Reinhard Diestel

Graph Theory

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9 Ramsey Theory for Graphs

In this chapter we set out from a type of problem which, on the face of it, appears to be similar to the theme of Chapter 7: what kind of substructures are necessarily present in every large enough graph?

The regularity lemma in Chapter 7.4 provides one possible answer to this question: every (large) graph G contains large random-like subgraphs. If we are looking for a concrete given subgraph H, on the other hand, our problem becomes more like Turán's theorem (7.1.1), Wagner's theorem (7.3.4), or Hadwiger's conjecture: we cannot expect an arbitrary graph G to contain a copy of H, but if it does not then this might have some interesting structural implications for G.

The kind of structural implication that will be typical for this chapter is simply that of containing some other (induced) subgraph. For example: given an integer r, does every large enough graph contain either a K^r or an induced $\overline{K^r}$? Does every large enough connected graph contain either a K^r or else a large induced path or star?

Despite its superficial similarity to extremal problems, the above type of question leads to a kind of mathematics with a distinctive flavour of its own. Indeed, the theorems and proofs in this chapter have more in common with similar results in algebra or geometry, say, than with most other areas of graph theory. The study of their underlying methods, therefore, is generally regarded as a combinatorial subject in its own right: the discipline of *Ramsey theory*.

In line with the subject of this book, we shall focus on results that are naturally expressed in terms of graphs. Even from the viewpoint of general Ramsey theory, however, this is not as much of a limitation as it might seem: graphs are a natural setting for Ramsey problems, and the material in this chapter brings out a sufficient variety of ideas and methods to convey some of the fascination of the theory as a whole.

9.1 Ramsey's original theorems

In its simplest version, Ramsey's theorem says that, given an integer $r \geqslant 0$, every large enough graph G contains either K^r or $\overline{K^r}$ as an induced subgraph. At first glance, this may seem surprising: after all, we need a proportion of about $(r-2)/(r-1)$ of all possible edges to force a K^r subgraph in G (Corollary 7.1.3), but neither G nor $\overline{G}$ can be expected to have more than half of all possible edges. However, as the Turán graphs illustrate well, squeezing many edges into G without creating a K^r imposes additional structure on G, which may help us find an induced $\overline{K^r}$.

So how could we go about proving Ramsey's theorem? Let us try to build a K^r or $\overline{K^r}$ in G inductively, starting with an arbitrary vertex $v_1 \in V_1 := V(G)$. If $|G|$ is large, there will be a large set $V_2 \subseteq V_1 \setminus \{v_1\}$ of vertices that are either all adjacent to v_1 or all non-adjacent to v_1. Accordingly, we may think of v_1 as the first vertex of a K^r or $\overline{K^r}$ whose other vertices all lie in V_2. Let us then choose another vertex $v_2 \in V_2$ for our K^r or $\overline{K^r}$. Since V_2 is large, it will have a subset V_3, still fairly large, of vertices that are all 'of the same type' with respect to v_2 as well: either all adjacent or all non-adjacent to it. We then continue our search for vertices inside V_3, and so on (Fig. 9.1.1).

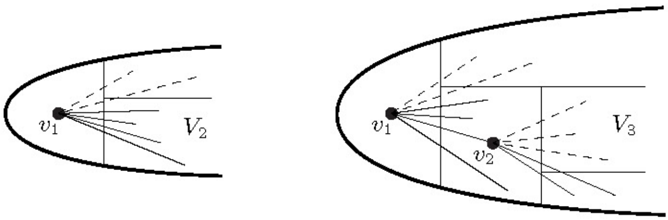

Fig. 9.1.1. Choosing the sequence $v_1, v_2, \ldots$

How long can we go on in this way? This depends on the size of our initial set V_1: each set V_i has at least half the size of its predecessor V_{i-1}, so we shall be able to complete s construction steps if G has order about 2^s. As the following proof shows, the choice of $s = 2r-3$ vertices v_i suffices to find among them the vertices of a K^r or $\overline{K^r}$.

[9.2.2] **Theorem 9.1.1.** (Ramsey 1930)
For every $r \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ such that every graph of order at least n contains either K^r or $\overline{K^r}$ as an induced subgraph.

Proof. The assertion is trivial for $r \leqslant 1$; we assume that $r \geqslant 2$. Let $n := 2^{2r-3}$, and let G be a graph of order at least n. We shall define a sequence $V_1, \ldots, V_{2r-2}$ of sets and choose vertices $v_i \in V_i$ with the following properties:

(i) $|V_i| = 2^{2r-2-i}$ $(i = 1, \ldots, 2r-2)$;

(ii) $V_i \subseteq V_{i-1} \setminus \{v_{i-1}\}$ $(i = 2, \ldots, 2r-2)$;

(iii) v_{i-1} is adjacent either to all vertices in V_i or to no vertex in V_i $(i = 2, \ldots, 2r-2)$.

Let $V_1 \subseteq V(G)$ be any set of 2^{2r-3} vertices, and pick $v_1 \in V_1$ arbitrarily. Then (i) holds for $i = 1$, while (ii) and (iii) hold trivially. Suppose now that V_{i-1} and $v_{i-1} \in V_{i-1}$ have been chosen so as to satisfy (i)–(iii) for $i-1$, where $1 < i \leqslant 2r-2$. Since

$$|V_{i-1} \setminus \{v_{i-1}\}| = 2^{2r-1-i} - 1$$

is odd, V_{i-1} has a subset V_i satisfying (i)–(iii); we pick $v_i \in V_i$ arbitrarily.

