategies that suffice to be examined to solve the hat problem, or the generalized hat problem.

2. Preliminaries

First, let us observe that we can confine to deterministic strategies (that is, strategies such that the decision of each player is determined uniquely by the hat colors of the other players). We can do this since for any randomized (not deterministic) strategy there exists a not worse deterministic one. It is true, because every randomized strategy is a convex combination of some deterministic strategies. The probability of winning is a linear function on the convex polyhedron corresponding to the set of all randomized strategies which can be achieved combining those deterministic strategies. It is well known that this function achieves its maximum on a vertex of the polyhedron which corresponds to a deterministic strategy.

Let $\{v_1, v_2, \dots, v_n\}$ mean a set of players. By $Sc = \{1, 2, \dots, q\}$ we denote the set of colors.

By a case for the hat problem with n players and q colors we mean a function $c\colon \{v_1, v_2, \dots, v_n\} \to \{1, 2, \dots, q\}$, where $c(v_i)$ means the hat color of player v_i. The set of all cases for the hat problem with n players and q colors we denote by $C(n, q)$, of course $|C(n, q)| = q^n$. If $c \in C(n, q)$, then to simplify notation, we write $c = c(v_1)c(v_2)\dots c(v_n)$ instead of $c = \{(v_1, c(v_1)), (v_2, c(v_2)), \dots, (v_n, c(v_n))\}$. For example, if a case $c \in C(4, 3)$ is such that $c(v_1) = 2$, $c(v_2) = 3$, $c(v_3) = 1$, and $c(v_4) = 2$, then we write $c = 2312$.

By a situation of a player v_i we mean a function $s_i\colon \{v_1, v_2, \dots, v_n\} \to Sc \cup \{0\}$, where $s_i(v_j) \in Sc$ if $i \neq j$, while $s_i(v_i) = 0$. The set of all possible situations of v_i in the hat problem with n players and q colors we denote by $St_i(n, q)$, of course $|St_i(n, q)| = q^{n-1}$. If $s_i \in St_i(n, q)$, then for simplicity of notation, we write $s_i = s_i(v_1)s_i(v_2)\dots s_i(v_n)$ instead of $s_i = \{(v_1, s_i(v_1)), (v_2, s_i(v_2)), \dots, (v_n, s_i(v_n))\}$. For example, if $s_2 \in St_2(4, 3)$ is such that $s_2(v_1) = 3$, $s_2(v_3) = 4$, and $s_2(v_4) = 2$, then we write $s_2 = 3042$.

We say that a case c corresponds to a situation s_i of player v_i if $c(v_j) = s_i(v_j)$, for every $j \neq i$. This implies that a case corresponds to a situation of v_i if every player excluding v_i in the case has a hat of the same color as in the situation. Of course, to every situation correspond exactly q cases.

By a guessing instruction of a player v_i we mean a function $g_i\colon St_i(n,q) \to Sc \cup \{*\}$, which for a given situation gives the color v_i guesses his hat is if $g_i(s_i) \neq *$, otherwise v_i passes. Thus a guessing instruction is a rule determining the behavior of a player in every situation.

Let c be a case, and let s_i be the situation (of player v_i) corresponding to this case. The guess of v_i in the case c is correct (wrong, respectively) if $g_i(s_i) = c(v_i)$ ($* \neq g_i(s_i) \neq c(v_i)$, respectively). By result of the case c we mean a win if at least one player guesses his hat color correctly, and no player guesses his hat color wrong, that is, $g_i(s_i) = c(v_i)$ (for some i) and there is no j such that $* \neq g_j(s_j) \neq c(v_j)$. Otherwise the result of the case c is a loss.

By a strategy we mean a sequence $(g_1, g_2, \ldots, g_n)$, where g_i is the guessing instruction of player v_i. The family of all strategies for the hat problem with n players and q colors we denote by $\mathcal{F}(n,q)$.

If $S \in \mathcal{F}(n,q)$, then the set of cases for which the team wins using the strategy S we denote by $W(S)$. Consequently, by the chance of success of the strategy S we mean the number $p(S) = \frac{|W(S)|}{|C(n,q)|}$. We define $h(n,q) = \max\{p(S) : S \in \mathcal{F}(n,q)\}$. We say that a strategy S is optimal for the hat problem with n players and q colors if $p(S) = h(n,q)$.

By solving the hat problem with n players and q colors we mean finding the number $h(n,q)$.

3. Hat problem with three players and three colors

In this section we solve the hat problem with three players and three colors.

We say that a strategy is symmetric if every player makes his decision on the basis of only numbers of hats of each color seen by him, and all players behave in the same way. A strategy is nonsymmetric if it is not symmetric.

The authors of [18] solved the hat problem with three players and three colors by giving a symmetric strategy found by computer, and proving that it is optimal. We solve this problem by proving the optimality of a nonsymmetric strategy found without using computer.

Let us consider the following strategy for the hat problem with three players and three colors.

Strategy 1

Let $S = (g_1, g_2, g_3) \in \mathcal{F}(3,3)$ be the strategy as follows:

$$g_1(s_1) = \begin{cases} s_1(v_3), & \text{if } s_1(v_2) \neq s_1(v_3), \\ *, & \text{otherwise;} \end{cases}$$

$$g_2(s_2) = \begin{cases} s_2(v_3), & \text{if } s_2(v_1) \neq s_2(v_3), \\ *, & \text{otherwise;} \end{cases}$$

$$g_3(s_3) = \begin{cases} s_3(v_1), & \text{if } s_3(v_1) = s_3(v_2), \\ *, & \text{otherwise.} \end{cases}$$

It means that players proceed as follows.

- **The player $\boldsymbol{v_1}$.** If v_2 and v_3 have hats of different colors, then he guesses he has a hat of the color v_3 has, otherwise he passes.
- **The player $\boldsymbol{v_2}$.** If v_1 and v_3 have hats of different colors, then he guesses he has a hat of the color v_3 has, otherwise he passes.
- **The player $\boldsymbol{v_3}$.** If v_1 and v_2 have hats of the same color, then he guesses he has a hat of the color they have, otherwise he passes.

All cases we present in table, where the symbol + means correct guess (success), − means wrong guess (loss), and blank square means passing.

No	The color of the hat of			The guess of			Result
	v_1	v_2	v_3	v_1	v_2	v_3	
1	1	1	1			+	+
2	1	1	2	−	−	−	−
3	1	1	3	−	−	−	−
4	1	2	1	+			+
5	1	2	2		+		+
6	1	2	3	−	−		−
7	1	3	1	+			+
8	1	3	2	−	−		−
9	1	3	3		+		+
10	2	1	1		+		+
11	2	1	2	+			+
12	2	1	3	−	−		−
13	2	2	1	−	−	−	−
14	2	2	2			+	+
15	2	2	3	−	−	−	−
16	2	3	1	−	−		−
17	2	3	2	+			+
18	2	3	3		+		+
19	3	1	1		+		+
20	3	1	2	−	−		−
21	3	1	3	+			+
22	3	2	1	−	−		−
23	3	2	2		+		+
24	3	2	3	+			+
25	3	3	1	−	−	−	−
26	3	3	2	−	−	−	−
27	3	3	3			+	+

For example, in the first case the player v_1 sees two hats of the same color, so he passes. By the same reason the player v_2 also passes. The player v_3 sees two hats of the first color, so he guesses he has a hat of the first color. Since v_3 has a hat of the first color, the guess is correct, and the result of the case is a win.