Among the $2r-3$ vertices $v_1, \ldots, v_{2r-3}$, there are $r-1$ vertices that show the same behaviour when viewed as v_{i-1} in (iii), being adjacent either to all the vertices in V_i or to none. Accordingly, these $r-1$ vertices and v_{2r-2} induce either a K^r or a $\overline{K^r}$ in G, because $v_i, \ldots, v_{2r-2} \in V_i$ for all i. □

The least integer n associated with r as in Theorem 9.1.1 is the *Ramsey number* $R(r)$ of r; our proof shows that $R(r) \leqslant 2^{2r-3}$. In Chapter 11 we shall use a simple probabilistic argument to show that $R(r)$ is bounded below by $2^{r/2}$ (Theorem 11.1.3). *Ramsey number $R(r)$*

In other words, the largest clique or independent set of vertices that a graph of order n must contain is, asymptotically, logarithmically small in n. As soon as we forbid some fixed induced subgraph, however, it may be much bigger, of size linear in n: The *Erdős-Hajnal conjecture* says that for every graph H there exists a constant $\delta_H > 0$ such that every graph G not containing an induced copy of H has a set of at least $|G|^{\delta_H}$ vertices that are either independent or span a complete subgraph in G. *Erdős-Hajnal conjecture*

It is customary in Ramsey theory to think of partitions as colourings: a *colouring* of (the elements of) a set X *with c colours*, or *c-colouring* for short, is simply a partition of X into c classes (indexed by the 'colours'). In particular, these colourings need not satisfy any non-adjacency requirements as in Chapter 5. Given a c-colouring of $[X]^k$, the set of all k-subsets of X, we call a set $Y \subseteq X$ *monochromatic* if all the elements of $[Y]^k$ have the same colour,[1] i.e. belong to the same of the c partition classes of $[X]^k$. Similarly, if $G = (V, E)$ is a graph and all the edges of $H \subseteq G$ have the same colour in some colouring of E, we call H a *monochromatic subgraph* of G, speak of a red (green, etc.) H in G, and so on. *c-colouring* *$[X]^k$* *monochromatic*

In the above terminology, Ramsey's theorem can be expressed as follows: for every r there exists an n such that, given any n-set X,

[1] Note that Y is called monochromatic, but it is the elements of $[Y]^k$, not of Y, that are (equally) coloured.

every 2-colouring of $[X]^2$ yields a monochromatic r-set $Y \subseteq X$. Interestingly, this assertion remains true for c-colourings of $[X]^k$ with arbitrary c and k—with almost exactly the same proof!

We first prove the infinite version, which is easier, and then deduce the finite version.

[12.1.1] **Theorem 9.1.2.** *Let k, c be positive integers, and X an infinite set. If $[X]^k$ is coloured with c colours, then X has an infinite monochromatic subset.*

Proof. We prove the theorem by induction on k, with c fixed. For $k = 1$ the assertion holds, so let $k > 1$ and assume the assertion for smaller values of k.

Let $[X]^k$ be coloured with c colours. We shall construct an infinite sequence $X_0, X_1, \dots$ of infinite subsets of X and choose elements $x_i \in X_i$ with the following properties (for all i):

(i) $X_{i+1} \subseteq X_i \setminus \{x_i\}$;

(ii) all k-sets $\{x_i\} \cup Z$ with $Z \in [X_{i+1}]^{k-1}$ have the same colour, which we *associate* with x_i.

We start with $X_0 := X$ and pick $x_0 \in X_0$ arbitrarily. By assumption, X_0 is infinite. Having chosen an infinite set X_i and $x_i \in X_i$ for some i, we c-colour $[X_i \setminus \{x_i\}]^{k-1}$ by giving each set Z the colour of $\{x_i\} \cup Z$ from our c-colouring of $[X]^k$. By the induction hypothesis, $X_i \setminus \{x_i\}$ has an infinite monochromatic subset, which we choose as X_{i+1}. Clearly, this choice satisfies (i) and (ii). Finally, we pick $x_{i+1} \in X_{i+1}$ arbitrarily.

Since c is finite, one of the c colours is associated with infinitely many x_i. These x_i form an infinite monochromatic subset of X. □

If desired, the finite version of Theorem 9.1.2 could be proved just like the infinite version above. However to ensure that the relevant sets are large enough at all stages of the induction, we have to keep track of their sizes, which involves a good deal of boring calculation. As long as we are not interested in bounds, the more elegant route is to deduce the finite version from the infinite 'by compactness', that is, using König's infinity lemma (8.1.2).

[9.3.3] **Theorem 9.1.3.** *For all $k, c, r \geqslant 1$ there exists an $n \geqslant k$ such that every n-set X has a monochromatic r-subset with respect to any c-colouring of $[X]^k$.*

(8.1.2) *Proof.* As is customary in set theory, we denote by $n \in \mathbb{N}$ (also) the
k, c, r set $\{0, \dots, n-1\}$. Suppose the assertion fails for some k, c, r. Then for every $n \geqslant k$ there exist an n-set, without loss of generality the set n, and a c-colouring $[n]^k \to c$ such that n contains no monochromatic r-set. Let
bad colouring us call such colourings *bad*; we are thus assuming that for every $n \geqslant k$

there exists a bad colouring of $[n]^k$. Our aim is to combine these into a bad colouring of $[\mathbb{N}]^k$, which will contradict Theorem 9.1.2.

For every $n \geqslant k$ let $V_n \neq \emptyset$ be the set of bad colourings of $[n]^k$. For $n > k$, the restriction $f(g)$ of any $g \in V_n$ to $[n-1]^k$ is still bad, and hence lies in V_{n-1}. By the infinity lemma (8.1.2), there is an infinite sequence $g_k, g_{k+1}, \dots$ of bad colourings $g_n \in V_n$ such that $f(g_n) = g_{n-1}$ for all $n > k$. For every $m \geqslant k$, all colourings g_n with $n \geqslant m$ agree on $[m]^k$, so for each $Y \in [\mathbb{N}]^k$ the value of $g_n(Y)$ coincides for all $n > \max Y$. Let us define $g(Y)$ as this common value $g_n(Y)$. Then g is a bad colouring of $[\mathbb{N}]^k$: every r-set $S \subseteq \mathbb{N}$ is contained in some sufficiently large n, so S cannot be monochromatic since g coincides on $[n]^k$ with the bad colouring g_n. □

The least integer n associated with k, c, r as in Theorem 9.1.3 is the *Ramsey number* for these parameters; we denote it by $R(k, c, r)$. *Ramsey number* $R(k,c,r)$

9.2 Ramsey numbers

Ramsey's theorem may be rephrased as follows: if $H = K^r$ and G is a graph with sufficiently many vertices, then either G itself or its complement $\overline{G}$ contains a copy of H as a subgraph. Clearly, the same is true for any graph H, simply because $H \subseteq K^h$ for $h := |H|$.