In the second case the player v_1 sees two hats of different colors, so he guesses he has a hat of the color v_3 has. Since v_1 and v_3 have hats of different colors, the guess is wrong, and the result of the case is a loss. Additionally, the player v_2 guesses his hat color wrong by the same reason as v_1. Moreover, the guess of v_3 is also wrong. The player v_3 sees two hats of the first color, so he guesses he has a hat of the first color. The guess is wrong, as v_3 has a hat of the second color.

In the fourth case the player v_1 sees two hats of different colors, so he guesses he has a hat of the color v_3 has. Since v_1 and v_3 have hats of the same color, the guess is correct. The player v_2 sees two hats of the same color, so he passes. The player v_3 sees two hats of different colors, so he passes. This implies that the result of the case is a win.

In the sixth case the player v_1 sees two hats of different colors, so he guesses he has a hat of the color v_3 has. Since v_1 and v_3 have hats of different colors, the guess is wrong, and the result of the case is a loss. Additionally, the player v_2 guesses his hat color wrong by reasons similar as v_1. The player v_3 passes, as he sees two hats of different colors.

Counting the plusses in the last column, we get the following observation.

Observation 2
Using Strategy 1 the team wins for 15 of 27 cases.

Now, we solve the hat problem with three players and three colors.

Fact 3
$h(3,3) = \frac{5}{9}$.

Proof. Since using Strategy 1 the team wins for 15 of 27 cases, we have $h(3,3) \geq \frac{15}{27} = \frac{5}{9}$. Suppose that $h(3,3) > \frac{5}{9}$, that is, there exists a strategy such that the team wins for more than 15 cases. Let S be any strategy for the hat problem with three players and three colors. Any guess made by any player in any situation is wrong in exactly two cases, because to any situation of any player correspond three cases, and in exactly two of them this player has a hat of a color different than the one he guesses. In the strategy S every player guesses his hat color in at most 5 situations, because if some player guesses his hat color in at least 6 situations, then the team loses for at least 12 cases, and wins for at most 15 cases, a contradiction. Any guess made by any player in any situation is correct in exactly one case, because to any situation of any player correspond three cases, and in exactly one of them this player has a hat of the color he guesses. There are three players, every one of them guesses his hat color in at most five cases, and every guess is correct in exactly one case. Therefore using the strategy S the team wins for at most 15 cases, a contradiction.

4. Hat problem with n players and q colors

Now we consider the generalized hat problem with n players and q colors. Noga Alon [2] has proven that for this problem there exists a strategy such that the chance of success is greater than or equal to

$$1 - \frac{1+(q-1)\log n}{n} - \Big(1 - \frac{1}{q}\Big)^n.$$

First we prove an upper bound on the number of cases for which the team wins using any strategy for the problem.

Theorem 4
If S is a strategy for the hat problem with n players and q colors, then

$$|W(S)| \leq n\left\lfloor \frac{q^n - |W(S)|}{q-1} \right\rfloor.$$

Proof. Any guess made by any player in any situation is wrong in exactly $q-1$ cases, because to any situation of any player correspond q cases, and in exactly $q-1$ of them this player has a hat of a color different than the one he guesses. Let us consider any player. The number of situations in which he guesses his hat color in the strategy S cannot be neither greater than nor equal to

$$\left\lfloor \frac{q^n - |W(S)|}{q-1} \right\rfloor + 1,$$

otherwise the number of cases in which he guesses his hat color wrong is greater than or equal to

$$(q-1)\left(\left\lfloor \frac{q^n - |W(S)|}{q-1} \right\rfloor + 1\right).$$

It is more than

$$(q-1)\left(\frac{q^n - |W(S)|}{q-1}\right) = q^n - |W(S)|.$$

This implies that the team loses for more than $q^n - |W(S)|$ cases, and therefore the number of cases for which the team wins is less than

$$|C(n,q)| - (q^n - |W(S)|) = q^n - q^n + |W(S)| = |W(S)|.$$

This is a contradiction, as $|W(S)|$ is the number of cases for which the team wins. Any guess made by any player in any situation is correct in exactly one case, because to any situation of any player correspond q cases, and in exactly one of them this player has a hat of the color he guesses. This implies that the number of cases for which the team wins using the strategy S is at most

$$n\left\lfloor \frac{q^n - |W(S)|}{q-1} \right\rfloor.$$

Now we give an equivalent upper bound on the chance of success of any strategy for the hat problem with n players and q colors, which is easy to prove.

Theorem 5
Let S be any strategy for the hat problem with n players and q colors. Then

$$p(S) \le \frac{n}{q^n}\left\lfloor \frac{q^n - q^n \cdot p(S)}{q-1} \right\rfloor.$$

Now we see that Fact 3 follows from Theorem 4, as well as from Theorem 5. We show that it follows from Theorem 4.

Proof of Fact 3. Since using Strategy 1 the team wins for 15 of 27 cases, by definition we get $h(3,3) \ge p(S) = \frac{15}{27} = \frac{5}{9}$. Now we prove that $h(3,3) \le \frac{5}{9}$. Let S be an optimal strategy for the hat problem with three players and three colors. By Theorem 4 we have

$$|W(S)| \le 3\left\lfloor \frac{27 - |W(S)|}{2} \right\rfloor.$$

This implies that

$$|W(S)| \le 3 \cdot \frac{27 - |W(S)|}{2} = 40.5 - \frac{3|W(S)|}{2}.$$

Now we easily get $|W(S)| \le \frac{81}{5} = 16.2$. Since $|W(S)|$ is an integer, we have $|W(S)| \le 16$. If $|W(S)| = 16$, then $16 \le 3\lfloor \frac{27-16}{2} \rfloor = 3 \cdot 5 = 15$, a contradiction. This implies that $|W(S)| \le 15$. Since $|C(3,3)| = 27$, we get $p(S) \le \frac{15}{27} = \frac{5}{9}$. Since S is an optimal strategy for the hat problem with three players and three colors, by definition we get $h(3,3) = p(S) \le \frac{5}{9}$.

The next result proven in [20, Proposition 3] is a corollary from Theorem 4 or 5.