However, if we ask for the *least* n such that every graph G of order n has the above property—this is the *Ramsey number* $R(H)$ of H—then the above question makes sense: if H has only few edges, it should embed more easily in G or $\overline{G}$, and we would expect $R(H)$ to be smaller than the Ramsey number $R(h) = R(K^h)$. *Ramsey number* $R(H)$

A little more generally, let $R(H_1, H_2)$ denote the least $n \in \mathbb{N}$ such that $H_1 \subseteq G$ or $H_2 \subseteq \overline{G}$ for every graph G of order n. For most graphs H_1, H_2, only very rough estimates are known for $R(H_1, H_2)$. Interestingly, lower bounds given by random graphs (as in Theorem 11.1.3) are often sharper than even the best bounds provided by explicit constructions. $R(H_1, H_2)$

The following proposition describes one of the few cases where exact Ramsey numbers are known for a relatively large class of graphs:

Proposition 9.2.1. *Let s, t be positive integers, and let T be a tree of order t. Then $R(T, K^s) = (s-1)(t-1) + 1$.*

(1.5.4) (5.2.3)

Proof. The disjoint union of $s-1$ graphs K^{t-1} contains no copy of T, while the complement of this graph, the complete $(s-1)$-partite graph K^{s-1}_{t-1}, does not contain K^s. This proves $R(T, K^s) \geqslant (s-1)(t-1)+1$.

Conversely, let G be any graph of order $n = (s-1)(t-1)+1$ whose complement contains no K^s. Then $s > 1$, and in any vertex colouring of G (in the sense of Chapter 5) at most $s-1$ vertices can have the same colour. Hence, $\chi(G) \geqslant \lceil n/(s-1) \rceil = t$. By Lemma 5.2.3, G has a subgraph H with $\delta(H) \geqslant t-1$, which by Corollary 1.5.4 contains a copy of T. □

As the main result of this section, we shall now prove one of those rare general theorems providing a relatively good upper bound for the Ramsey numbers of a large class of graphs, a class defined in terms of a standard graph invariant. The theorem deals with the Ramsey numbers of sparse graphs: it says that the Ramsey number of graphs H with bounded maximum degree grows only linearly in $|H|$—an enormous improvement on the exponential bound from the proof of Theorem 9.1.1.

Theorem 9.2.2. (Chvátal, Rödl, Szemerédi & Trotter 1983)
For every positive integer Δ there is a constant c such that

$$R(H) \leqslant c\,|H|$$

(7.1.1) *for all graphs H with $\Delta(H) \leqslant \Delta$.*

(7.4.1)
(7.5.2)
(9.1.1)

Proof. The basic idea of the proof is as follows. We wish to show that $H \subseteq G$ or $H \subseteq \overline{G}$ if $|G|$ is large enough (though not too large). Consider an ϵ-regular partition of G, as provided by the regularity lemma. If enough of the ϵ-regular pairs in this partition have high density, we may hope to find a copy of H in G. If most pairs have low density, we try to find H in $\overline{G}$. Let R, R' and R'' be the regularity graphs of G whose edges correspond to the pairs of density $\geqslant 0$; $\geqslant 1/2$; $< 1/2$ respectively.[2] Then R is the edge-disjoint union of R' and R''.

Now to obtain $H \subseteq G$ or $H \subseteq \overline{G}$, it suffices by Lemma 7.5.2 to ensure that H is contained in a suitable 'inflated regularity graph' R'_s or R''_s. Since $\chi(H) \leqslant \Delta(H)+1 \leqslant \Delta+1$, this will be the case if $s \geqslant \alpha(H)$ and we can find a $K^{\Delta+1}$ in R' or in R''. But that is easy to ensure: we just need that $K^r \subseteq R$, where r is the Ramsey number of $\Delta+1$, which will follow from Turán's theorem because R is dense.

Δ, d For the formal proof let now $\Delta \geqslant 1$ be given. On input $d := 1/2$
ϵ_0, m and Δ, Lemma 7.5.2 returns an ϵ_0. Let $m := R(\Delta+1)$ be the Ramsey
ϵ number of $\Delta+1$. Let $\epsilon \leqslant \epsilon_0$ be positive but small enough that for $k = m$ (and hence for all $k \geqslant m$)

$$2\epsilon < \frac{1}{m-1} - \frac{1}{k}\,; \tag{1}$$

M then in particular $\epsilon < 1$. Finally, let M be the integer returned by the regularity lemma (Theorem 7.4.1) on input ϵ and m.

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[2] In our formal proof later we shall define R'' a little differently, so that it complies properly with our definition of a regularity graph.

All the quantities defined so far depend only on Δ. We shall prove the theorem with

$$c := \frac{2^{\Delta+1} M}{1-\epsilon}\,. \qquad c$$

Let H with $\Delta(H) \leqslant \Delta$ be given, and let $s := |H|$. Let G be an arbitrary graph of order $n \geqslant c|H|$; we show that $H \subseteq G$ or $H \subseteq \overline{G}$. s G, n

By Theorem 7.4.1, G has an ϵ-regular partition $\{V_0, V_1, \dots, V_k\}$ with exceptional set V_0 and $|V_1| = \ldots = |V_k| =: \ell$, where $m \leqslant k \leqslant M$. Then k ℓ

$$\ell = \frac{n - |V_0|}{k} \geqslant n\,\frac{1-\epsilon}{M} \geqslant cs\,\frac{1-\epsilon}{M} \geqslant 2^{\Delta+1}s = 2s/d^{\Delta}. \qquad (2)$$

Let R be the regularity graph with parameters $\epsilon, \ell, 0$ corresponding to this partition. By definition, R has k vertices and R

$$\begin{aligned}
\|R\| &\geqslant \binom{k}{2} - \epsilon k^2 \\
&= \tfrac{1}{2}k^2\Big(1 - \frac{1}{k} - 2\epsilon\Big) \\
&\underset{(1)}{>} \tfrac{1}{2}k^2\Big(1 - \frac{1}{k} - \frac{1}{m-1} + \frac{1}{k}\Big) \\
&= \tfrac{1}{2}k^2\,\frac{m-2}{m-1} \\
&\geqslant t_{m-1}(k)
\end{aligned}$$

edges. By Theorem 7.1.1, therefore, R has a subgraph $K = K^m$. K

We now colour the edges of R with two colours: red if the edge corresponds to a pair (V_i, V_j) of density at least $1/2$, and green otherwise. Let R' be the spanning subgraph of R formed by the red edges, and R'' the spanning subgraph of R formed by the green edges and those whose corresponding pair has density exactly $1/2$. Then R' is a regularity graph of G with parameters ϵ, ℓ and $1/2$. And R'' is a regularity graph of $\overline{G}$, with the same parameters: as one easily checks, every pair (V_i, V_j) that is ϵ-regular for G is also ϵ-regular for $\overline{G}$.