Corollary 6 ([20, Proposition 3])
If S is a strategy for the hat problem with n players and q colors, then

$$p(S) \le \frac{n}{n+q-1}.$$

Proof. By Theorem 4 we have

$$|W(S)| \le n\left\lfloor \frac{q^n - |W(S)|}{q-1} \right\rfloor.$$

This implies that

$$|W(S)| \le n \cdot \frac{q^n - |W(S)|}{q-1} = \frac{nq^n}{q-1} - |W(S)|\left(\frac{n}{q-1}\right).$$

Consequently,

$$|W(S)|\Big(1+\frac{n}{q-1}\Big) \le \frac{nq^n}{q-1} \iff |W(S)| \le \frac{q-1}{n+q-1}\cdot\frac{nq^n}{q-1}$$
$$\iff p(S)=\frac{|W(S)|}{q^n} \le \frac{n}{n+q-1}.$$

Now we show that the previous corollary is weaker than Theorem 4, that is, Theorem 4 does not follow from Corollary 6. Let S be any strategy for the hat problem with three players and three colors. By Theorem 4 we have $|W(S)| \le 15$ (it is shown in the proof of Fact 3 using Theorem 4). Thus

$$p(S)=\frac{|W(S)|}{|C(3,3)|} \le \frac{15}{3^3}=\frac{5}{9}.$$

By Corollary 6 we get

$$p(S) \le \frac{n}{n+q-1}=\frac{3}{5}.$$

Since $\frac{3}{5} > \frac{5}{9}$, Corollary 6 is weaker than Theorem 4.

Now let us consider the hat problem with two colors ($q=2$), and any strategy S for this problem. By Corollary 6 we get the upper bound

$$p(S) \le \frac{n}{n+1}$$

previously given in [15], which is sharp for $n=2^k-1$, where k is a positive integer.

5. Number of strategies that suffice to be examined

In this section we consider the number of strategies the examination of which suffices to solve the hat problem, and the generalized hat problem with q colors.

First, we count all possible strategies for the hat problem. We have n players, there are 2^{n-1} possible situations of each one of them, and in each situation there are three possibilities of behavior (to guess the first color, to guess the second color, or to pass). This implies that the number of possible strategies is equal to

$$\left(3^{2^{n-1}}\right)^n.$$

Now we prove that it is not necessary to examine every strategy to solve the problem.

Fact 7

To solve the hat problem with n players, it suffices to examine

$$\left(3^{2^{n-1}-2}\right)^n=\left(3^{2^{n-1}}\right)^n\cdot\frac{1}{9^n}$$

strategies.

Proof. Let S be an optimal strategy for the hat problem with n players. If in this strategy no player guesses his hat color, then obviously $p(S) = 0$. This is a contradiction to the optimality of S. Thus in the strategy S some player guesses his hat color. Without loss of generality we assume that this player is v_1, and he guesses his hat color in the situation $011\ldots1$. Additionally, without loss of generality we assume that in this situation he guesses he has a hat of the second color. This guess is wrong in the case $11\ldots1$, causing the loss of the team. Thus the result of this case cannot be made worse. If some player other than v_1, say v_i, guesses he has the second color when he sees only hats of the first color, then his guess is wrong in the case $11\ldots1$, and is correct in the case when v_i has the second color and all the remaining vertices have the first color. Since it cannot make worse the chance of success, we may assume that every player excluding v_i guesses he has a hat of the second color when he sees hats only of the first color. Assume that some player, say v_i, guesses his hat color when he sees one hat of the second color and $n-2$ hats of the first color. If in this situation he guesses he has a hat of the first color, then in the case corresponding to that situation, and in which he has a hat of the first color, his guess is correct, as well as the guess of the player who has a hat of the second color. Since it cannot improve the chance of success, we may assume that in this situation v_i does not guess he has a hat of the first color. If in that situation he guesses he has a hat of the second color, then in the case corresponding to that situation, and in which he has a hat of the first color, his guess is wrong, while at the same time the guess of the player who has a hat of the second color is correct. Since it makes the guess of this player pointless, we may assume that in that situation v_i does not guess he has a hat of the second color. This implies that we may assume that every player who sees one hat of the second color and $n-2$ hats of the first color, passes. Now we conclude that for each player we can assume his behavior in two situations. This implies that for each player there are two situations less to consider. In this way we get the desired number.

Now, we count all possible strategies for the generalized hat problem with q colors. We have n players, there are q^{n-1} possible situations of each one of them, and in each situation there are $q+1$ possibilities of behavior (to guess one of the q colors, or to pass). This implies that the number of possible strategies is equal to

$$\left((q+1)^{q^{n-1}}\right)^n.$$

Now we prove that it is not necessary to examine every strategy to solve the problem.

Fact 8

To solve the hat problem with n players and q colors, it suffices to examine

$$\left((q+1)^{q^{n-1}-1}\right)^n = \left((q+1)^{q^{n-1}}\right)^n \cdot \frac{1}{(q+1)^n}$$

strategies.

Proof. Let S be an optimal strategy for the hat problem with n players and q colors. If in this strategy no player guesses his hat color, then obviously $p(S) = 0$. This is a contradiction to the optimality of S. Thus in the strategy S some player guesses his hat color. Without loss of generality we assume that this player is v_1, and he guesses his hat color in the situation $011\dots1$. Additionally, without loss of generality we assume that in this situation he guesses he has a hat of the second color. Let v_i be any player other than v_1. If in this situation v_i guesses he has a hat of the first color, then in the case corresponding to that situation, and in which he has a hat of the first color, his guess is correct, as well as the guess of v_1. Since it cannot improve the chance of success, we may assume that in this situation v_i does not guess he has a hat of the first color. If in that situation v_i guesses he has a hat of any color other than the first, then in the case corresponding to that situation, and in which he has a hat of the first color, his guess is wrong, while at the same time the guess of v_1 is correct. Since it makes the guess of v_1 pointless, we may assume that in that situation v_i does not guess any color other that the first. This implies that we may assume that every player other than v_1 in the situation in which v_1 has a hat of the second color, and all the remaining players have hats of the first color, passes. Now we conclude that for each player we can assume his behavior in one situation. This implies that for each player there is one situation less to consider. In this way we get the desired number.

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2.9 A modified hat problem

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A modified hat problem

Abstract. The topic of our paper is the hat problem in which each of n players is randomly fitted with a blue or red hat. Then everybody can try to guess simultaneously his own hat color by looking at the hat colors of the other players. The team wins if at least one player guesses his hat color correctly, and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of a win. There are known many variations of the hat problem. In this paper we consider a variation in which there are $n \geqslant 3$ players, and blue and red hats. Players do not have to guess their hat colors simultaneously. In this variation of the hat problem players guess their hat colors by coming to the basket and throwing the proper card into it. Every player has got two cards with his name and the sentence "I have got a red hat" or "I have got a blue hat". If someone wants to resign from answering, then he does not do anything. The team wins if at least one player guesses his hat color correctly, and no one guesses his hat color wrong; otherwise the team loses. Is there a strategy such that the team always succeeds? We give an optimal strategy for the problem which always succeeds. Additionally, we prove in which step the team wins using the strategy. We also prove what is the greatest possible number of steps that are needed for the team to win using the strategy.

2000 Mathematics Subject Classification: 91A12.

Key words and phrases: hat problem, variation of hat problem.

1. Introduction. In the hat problem, a team of n players enters a room and a blue or red hat is randomly placed on the head of each player. Each player can see the hats of all of the other players but not his own. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats, each player must simultaneously guess the color of his own hat or pass. The team wins if at least one player guesses his hat color correctly and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of a win.

The hat problem with seven players, called the "seven prisoners puzzle", was formulated by T. Ebert in his Ph.D. Thesis [12]. The hat problem was also the subject of articles in The New York Times [21], Die Zeit [6], and abcNews [20]. It is also a one of subjects of the webpage [4].

The hat problem with 2^k-1 players was solved in [14], and for 2^k players in [11]. The problem with n players was investigated in [7]. The hat problem and Hamming codes were the subject of [8].

There are known many variations of the hat problem. For example the generalized hat problem with n people and q colors was investigated in [19]. In the papers [1, 10, 18] there was considered a variation in which passing is not allowed, thus everybody has to guess his hat color. The aim is to maximize the number of correct guesses. The authors of [16] investigated several variations of the hat problem in which the aim is to design a strategy guaranteeing desired number of correct guesses. In [17] there was considered a variation in which the probabilities of getting hats of each colors do not have to be equal. The authors of [2] investigated a problem similar to the hat problem. There are n players which have random bits on foreheads, and they have to vote on the parity of the n bits.

The hat problem and its variations have many applications and connections to different areas of science, for example: information technology [5], linear programming [16], genetic programming [9], economy [1, 18], biology [17], approximating Boolean functions [2], and autoreducibility of random sequences [3, 12–15].

In this paper we consider a variation in which there are $n \geqslant 3$ players, and blue and red hats. Players do not have to guess their hat colors simultaneously. In this variation of the hat problem players guess their hat colors by coming to the basket and throwing the proper card into it. Every player has got two cards with his name and the sentence "I have got a red hat" or "I have got a blue hat". If someone wants to resign from answering, then he does not do anything. The team wins if at least one player guesses his hat color correctly, and no one guesses his hat color wrong; otherwise the team loses. Is there a strategy such that the team always succeeds? We give an optimal strategy for the problem which always succeeds. Additionally, we prove in which step the team wins using the strategy. We also prove what is the greatest possible number of steps that are needed for the team to win using the strategy.

2. Modified hat problem. Let us consider a modified hat problem which we define as follows. There are $n \geqslant 3$ players and two colors (red and blue) in which players do not have to guess their hat colors simultaneously. Players guess their hat colors by coming to the basket and throwing the proper card into it. Every player has got two cards with his name and the sentence "I have got a red hat" or "I have got a blue hat". If someone wants to resign from answering, then he does not do anything. The team wins if at least one player guesses his hat color correctly, and no one guesses his hat color wrong; otherwise the team loses. Is there a strategy such that everybody wins?

We give an optimal strategy for the problem which always succeeds. Additionally, we prove in which step the team wins using the strategy. We also prove what is the greatest possible number of steps that are needed for the team to win using the strategy.

Let us consider the following strategy for the modified hat problem.

STRATEGY 1 Players proceed as follows.

Step 1 (one minute after the beginning)
Only these players who see the hats of one color only come to the basket. There are three possibilities:

- Only one player comes to the basket. Then he guesses he has a hat of the color differing from the one he sees.
- More than one player come to the basket. Then every one of them guesses he has a hat of the color which he sees.
- No player comes to the basket. Then we execute Step 2.

Let i be a positive integer.

Step $2i$ ($2i$ minutes after the beginning)
Only these players who see exactly i blue hats come to the basket. There are two possibilities:

- At least one player comes to the basket. Then every one of them guesses he has a blue hat.
- No player comes to the basket. Then we execute Step $2i+1$.

Step $2i+1$ ($(2i+1)$ minutes after the beginning)
Only these players who see exactly i red hats come to the basket. There are two possibilities:

- At least one player comes to the basket. Then every one of them guesses he has a red hat.
- No player comes to the basket. Then we execute Step $2i+2$.

Now we prove that this strategy always succeeds.

THEOREM 2.1 *Strategy 1 always succeeds for the modified hat problem.*

PROOF If all players have hats of the same color, then in Step 1 every player guesses his hat color correctly. Thus the team wins. If one player has a hat of some color, while the remaining $n-1$ players have hats of another color, then in Step 1 only the player which has a hat of the unique color guesses its color, and the guess is correct. Therefore the team wins.

If there are $n=3$ players, then the team wins in Step 1, as there is no other possibility.

Now assume that there are $n \geqslant 4$ players, and at least two of them have red hats and at least two of them have blue hats. Let i be a positive integer. Now we prove that if i is odd (even, respectively), then if in the executed Step i no player comes to the basket, then every player sees at least $(i+1)/2$ red hats ($(i/2+1)$ blue hats, respectively). We prove that by induction. First, assume that $i = 1$. Since no player sees hats only of the one color (as no player comes to the basket in Step 1), every player sees at least one hat of each color. Now assume that $i = 2$. Since every player sees at least one blue hat (as no player has come to the basket in Step 1) and no player sees exactly one blue hat (as no player comes to the basket in Step 2), it follows that every player sees at least two blue hats. Let k be a positive integer. Assume that if no player comes to the basket in Step $2k - 1$, then every player sees at least k red hats, and if no player comes to the basket in Step $2k$, then every player sees at least $k + 1$ blue hats. First, we prove that if in the executed Step $2k + 1$ no player comes to the basket, then every player sees at least $k + 1$ red hats. Since every player sees at least k red hats (as no player has come to the basket in Step $2k - 1$) and no player sees exactly k red hats (as no player comes to the basket in Step $2k + 1$), it follows that every player sees at least $k + 1$ red hats. Now, we prove that if in the executed Step $2k + 2$ no player comes to the basket, then every player sees at least $k + 2$ blue hats. Since every player sees at least $k + 1$ blue hats (as no player has come to the basket in Step $2k$) and no player sees exactly $k + 1$ blue hats (as no player comes to the basket in Step $2k + 2$), it follows that every player sees at least $k + 2$ blue hats.

Now we prove that if some player guesses his hat color in any Step i, then his guess is correct. First assume that $i = 2$. Since every player sees at least one blue hat (as no player has come to the basket in Step 1), there are at least two blue hats (as particularly the player who has a blue hat also sees at least one blue hat). Some player comes to the basket in Step 2, thus he sees exactly one blue hat. This implies that he has a blue hat, and therefore his guess that he has a blue hat is correct. Now assume that $i \geqslant 4$ is an even integer, that is $i = 2k + 2$, where k is a positive integer. First, assume that some player comes to the basket in the executed Step $2k + 2$. No player has come to the basket in Steps $2k$ and $2k + 1$, thus every player sees at least $k + 1$ blue hats and at least $k + 1$ red hats. Since every player who has a blue hat sees at least $k + 1$ blue hats, there are at least $k + 2$ blue hats. The person who comes to the basket in Step $2k + 2$ sees exactly $k + 1$ blue hats. This implies that he has a blue hat, and therefore his guess that he has a blue hat is correct. Now, assume that some player comes to the basket in the executed Step $2k + 1$. Since every player sees at least k red hats (as no player has come to the basket in Step $2k - 1$), there are at least $k + 1$ red hats. The person who comes to the basket in Step $2k + 1$ sees exactly k red hats. This implies that he has a red hat, and therefore his guess that he has a red hat is correct. ∎

Let us consider the numbers of red and blue hats on the heads of players. If there are less red hats than blue hats, then let x mean the number of red hats. Otherwise let it mean the number of blue hats.