By definition of m, our graph K contains a red or a green K^r, for $r := \chi(H) \leqslant \Delta + 1$. Correspondingly, $H \subseteq R'_s$ or $H \subseteq R''_s$. Since $\epsilon \leqslant \epsilon_0$ and $\ell \geqslant 2s/d^{\Delta}$ by (2), both R' and R'' satisfy the requirements of Lemma 7.5.2, so $H \subseteq G$ or $H \subseteq \overline{G}$ as desired. □

So far in this section, we have been asking what is the least order of a complete graph G such that every 2-colouring of its edges yields a monochromatic copy of some given graph H. Rather than keeping G complete and focusing on its order, let us now consider its structure too, i.e., minimize G with respect to the subgraph relation. Given a graph H, Ramsey-minimal

let us call a graph G *Ramsey-minimal* for H if G is minimal with the property that every 2-colouring of its edges yields a monochromatic copy of H.

What do such Ramsey-minimal graphs look like? Are they unique? The following result, which we include for its pretty proof, answers the second question for some H:

Proposition 9.2.3. *If T is a tree but not a star, then infinitely many graphs are Ramsey-minimal for T.*

(1.5.4)
(5.2.3)
(5.2.5)
(=11.2.2)

Proof. Let $|T| =: r$. We show that for every $n \in \mathbb{N}$ there is a graph of order at least n that is Ramsey-minimal for T.

By Theorem 5.2.5, there exists a graph G with chromatic number $\chi(G) > r^2$ and girth $g(G) > n$. If we colour the edges of G red and green, then the red and the green subgraph cannot both have an r-(vertex-)colouring in the sense of Chapter 5: otherwise we could colour the vertices of G with the pairs of colours from those colourings and obtain a contradiction to $\chi(G) > r^2$. So let $G' \subseteq G$ be monochromatic with $\chi(G') > r$. By Lemma 5.2.3, G' has a subgraph of minimum degree at least r, which contains a copy of T by Corollary 1.5.4.

Let $G^* \subseteq G$ be Ramsey-minimal for T. Clearly, G^* is not a forest: the edges of any forest can be 2-coloured (partitioned) so that no monochromatic subforest contains a path of length 3, let alone a copy of T. (Here we use that T is not a star, and hence contains a P^3.) So G^* contains a cycle, which has length $g(G) > n$ since $G^* \subseteq G$. In particular, $|G^*| > n$ as desired. □

9.3 Induced Ramsey theorems

Ramsey's theorem can be rephrased as follows. For every graph $H = K^r$ there *exists* a graph G such that every 2-colouring of the edges of G yields a monochromatic $H \subseteq G$; as it turns out, this is witnessed by any large enough complete graph as G. Let us now change the problem slightly and ask for a graph G in which every 2-edge-colouring yields a monochromatic *induced* $H \subseteq G$, where H is now an arbitrary given graph.

This slight modification changes the character of the problem dramatically. What is needed now is no longer a simple proof that G is 'big enough' (as for Theorem 9.1.1), but a careful construction: the construction of a graph that, however we bipartition its edges, contains an induced copy of H with all edges in one partition class. We shall call such a graph a *Ramsey graph* for H.

Ramsey graph

The fact that such a Ramsey graph exists for every choice of H is one of the fundamental results of graph Ramsey theory. It was proved around 1973, independently by Deuber, by Erdős, Hajnal & Pósa, and by Rödl.

Theorem 9.3.1. *Every graph has a Ramsey graph. In other words, for every graph H there exists a graph G that, for every partition $\{E_1, E_2\}$ of $E(G)$, has an induced subgraph H with $E(H) \subseteq E_1$ or $E(H) \subseteq E_2$.*

We give two proofs. Each of these is highly individual, yet each offers a glimpse of true Ramsey theory: the graphs involved are used as hardly more than bricks in the construction, but the edifice is impressive.

First proof. In our construction of the desired Ramsey graph we shall repeatedly replace vertices of a graph $G = (V, E)$ already constructed by copies of another graph H. For a vertex set $U \subseteq V$ let $G[U \to H]$ denote the graph obtained from G by replacing the vertices $u \in U$ with copies $H(u)$ of H and joining each $H(u)$ completely to all $H(u')$ with $uu' \in E$ and to all vertices $v \in V \setminus U$ with $uv \in E$ (Fig. 9.3.1). Formally,

$G[U \to H]$

$H(u)$

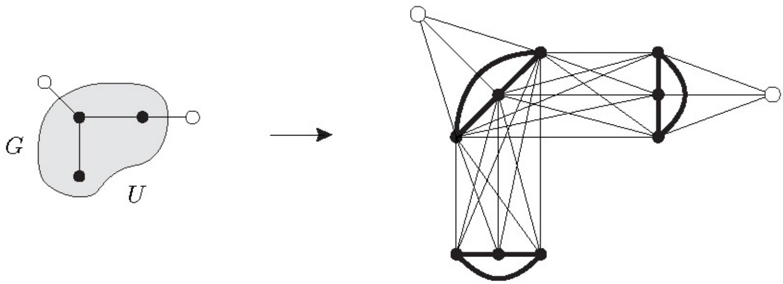

Fig. 9.3.1. A graph $G[U \to H]$ with $H = K^3$

$G[U \to H]$ is the graph on

$$(U \times V(H)) \cup ((V \setminus U) \times \{\emptyset\})$$

in which two vertices (v, w) and (v', w') are adjacent if and only if either $vv' \in E$, or else $v = v' \in U$ and $ww' \in E(H)$.[3]

We prove the following formal strengthening of Theorem 9.3.1:

For any two graphs H_1, H_2 there exists a graph $G = G(H_1, H_2)$ such that every edge colouring of G with the colours 1 and 2 yields either an induced $H_1 \subseteq G$ with all its edges coloured 1 or an induced $H_2 \subseteq G$ with all its edges coloured 2. $(*)$

$G(H_1, H_2)$

[3] The replacement of $V \setminus U$ by $(V \setminus U) \times \{\emptyset\}$ is just a formal device to ensure that all vertices of $G[U \to H]$ have the same form (v, w), and that $G[U \to H]$ is formally disjoint from G.