Now we prove in which step the team wins using Strategy 1.

FACT 2.2 *If there are less than two blue hats or less than two red hats, then Strategy 1 succeeds in Step* 1*. Otherwise, for x defined above, Strategy 1 succeeds in Step* $2x - 1$ *if there are more blue hats than red hats, and otherwise in Step* $2x - 2$.

PROOF If there are less than two blue hats or less than two red hats, then from the proof of Theorem 2.1 we know that the team wins in Step 1. Now assume that there are at least two hats of each color. First, let us assume that there are more blue hats than red hats. Thus there are exactly x red hats and more than x blue hats. From the proof of Theorem 2.1 we know that some player having a red (blue, respectively) hat would guess his hat color correctly in Step $2x-1$ ($2x$ or further, respectively). This implies that the team wins in Step $2x-1$. Now, assume that the number of blue hats is smaller than or equal to the number of red hats. Thus there are exactly x blue hats and at least x red hats. From the proof of Theorem 2.1 we know that some player having a blue (red, respectively) hat would guess his hat color correctly in Step $2x-2$ ($2x-1$ or further, respectively). This implies that the team wins in Step $2x-2$. ■

Now we prove what is the greatest possible number of steps that are needed for the team to win using Strategy 1.

COROLLARY 2.3 *The greatest possible number of step in which the team wins using Strategy 1 is $n-2$.*

PROOF Let n mean the number of players. From Fact 2.2 we know that if there are less than two blue hats or less than two red hats, then the team wins in Step 1. Since $n \geqslant 3$, we have $n-2 \geqslant 1$. Now assume that there are at least two blue hats and at least two red hats. If there are more blue hats than red hats, then obviously $x < n/2$, that is, $x \leqslant (n-1)/2$. By Fact 2.2, the team wins in Step $2x-1$. We have $2x-1 \leqslant n-1-1 = n-2$. Now assume that the number of blue hats is smaller than or equal to the number of red hats. Obviously, $x \leqslant n/2$. By Fact 2.2, the team wins in Step $2x-2$. We have $2x-2 \leqslant n-2$. ■

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2.10 On the hat problem, its variations, and their applications

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On the hat problem, its variations, and their applications

Abstract. The topic of our paper is the hat problem in which each of n players is randomly fitted with a blue or red hat. Then everybody can try to guess simultaneously his own hat color by looking at the hat colors of the other players. The team wins if at least one player guesses his hat color correctly, and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of a win. There are known many variations of the hat problem. In this paper we give a comprehensive list of variations considered in the literature. We describe the applications of the hat problem and its variations, and their connections to different areas of science. We give the full bibliography of any papers, books, and electronic publications about the hat problem.

1. Introduction

In the hat problem, a team of n players enters a room and a blue or red hat is randomly placed on the head of each player. Each player can see the hats of all of the other players but not his own. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats, each player must simultaneously guess the color of his own hat or pass. The team wins if at least one player guesses his hat color correctly and no one guesses his hat color wrong; otherwise the team loses. The aim is to maximize the probability of a win.

The hat problem with seven players, called the "seven prisoners puzzle", was formulated by T. Ebert in his Ph.D. Thesis [20]. The hat problem was also the subject of articles in The New York Times [46], Die Zeit [9], and abcNews [44]. It is also a one of subjects of the webpage [7].

The hat problem with $2^k - 1$ players was solved in [22], and for 2^k players in [17]. The problem with n players was investigated in [11]. The hat problem and Hamming codes were the subject of [12].

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There are known many variations of the hat problem. For example the generalized hat problem with n players and q colors was investigated in [40]. In the papers [1, 15, 35] there was considered a variation in which passing is not allowed, thus everybody has to guess his hat color. The aim is to maximize the number of correct guesses. The authors of [25] investigated several variations of the hat problem in which the aim is to design a strategy guaranteeing desired number of correct guesses. In [30] there was considered a variation in which the probabilities of getting hats of each colors do not have to be equal. The authors of [5] investigated a problem similar to the hat problem. There are n players which have random bits on foreheads, and they have to vote on the parity of the n bits. The hat problem on a graph is as follows. There is a graph, where vertices correspond to players and a player can see each player to whom he is connected by an edge. This variation of the hat problem was first considered in [38]. There were proven some general theorems about the hat problem on a graph, and the problem was solved on trees. Additionally, there was considered the hat problem on a graph such that the only known information are degrees of vertices. In [39] the problem was solved on the cycle C_4. Further results about the hat problem on a graph were established by Uriel Feige [24]. For example, there the problem was solved for bipartite graphs, and planar graphs containing a triangle. Based on these and some other results, the author conjectured that for every graph there is an optimal strategy in which all vertices which do not belong to the maximum clique always pass.

The hat problem and its variations have many applications and connections to different areas of science, for example: information technology [8], linear programming [25], genetic programming [14], economy [1, 35], biology [30], approximating Boolean functions [5], and autoreducibility of random sequences [6, 20–23].

In this paper we give a comprehensive list of variations of the hat problem considered in the literature. We also present what is already known about each variation. For some variations we give a strategy which solves the problem. Next we describe the applications of the hat problem and its variations, and their connections to different areas of science. We give the full bibliography of any papers, books, and electronic publications about the hat problem.

2. Applications of the hat problem

In this section we present applications of the hat problem and its variations. We also consider their connections to different areas of science.

Information technology. The paper [8] shows the strong connection between the hat problem and the following problem. In storing or transmitting digital data, there is always some risk of distortion: a 0 might accidentally flip to 1 or vice versa. One way to deal with this problem is to introduce some redundancy into the transmission – for instance, by sending each bit multiple times. However, transmitting too many extra bits is costly and ineffective. We need to protect k bits of data against the possibility of an error by using the minimal number of additional "check bits". Note that the method must not only detect the error, but also determine its precise location, so that the user can recover the original message every time. This problem has been solved using Hamming codes – codes which detect and correct errors. So called covering codes are strongly related to Hamming codes. The website [41] contains up-to-date data on the best known covering codes. The coding theory (for more information, see [47]) was inaugurated by Richard Hamming. He realized that there is a way to use as few bits as possible and still receive the correct message, but he was unable to explicitly prove it [42]. The work of Hamming piqued the interest of other mathematicians, such as Claude Shannon who worked on the information theory aspects of coding to achieve clear data transmission. Some of work of Shannon provides us with high sound quality of compact discs. Even though compact discs may have visible scratches and thumb prints, a compact disc player still reads the song accurately. This is because of the error-correcting capabilities built into the compact discs. The hat problem with $2^k - 1$ or 2^k players has been solved using the Hamming codes. The hat problem with $n \notin \{2^k - 1, 2^k\}$ players, and the generalized hat problem with any number of players and at least three colors are unsolved. These hat problems may have further connections to and applications in information technology.