This formal strengthening makes it possible to apply induction on $|H_1| + |H_2|$, as follows.

If either H_1 or H_2 has no edges (in particular, if $|H_1| + |H_2| \leqslant 1$), then (∗) holds with $G = \overline{K^n}$ for large enough n. For the induction step, we now assume that both H_1 and H_2 have at least one edge, and that (∗) holds for all pairs (H_1', H_2') with smaller $|H_1'| + |H_2'|$.

x_i For each $i = 1, 2$, pick a vertex $x_i \in H_i$ that is incident with an
H_i', H_i'' edge. Let $H_i' := H_i - x_i$, and let H_i'' be the subgraph of H_i' induced by the neighbours of x_i.

We shall construct a sequence $G^0, \dots, G^n$ of disjoint graphs; G^n will be the desired Ramsey graph $G(H_1, H_2)$. Along with the graphs G_i, we shall define subsets $V^i \subseteq V(G^i)$ and a map

$$f \colon V^1 \cup \ldots \cup V^n \to V^0 \cup \ldots \cup V^{n-1}$$

such that

$$f(V^i) = V^{i-1} \tag{1}$$

f^i for all $i \geqslant 1$. Writing $f^i := f \circ \ldots \circ f$ for the i-fold composition of f, and f^0 for the identity map on $V^0 = V(G^0)$, we thus have $f^i(v) \in V^0$
origin for all $v \in V^i$. We call $f^i(v)$ the *origin* of v.

The subgraphs $G^i[V^i]$ will reflect the structure of G^0 as follows:

Vertices in V^i with different origins are adjacent in G^i if and only if their origins are adjacent in G^0. (2)

Assertion (2) will not be used formally in the proof below. However, it can help us to visualize the graphs G^i: every G^i (more precisely, every $G^i[V^i]$; for $i \geqslant 1$ there will also be some vertices $x \in G^i - V^i$) is essentially an inflated copy of G^0 in which every vertex $w \in G^0$ has been replaced by the set of all vertices in V^i with origin w, and the map f links vertices with the same origin across the various G^i.

By the induction hypothesis, there are Ramsey graphs

G_1, G_2
$$G_1 := G(H_1, H_2') \quad \text{and} \quad G_2 := G(H_1', H_2)\,.$$

G^0, V^0 Let G^0 be a copy of G_1, and set $V^0 := V(G^0)$. Let $W_0', \dots, W_{n-1}'$ be the
W_i' subsets of V^0 spanning an H_2' in G^0. Thus, n is defined as the number
n of induced copies of H_2' in G^0, and we shall construct a graph G^i for
W_i'' every set W_{i-1}', $i = 1, \dots, n$. For $i = 0, \dots, n-1$, let W_i'' be the image of $V(H_2'')$ under some isomorphism $H_2' \to G^0[W_i']$.

Assume now that $G^0, \dots, G^{i-1}$ and $V^0, \dots, V^{i-1}$ have been defined for some $i \geqslant 1$, and that f has been defined on $V^1 \cup \ldots \cup V^{i-1}$ and satisfies (1) for all $j \leqslant i$. We construct G^i from G^{i-1} in two steps. For
U^{i-1} the first step, consider the set U^{i-1} of all the vertices $v \in V^{i-1}$ whose origin $f^{i-1}(v)$ lies in W_{i-1}''. (For $i = 1$, this gives $U^0 = W_0''$.) Expand

G^{i-1} to a new graph $\tilde{G}^{i-1}$ (disjoint from G^{i-1}) by replacing every vertex $u \in U^{i-1}$ with a copy $G_2(u)$ of G_2, i.e. let $G_2(u)$

$$\tilde{G}^{i-1} := G^{i-1}[U^{i-1} \to G_2]$$ $\tilde{G}^{i-1}$

(see Figures 9.3.2 and 9.3.3). Set $f(u') := u$ for all $u \in U^{i-1}$ and

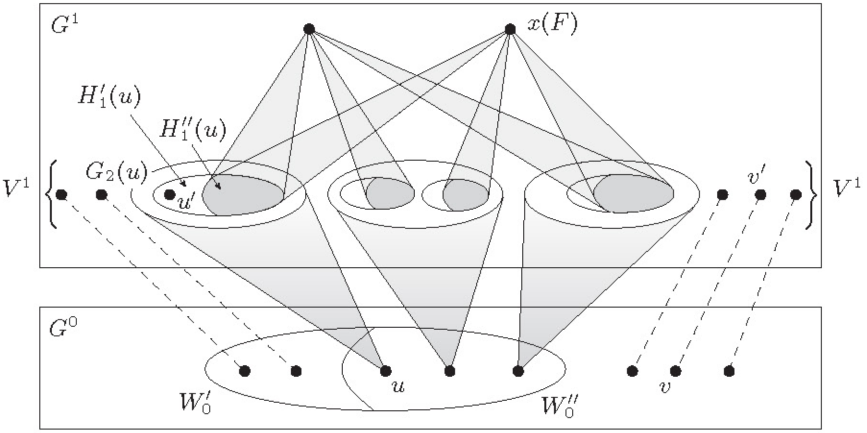

Fig. 9.3.2. The construction of G^1

$u' \in G_2(u)$, and $f(v') := v$ for all $v' = (v, \emptyset)$ with $v \in V^{i-1} \setminus U^{i-1}$. (Recall that $(v, \emptyset)$ is simply the unexpanded copy of a vertex $v \in G^{i-1}$ in $\tilde{G}^{i-1}$.) Let V^i be the set of those vertices v' or u' of $\tilde{G}^{i-1}$ for which V^i f has thus been defined, i.e. the vertices that either correspond directly to a vertex v in V^{i-1} or else belong to an expansion $G_2(u)$ of such a vertex u. Then (1) holds for i. Also, if we assume (2) inductively for $i-1$, then (2) holds again for i (in $\tilde{G}^{i-1}$). The graph $\tilde{G}^{i-1}$ is already the essential part of G^i: the part that looks like an inflated copy of G^0.