Genetic programming. In [14] the authors try to solve the hat problem with $n \notin \{2^k - 1, 2^k\}$ players using genetic programming. The aim is not only to solve the hat problem, but also to learn the way in which the genetic programming works, and what is its effectiveness, because the hat problem seems to be a typical one to solve using genetic programming. As a result it can help us in solving another, even practical problems using genetic programming.

Biology. In [30] it is shown that one of the most important problems in cell biology is to understand functionality of each and every gene of any living organism. A mammoth project, called the Deletion Project, is underway to study the DNA of the yeast organism. The genome of yeast organism has been completely mapped out. It has about 6000 genes. Experiments on yeast cells, under the project, have the following basic operations:

1. removal of a gene from the cell;
2. placement of the cell in a chamber at a set temperature;
3. examination of every one of the remaining cells to determine whether or not it is active.

The data vector generated is of order 1×6000. Every entry in the vector, except one, is 0 (inactive) or 1 (active). The missing entry corresponds to the deleted gene. Steps 1, 2, and 3 should be repeated with respect to every gene. Thus, at the set temperature, we will have 6000 binary data vectors, each vector having exactly one blank space. The whole cell is also placed in the chamber without removing any of its genes. The data vector generated will not have any blanks. Using all these data vectors, one has to guess what would have been the role of the deleted gene had it been present in the cell. It can be hoped that the hat problem might have some pointers.

Mathematics: the autoreducibility of random sequences. In the Ph.D. Thesis of Todd Ebert [20] and in [23] it can be read that the autoreducibility of random sequences is the problem of deducing a property of a random binary sequence when some of the bits of the sequence upon which the property depends are not known. This occurs quite often in practice when, due to time and other resource constrains, a decision is made using only partial information. This consideration is closely related to complexity theory since a decision must be made before a limited resource such as time has been exhausted. In [22, 23] the authors use the hat problem to investigate the autoreducibility of random sequences. The problem of autoreducibility of random sets, which is strongly connected to the problem of autoreducibility of random sequences, was investigated in [6, 21].

Cellular automata. It can be seen that a similarity exists between the hat problem on a graph and so called cellular automata.

First, let us consider asynchronous threshold networks studied by Noga Alon in [2]. There is a graph G with an initial sign $s(v) \in \{-1, 1\}$ for every vertex v. When v becomes active, it changes its sign to $s'(v)$ which is the sign of majority of its neighbors (we define $s'(v) = 1$ if there is a tie). We say that G is in a stable state if $s(v) = s'(v)$ for every vertex v. The timing is synchronus if all vertices become active simultaneously. The timing is asynchronous when only one vertex becomes active at a time. Alon has proven that for every threshold network with all positive edge weights there is an asynchronous run with at most one sign change per vertex which leads the network to a stable state.

The problem above is connected to societies with symmetric influences introduced by Svatjopluk Poljak and Miroslav Sura [43]. The authors proposed a simple model of society with a symmetric function $w(u, v)$ measuring the influence of the opinion of member v on that of member u. The opinions are chosen from the set $\{0, 1, \ldots, p\}$ for some positive integer p. At each step everyone accepts the majority opinion (with respect to w) of the other members (if there are two or more majority opinions, then he accepts the highest one). Obviously, the behavior of such a society is periodic after some initial time. The authors have proven that the length of the period is either one or two. They also concluded that if the influence function w is not symmetric, then the period can be arbitrarily large.

Another model of social influences was introduced by French [26] and Harary [31]. The main differences between their model and the one of Poljak and Sura are that the “opinions” of the members $u \in V$ are real numbers, influences $w(u, v)$ between members are nonnegative real numbers, and the opinion of a member u is the average opinion of the others. For a survey on this topic, see the book [45].

For more information about cellular automata, see [18].

From now to the end of this section we consider variations of the hat problem.

Linear programming. One of the theorems about the hat problem proved in [25] can be represented as a special case of the well known fact that linear programs with integer constraints and a totally unimodular constraint matrix always have integer optimal solutions. The connection between total unimodularity and the solution of integer programs was apparently first shown in [34]. It can be hoped that the hat problem has further connection to and application in linear programming.

Economy. Nicole Immorlica in her Ph.D. Thesis [35] and the authors of [1] project auctions in which the aim is to maximize the profit of the seller. During investigating this problem, they consider a variation of the hat problem in which everybody has to guess his hat color and we are interested in guaranteeing as much correct guesses as possible. This problem is related to the auction problem as follows. Consider the case where are only two types of bidders, those with high valuation for the item, h; and those with a low valuation for the item, l. Mapping h to the color red and l to the color blue, a solution of the hat problem would offer half of the h bids at a price h and half of the l bids at a price l, thus the profit of such an auction would be at least half of optimal revenue.

Mathematics: approximating a Boolean function. The authors of [5] consider the problem of approximating a Boolean function $f\colon \{0,1\}^n \to \{0,1\}$ by the sign of an integer polynomial p of degree k. We say that a polynomial $p(x)$ predicts the value of $f(x)$ if, whenever $p(x) \geq 0$, $f(x) = 1$, and whenever $p(x) < 0$, $f(x) = 0$. A low-degree polynomial p is a good approximator for f if it predicts f at almost all points. Given a positive k, and a Boolean function f, the problem is how good is the best degree k approximator to f. To investigate this problem, the authors use the problem similar to the hat problem in which every one from an odd number of players has 0 or 1 on his forehead. Everybody has to guess the parity of the bits. The game is won if more than half of all guesses are correct.

3. Variations of hat problem

Now, we give a comprehensive list of variations of the hat problem considered in literature. We also present what is already known about each variation. For some variations we give a strategy which solves the problem.

(1) “The generalized hat problem with n players and q colors” was first investigated in [40]. Every one of n players has got a hat of one from q possible colors, and the probabilities of getting hats of all colors are equal. We say that a strategy is symmetric if every player makes his decision on the basis of only numbers of hats of each color seen by him, and all players behave in the same way. A strategy is nonsymmetric if it is not symmetric. The authors of [30] solved the hat problem with three players and three colors by giving a symmetric strategy found by computer, and proving that it is optimal. In [37] the problem was solved by proving the optimality of a nonsymmetric strategy found without using computer. There were also proven some upper bounds on the effectiveness of any strategy for the generalized hat problem with n players and q colors. Additionally, there were considered the numbers of strategies that suffice to be verified to solve the hat problem, or the generalized hat problem. N. Alon [3] proved a lower bound on the maximum chance of success for the generalized hat problem.

(2) There are n players and two colors. Everybody has to guess his hat color. The aim is to find a strategy guaranteeing as many correct guesses as possible. It is known that guaranteeing $\lfloor n/2 \rfloor$ correct guesses is the best possible. The following strategy is optimal. Have players paired up. If the number of players is odd, then the unpaired one always guesses he has, let us say, a blue hat. In each pair one player guesses he has a hat of the same color as the other player, while the other player guesses he has a hat of the color another than the first player, see [13, 15, 32, 49, 50].

(3) It differs from the previous problem only in that there are $q \geq 3$ colors. It has been proven that guaranteeing $\lfloor n/q \rfloor$ correct guesses is the best possible. The following strategy is optimal. Number players 1 to n, and colors 1 to q. The ith player guesses as if the sum of colors of all hats (including own) is congruent to i modulo q, see [15].

(4) It differs from the previous problem only in that there is a directed graph G determining players seen by each player – if there is an arc from u to v, then the player u can see the player v. Optimal strategy for this problem is not known. There exist some lower and upper bounds on the number $t(G)$ which means the maximum number of correct guesses that can be guaranteed. For a directed graph G, let $c(G)$ denote the maximal number of vertex-disjoint cycles in G, and let $F(G)$ denote the minimum number of vertices whose removal from G makes the graph acyclic. Then $c(G) \leq t(G) \leq F(G)$, see [15].

(5) It differs from the previous problem only in that there is also a graph H determining each player to guess the hat color of the particular player (possibly own) – if there is an arc from u to v, then the player u has to guess the hat color of the player v. Let $t_q(G, H)$ mean the maximum number of correct guesses that can be guaranteed when there are q colors. There is known only the fact that $t_q(G, H) > 0$ if and only if there is a vertex of H whose outdegree is greater than 1, or there is a directed cycle in the union of G and H, see [15].

(6) It differs from the previous problem only in that there are $a_1, a_2, \ldots, a_q$ hats of the color $1, 2, \ldots, q$, respectively. There are few facts known for the variation, one of them is as follows. By $t(n; a_1, a_2, \ldots, a_q)$ let us denote the maximum number of correct guesses that can be guaranteed when there are n players, and a_1 hats of the first color, a_2 hats of the second color, and so on up through a_q hats of qth color. Of course, we need $a_1 + a_2 + \ldots + a_q \geq n$ to ensure that we have enough hats. Without loss of generality we may assume that $0 < a_i \leq n$ for all i. It is easy to notice that if $a_1 + a_2 + \ldots + a_q = n$, then $t(n; a_1, a_2, \ldots, a_q) = n$, see [15].

(7) There are n players standing in a line and two colors. Everybody can see the hat colors of players before him, but neither his nor those of players behind him. Players have to guess their hat colors sequentially, starting from the back of the line. Everybody can hear the answer called out by each player. We are interested in a strategy guaranteeing as many correct guesses as possible. The following strategy is optimal. If the last player sees an odd number of red hats in front of him, then he guesses he has a red hat. Otherwise he guesses he has a blue hat. Player $n-1$ will deduce his own hat color from the information said by

the last player. Similar reasoning applies to each player going up the line. Player i sums the number of red hats he sees and red guesses he hears. If the sum is odd, then he guesses he has a red hat. Otherwise he guesses he has a blue hat. Of course, it is not possible to guarantee the correctness of the guess of the player who guesses as first, thus guaranteeing $n-1$ correct guesses is the best possible, see [4, 19, 27, 49].

(8) It differs from the previous problem only in that there are $q \geq 3$ colors. Now also the maximum number of correct guesses that can be guaranteed is $n-1$. By $v_1, v_2, \ldots, v_n$ let us denote players, and by $1, 2, \ldots, q$ let us denote colors. Let y_i represent the hat color of player v_i, and let us define $Y_i = \sum_{j=i}^{n} y_j \bmod \mathrm{q}$. The following strategy is optimal. Player v_1 guesses he has a hat of the color $Y_2 = \sum_{i=2}^{n} y_i \bmod \mathrm{q}$. For each $i > 1$ player v_i can see the values $y_{i+1}, \ldots, y_n$, and has heard the values Y_2 and $y_2, \ldots, y_{i-1}$. As an effect, he solves the expression for Y_2 to get y_i. As the result, $n-1$ players guess their hat colors correctly, see [4, 19].

(9) It differs from the two previous problems only in that the seeing radius and/or the hearing radius are limited (there are $q \geq 2$ colors). The seeing radius of a player is the maximum number of players that he can see ahead of him. The hearing radius of a player is the maximum number of players ahead of him that can hear him. We assume that the seeing (hearing, respectively) radius is the same for all players, and we denote it by s (h, respectively). For this variation it is known only that the maximum number of correct guesses that can be guaranteed is $n - \lceil n/(\min(s,h)+1) \rceil$, see [4].

(10) There are n players and two colors. There is also a clock and as every minute elapses, everybody can guess his hat color. Time elapses after n minutes, and everybody who has not tried to guess his hat color loses. If some player guesses his hat color wrong, then all players lose. Is there a strategy such that everybody wins? No, although we can try to find a strategy such that as many players as possible wins, see [27].

(11) It differs from the previous problem only in that there is an additional player who comes to the team and says "somebody has a blue hat" or "everybody has a red hat" or something else. Does it can help to guarantee that everybody will win? Assume that the additional player says that somebody has a blue hat. Let us consider the following strategy. Everybody counts blue hats he sees. After k minutes, if nobody has tried to guess his hat color, then everybody who sees $k-1$ red hats guesses he has a red hat. If at least two players have a red hat, then the information from the additional player that somebody has a red hat is a fact known by everybody. Paradoxically, it has a value. The information from the additional player is called common knowledge. That is, everybody knows it, and everybody knows that everybody knows it, and everybody knows that everybody knows that everybody knows it, etc. Players can use this meta-information to derive their own hat colors, see [10, 27].

(12) There are three players, A, B, and C. There are four green and four red stamps. Players are blindfolded, and two stamps are pasted on the head of each

player. After removing the blindfolds, A, B, and C are asked in turn about colors of own stamps. No player knows the answer. Now A is asked once more. He again does not know the answer. Now B is asked, and he replies "yes". What are the colors of the stamps of B? The answer is that he has one green, and one red stamp, see [29].

(13) There are three players and two colors. Everybody has to simultaneously guess his hat color or pass. The team wins if at least one player guesses his hat color correctly and nobody guesses his hat color wrong. The probabilities of the eight cases which can appear does not have to be the same. How does it influence the strategy which should be applied by the team? It has been proven (using computer) that to solve the problem it suffices to calculate the chance of success for a family of twelve strategies, see [30].

(14) It differs from the previous variation only in that there are n players and $q \geq 2$ colors, see [40].

(15) In the "Gabay – O'Connor hat problem" there are an infinite number of players numbered $1, 2, \ldots,$ and two colors. Everybody has to guess his hat color. The team wins if only finite number of guesses are wrong. Is there a strategy guaranteeing that the team will win? Yes, but only if the Axiom of Choice holds, see [32, 33, 51].

(16) The variation called "All right or all wrong" differs from the previous problem only in that the team wins if and only if all guesses are correct or all guesses are wrong. Similarly as for the previous variation, the win of the team can be guaranteed if and only if the Axiom of Choice holds, see [51].

(17) There are ten players and every one of them has a digit from 0 to 9 written on the forehead. Everybody has to guess his digit. The team wins if at least one player does it correctly. The aim is to find a strategy guaranteeing that the team will win. Let us consider the following strategy. Number players 0 to $n-1$. Let s be the sum of the numbers on the foreheads of all players, modulo n. Now let player k guess that $s = k$, that is, guess that his own number is k minus the sum of the numbers he sees, modulo n. This will ensure that player s will be correct, see [51].