In the second step we now extend $\tilde{G}^{i-1}$ to the desired graph G^i by adding some further vertices $x \notin V^i$. Let $\mathcal{F}$ denote the set of all families $\mathcal{F}$ F of the form

$$F = \left(H_1'(u) \mid u \in U^{i-1} \right),$$

where each $H_1'(u)$ is an induced subgraph of $G_2(u)$ isomorphic to H_1'. $H_1'(u)$ (Less formally: $\mathcal{F}$ is the collection of ways to select simultaneously from each $G_2(u)$ exactly one induced copy of H_1'.) For each $F \in \mathcal{F}$, add a vertex $x(F)$ to $\tilde{G}^{i-1}$ and join it, for every $u \in U^{i-1}$, to all the vertices in $x(F)$ the image $H_1''(u) \subseteq H_1'(u)$ of H_1'' under some isomorphism from H_1' to $H_1''(u)$ the $H_1'(u) \subseteq G_2(u)$ selected by F (Fig. 9.3.2). Denote the resulting graph by G^i. This completes the inductive definition of the graphs $G^0, \ldots, G^n$. G^i

Let us now show that $G := G^n$ satisfies $(*)$. To this end, we prove the following assertion $(**)$ about G^i for $i = 0, \ldots, n$:

> *For every edge colouring with the colours 1 and 2, G^i contains either an induced H_1 coloured 1, or an induced H_2 coloured 2, or an induced subgraph H coloured 2 such that $V(H) \subseteq V^i$ and the restriction of f^i to $V(H)$ is an isomorphism between H and $G^0[W'_k]$ for some $k \in \{i, \dots, n-1\}$.* (∗∗)

Note that the third of the above cases cannot arise for $i = n$, so (∗∗) for n is equivalent to (∗) with $G := G^n$.

For $i = 0$, (∗∗) follows from the choice of G^0 as a copy of $G_1 = G(H_1, H'_2)$ and the definition of the sets W'_k. Now let $1 \leqslant i \leqslant n$, and assume (∗∗) for smaller values of i.

Let an edge colouring of G^i be given. For each $u \in U^{i-1}$ there is a copy of G_2 in G^i:

$$G^i \supseteq G_2(u) \simeq G(H'_1, H_2)\,.$$

If $G_2(u)$ contains an induced H_2 coloured 2 for some $u \in U^{i-1}$, we are done. If not, then every $G_2(u)$ has an induced subgraph $H'_1(u) \simeq H'_1$ coloured 1. Let F be the family of these graphs $H'_1(u)$, one for each $u \in U^{i-1}$, and let $x := x(F)$. If, for some $u \in U^{i-1}$, all the x–$H''_1(u)$ edges in G^i are also coloured 1, we have an induced copy of H_1 in G^i and are again done. We may therefore assume that each $H''_1(u)$ has a vertex y_u for which the edge xy_u is coloured 2. The restriction $y_u \mapsto u$ of f to

x $\quad y_u$ $\quad \hat{U}^{i-1}$

$$\hat{U}^{i-1} := \{\, y_u \mid u \in U^{i-1} \,\} \subseteq V^i$$

extends by $(v, \emptyset) \mapsto v$ to an isomorphism from

$\hat{G}^{i-1}$

$$\hat{G}^{i-1} := G^i \left[\hat{U}^{i-1} \cup \{ (v, \emptyset) \mid v \in V(G^{i-1}) \setminus U^{i-1} \} \right]$$

to G^{i-1}, and so our edge colouring of G^i induces an edge colouring of G^{i-1}. If this colouring yields an induced $H_1 \subseteq G^{i-1}$ coloured 1 or an induced $H_2 \subseteq G^{i-1}$ coloured 2, we have these also in $\hat{G}^{i-1} \subseteq G^i$ and are again home.

H' $\quad \hat{H}'$

By (∗∗) for $i-1$ we may therefore assume that G^{i-1} has an induced subgraph H' coloured 2, with $V(H') \subseteq V^{i-1}$, and such that the restriction of f^{i-1} to $V(H')$ is an isomorphism from H' to $G^0[W'_k] \simeq H'_2$ for some $k \in \{i-1, \dots, n-1\}$. Let $\hat{H}'$ be the corresponding induced subgraph of $\hat{G}^{i-1} \subseteq G^i$ (also coloured 2); then $V(\hat{H}') \subseteq V^i$,

$$f^i(V(\hat{H}')) = f^{i-1}(V(H')) = W'_k\,,$$

and $f^i\colon \hat{H}' \to G^0[W'_k]$ is an isomorphism.

If $k \geqslant i$, this completes the proof of (∗∗) with $H := \hat{H}'$; we therefore assume that $k < i$, and hence $k = i - 1$ (Fig. 9.3.3). By definition of U^{i-1} and $\hat{G}^{i-1}$, the inverse image of W''_{i-1} under the isomorphism

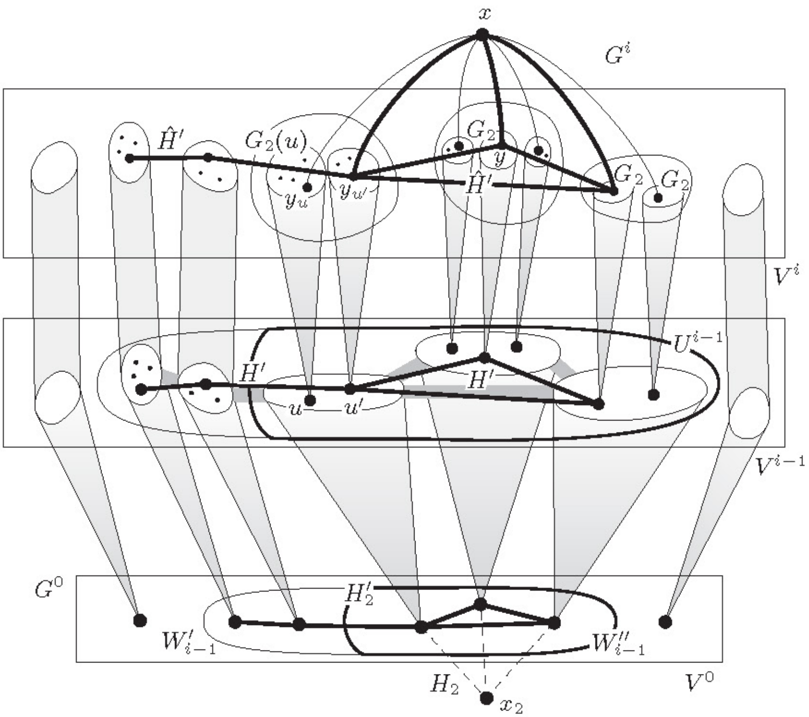

Fig. 9.3.3. A monochromatic copy of H_2 in G^i

$f^i\colon \hat{H}' \to G^0[W'_{i-1}]$ is a subset of $\hat{U}^{i-1}$. Since x is joined to precisely those vertices of $\hat{H}'$ that lie in $\hat{U}^{i-1}$, and all these edges xy_u have colour 2, the graph $\hat{H}'$ and x together induce in G^i a copy of H_2 coloured 2, and the proof of $(**)$ is complete. □

Let us return once more to the reformulation of Ramsey's theorem considered at the beginning of this section: for every graph H there exists a graph G such that every 2-colouring of the edges of G yields a monochromatic $H \subseteq G$. The graph G for which this follows at once from Ramsey's theorem is a sufficiently large complete graph. If we ask, however, that G shall not contain any complete subgraphs larger than those in H, i.e. that $\omega(G) = \omega(H)$, the problem again becomes difficult—even if we do not require H to be induced in G.

Our second proof of Theorem 9.3.1 solves both problems at once: given H, we shall construct a Ramsey graph for H with the same clique number as H.

For this proof, i.e. for the remainder of this section, let us view bipartite graphs P as triples (V_1, V_2, E), where V_1 and V_2 are the two vertex classes and $E \subseteq V_1 \times V_2$ is the set of edges. The reason for this more explicit notation is that we want embeddings between bipartite *bipartite* graphs to respect their bipartitions: given another bipartite graph $P' = (V_1', V_2', E')$, an injective map $\varphi\colon V_1 \cup V_2 \to V_1' \cup V_2'$ will be called an *embedding* of P in P' if $\varphi(V_i) \subseteq V_i'$ for $i = 1, 2$ and $\varphi(v_1)\varphi(v_2)$ is an edge of P' if and only if v_1v_2 is an edge of P. (Note that such embeddings are 'induced'.) Instead of $\varphi\colon V_1 \cup V_2 \to V_1' \cup V_2'$ we may simply write $\varphi\colon P \to P'$.

embedding $P \to P'$

We need two lemmas.

Lemma 9.3.2. *Every bipartite graph can be embedded in a bipartite graph of the form* $(X, [X]^k, E)$ *with* $E = \{\, xY \mid x \in Y \,\}$.

E

Proof. Let P be any bipartite graph, with vertex classes $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_m\}$, say. Let X be a set with $2n + m$ elements, say

$$X = \{x_1, \dots, x_n,\ y_1, \dots, y_n,\ z_1, \dots, z_m\}\,;$$

we shall define an embedding $\varphi\colon P \to (X, [X]^{n+1}, E)$.

Let us start by setting $\varphi(a_i) := x_i$ for all $i = 1, \dots, n$. Which $(n+1)$-sets $Y \subseteq X$ are suitable candidates for the choice of $\varphi(b_i)$ for a given vertex b_i? Clearly those adjacent exactly to the images of the neighbours of b_i, i.e. those satisfying

$$Y \cap \{x_1, \dots, x_n\} = \varphi(N_P(b_i))\,. \tag{1}$$

Since $d(b_i) \leqslant n$, the requirement of (1) leaves at least one of the $n+1$ elements of Y unspecified. In addition to $\varphi(N_P(b_i))$, we may therefore include in each $Y = \varphi(b_i)$ the vertex z_i as an 'index'; this ensures that $\varphi(b_i) \neq \varphi(b_j)$ for $i \neq j$, even when b_i and b_j have the same neighbours in P. To specify the sets $Y = \varphi(b_i)$ completely, we finally fill them up with 'dummy' elements y_j until $|Y| = n + 1$. □

Our second lemma already covers the bipartite case of the theorem: it says that every bipartite graph has a Ramsey graph—even a bipartite one.

Lemma 9.3.3. *For every bipartite graph P there exists a bipartite graph P' such that for every 2-colouring of the edges of P' there is an embedding $\varphi\colon P \to P'$ for which all the edges of $\varphi(P)$ have the same colour.*

(9.1.3) *Proof.* We may assume by Lemma 9.3.2 that P has the form $(X, [X]^k, E)$
P, X, k, E with $E = \{\, xY \mid x \in Y \,\}$. We show the assertion for the graph $P' :=$
P', X', k' $(X', [X']^{k'}, E')$, where $k' := 2k - 1$, X' is any set of cardinality

$$|X'| = R\left(k',\, 2\tbinom{k'}{k},\, k\,|X| + k - 1\right),$$

(this is the Ramsey number defined after Theorem 9.1.3), and

$$E' := \{\, x'Y' \mid x' \in Y' \,\}.$$ E'

Let us then colour the edges of P' with two colours α and β. Of the $|Y'| = 2k-1$ edges incident with a vertex $Y' \in [X']^{k'}$, at least k must have the same colour. For each Y' we may therefore choose a fixed k-set $Z' \subseteq Y'$ such that all the edges $x'Y'$ with $x' \in Z'$ have the same colour; we shall call this colour *associated* with Y'. α, β Z' *associated*