(18) The variation called "The color-blind prisoner" differs from the previous problem in that numbers are written in red, one player has a green skin, and one another player does not distinguish green and red. Thus he decides about his guess on the basis of only eight digits. Now it is not possible to guarantee that the team will win, see [51].

(19) In the variation called "Numbers and hats" there are n players, and every one of them has a distinct real number written on the forehead. Everybody has to choose a blue or red hat for himself. The aim is for the hat colors to alternate in the order determined by the real numbers. There is a strategy guaranteeing that the team will win, but it is very long and complicated, see [51].

(20) In the "Voting puzzle 1" there are an odd number of players, say n. Every

one of them has a random bit written on the forehead. Players have to vote on the parity of the bits (by voting 0 or 1).The result of the voting is the bit chosen more often. Players win if the result of the voting is equal to the parity of the bits. The aim is to maximize the chance of success. Optimal strategy gives the chance of success equaling $n/(n+1)$. For the strategy, see [5].

(21) The "Voting puzzle 2" differs from the previous problem only in that everybody can make as many votes as he wants. Optimal strategy gives the chance of success equaling $(2^n - 1)/2^n$. For the strategy, see [5].

(22) The "Voting puzzle 3" is as follows. Let S be a set of randomly chosen n bits. There are $\binom{n}{k}$ players, every one of them can see another k-element subset of S. Players participate in a voting, the result of which should be the parity of the bits. Everybody has to make an integer number of votes. If their sum is positive, then the result of the voting is 0. If it is negative, then the result is 1. If the sum is zero, then the result of the voting is not defined. A strategy, based on approximating a Boolean function, guarantees that the team will win, see [5].

(23) In the variation called "Not distinguishable players" there are n players and $q \geq 2$ colors. Every player can see everybody excluding him, but cannot distinguish them. Thus everybody makes his guess on the basis of only numbers of hats of each color seen by him. Every player guesses his hat color or passes. The team wins if at least one player guesses his hat color correctly and nobody guesses his hat color wrong. It has been proven that for large n the maximum chance of success is approximately $(1 + (1/3)^{q-1})/2$, for details see [28].

(24) It differs from the previous variation only in that all players have to behave in the same way, see [40].

(25) The variation called "Players do not distinguish colors 1" is as follows. There are n color-blind players and two colors. Before fitting players with hats somebody says players what will be the probability of getting a blue hat, and what of a red hat. By q let us denote the probability of getting a blue hat. It is known that for large n the maximum chance of success is approximately $(1-q)^{(1-q)/q} - (1-q)^{1/q}$, see [28].

(26) The variation "Players do not distinguish colors 2" differs from the previous problem only in that later (after fitting with hats) somebody says what was the probability of getting a blue hat, and what of a red hat (somebody says how many blue and how many red hats were placed). It is known that, comparing to the previous variation, it does not change the chance of success of optimal strategy, see [28].

(27) In the variation called "Crowns of the Minotaur" there are three players and every one of them is fitted by the Minotaur with a blue or red crown. Every player bets zero or more points on guessing his crown color. A player wins or loses as many points as he has bet, depending on the accuracy of his guess. Then the won and the lost points are added separately, and the team wins if there are more won than lost points. It is known that the maximum chance of success is equal to 7/8. The following strategy is optimal. At first, number players who is first,

second, and third. The first player bets one point for red. If the second player sees that the first has a blue crown, then he bets two points for red, otherwise passes. If the third player sees that the first two have both blue crowns, then he bets four points for red, otherwise passes. Unless every player has a blue crown (chance 1/8), everybody wins, see [48].

(28) In "The discarded hat variation" there are $4k-1$ players, and $2k$ blue and $2k$ red hats. Every player is fitted with a hat, and one hat is taken away. Then everybody has to guess his hat color. The aim is to guarantee as many correct guesses as possible. It is known that guaranteeing $3k-1$ correct guesses is the best possible. For an optimal strategy, involving cyclic arrangement of players, see [25].

(29) In "The everywhere balanced variation" there are n players and $q \geq 2$ colors. Let $\{c_1, c_2, \ldots, c_q\}$ be the set of colors, and let H_i mean the set of players having a hat of color c_i. Nobody knows neither to which set he belongs nor what are the cardinalities of sets H_i. The aim is to find a strategy guaranteeing that in every set H_i the number of players guessing their hat colors correctly is between $\lfloor |H_i|/q \rfloor$ and $\lceil |H_i|/q \rceil$. For such strategy (a complicated one), see [25].

(30) The variation "Hat problem on a directed graph asking for at least one correct guess" is as follows. There are n players and two colors. We have a directed graph G determining players seen by each player – if there is an arc from u to v, then the player u can see the player v. What subgraph has to have the visibility graph to ensure the existence of a strategy guaranteeing at least one correct guess? It has to have a cycle as a subgraph, for details see [32].

(31) It differs from the previous problem in that there are n players and n colors. It is known that now the visibility graph has to be complete, see [32].

(32) It differs from the two previous problems in that there are n players and q colors. What is the maximum number of correct guesses that can be guaranteed? The answer is $\lfloor n/q \rfloor$, see [32].

(33) There are n players and $q \geq 2$ colors. Players are allowed more than one round in which to guess their hat colors. During each round everybody must simultaneously say "My hat color is i", "My hat color is not i", or "Pass", where i is one of the colors. However, if everybody passes in any round, then the team loses. The rounds continue, with each player making a guess or passing, as long as no incorrect guess is made and at least one player guesses his hat color correctly. Then the team wins. It has been proven that the maximum chance of success is $n(q-1)/(1+n(q-1))$, see [16].

(34) In the variation called "Zero-information strategies" there are n players and two colors. Everybody has to simultaneously guess his hat color or pass. The team wins if at least one player guesses his hat color correctly and nobody guesses his hat color wrong. Every player makes his decision without access to any information. Now a winning probability of 1/4 is asymptotically attainable and optimal, see [40].

(35) "The hat problem on a graph" is as follows. There is a graph, where vertices correspond to players and a player can see each player to whom he is connected by an edge. This variation of the hat problem was first considered in [38]. There were proven some general theorems about the hat problem on a graph, and the problem was solved on trees. Additionally, there was considered the hat problem on a graph such that the only known information are degrees of vertices. In [39] the problem was solved on the cycle C_4. Further results about the hat problem on a graph were established by Uriel Feige [24]. For example, there the problem was solved for bipartite graphs, and planar graphs containing a triangle. Based on these and some other results, the author conjectured that for every graph there is an optimal strategy in which all vertices which do not belong to the maximum clique always pass.

(36) "The modified hat problem" is as follows. There are $n \geq 3$ players. Everyone of them is randomly fitted with a blue or red hat. Players do not have to guess their hat colors simultaneously. In this variation of the hat problem players guess their hat colors by coming to the basket and throwing the proper card into it. Every player has got two cards with his name and the sentence "I have got a blue hat" or "I have got a red hat". If someone wants to resign from answering, then he does not do anything. The problem was investigated in [36]. There was given an optimal strategy for the problem which always succeeds.

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