The sets Z' can lie within their supersets Y' in $\binom{k'}{k}$ ways, as follows. Let X' be linearly ordered. Then for every $Y' \in [X']^{k'}$ there is a unique order-preserving bijection $\sigma_{Y'}\colon Y' \to \{1,\dots,k'\}$, which maps Z' to one of $\binom{k'}{k}$ possible images. $\sigma_{Y'}$

We now colour $[X']^{k'}$ with the $2\binom{k'}{k}$ elements of the set

$$[\{1,\dots,k'\}]^k \times \{\alpha,\beta\}$$

as colours, giving each $Y' \in [X']^{k'}$ as its colour the pair $(\sigma_{Y'}(Z'),\gamma)$, where γ is the colour α or β associated with Y'. Since $|X'|$ was chosen as the Ramsey number with parameters k', $2\binom{k'}{k}$ and $k\,|X|+k-1$, we know that X' has a monochromatic subset W of cardinality $k\,|X|+k-1$. All Z' with $Y' \subseteq W$ thus lie within their Y' in the same way, i.e. there exists an $S \in [\{1,\dots,k'\}]^k$ such that $\sigma_{Y'}(Z') = S$ for all $Y' \in [W]^{k'}$, and all $Y' \in [W]^{k'}$ are associated with the same colour, say with α. W α

We now construct the desired embedding φ of P in P'. We first define φ on $X =: \{x_1,\dots,x_n\}$, choosing images $\varphi(x_i) =: w_i \in W$ so that $w_i < w_j$ in our ordering of X' whenever $i < j$. Moreover, we choose the w_i so that exactly $k-1$ elements of W are smaller than w_1, exactly $k-1$ lie between w_i and w_{i+1} for $i = 1,\dots,n-1$, and exactly $k-1$ are bigger than w_n. Since $|W| = kn+k-1$, this can indeed be done (Fig. 9.3.4). $\varphi|_X$ x_i, w_i, n

We now define φ on $[X]^k$. Given $Y \in [X]^k$, we wish to choose $\varphi(Y) =: Y' \in [X']^{k'}$ so that the neighbours of Y' among the vertices in $\varphi(X)$ are precisely the images of the neighbours of Y in P, i.e. the k vertices $\varphi(x)$ with $x \in Y$, and so that all these edges at Y' are coloured α. To find such a set Y', we first fix its subset Z' as $\{\,\varphi(x) \mid x \in Y\,\}$ (these are k vertices of type w_i) and then extend Z' by $k'-k$ further vertices $u \in W \setminus \varphi(X)$ to a set $Y' \in [W]^{k'}$, in such a way that Z' lies correctly within Y', i.e. so that $\sigma_{Y'}(Z') = S$. This can be done, because $k-1 = k'-k$ other vertices of W lie between any two w_i. Then $\varphi|_{[X]^k}$

$$Y' \cap \varphi(X) = Z' = \{\,\varphi(x) \mid x \in Y\,\},$$

so Y' has the correct neighbours in $\varphi(X)$, and all the edges between Y' and these neighbours are coloured α (because those neighbours lie in Z' and Y' is associated with α). Finally, φ is injective on $[X]^k$: the images Y' of different vertices Y are distinct, because their intersections with $\varphi(X)$ differ. Hence, our map φ is indeed an embedding of P in P'. □

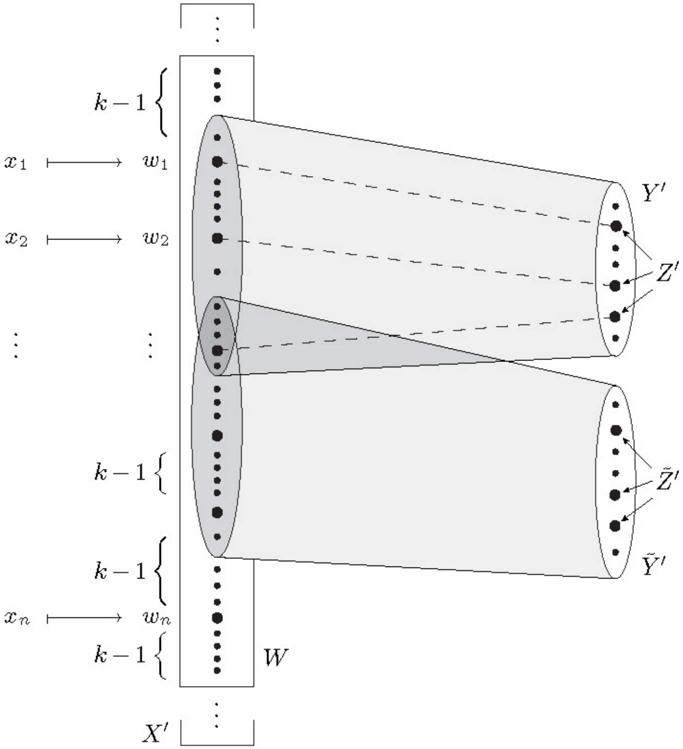

Fig. 9.3.4. The graph of Lemma 9.3.3

Second proof of Theorem 9.3.1. Let H be given as in the theorem, and let $n := R(r)$ be the Ramsey number of $r := |H|$. Then, for every 2-colouring of its edges, the graph $K = K^n$ contains a monochromatic copy of H—although not necessarily induced.

r, n

K

We start by constructing a graph G^0, as follows. Imagine the vertices of K to be arranged in a column, and replace every vertex by a row of $\binom{n}{r}$ vertices. Then each of the $\binom{n}{r}$ columns arising can be associated with one of the $\binom{n}{r}$ ways of embedding $V(H)$ in $V(K)$; let us furnish this column with the edges of such a copy of H. The graph G^0 thus arising consists of $\binom{n}{r}$ disjoint copies of H and $(n-r)\binom{n}{r}$ isolated vertices (Fig. 9.3.5).

In order to define G^0 formally, we assume that $V(K) = \{1, \dots, n\}$ and choose copies $H_1, \dots, H_{\binom{n}{r}}$ of H in K with pairwise distinct vertex sets. (Thus, on each r-set in $V(K)$ we have one fixed copy H_j of H.) We then define

G^0

$$V(G^0) := \left\{ (i,j) \mid i = 1, \dots, n;\ j = 1, \dots, \tbinom{n}{r} \right\}$$

and

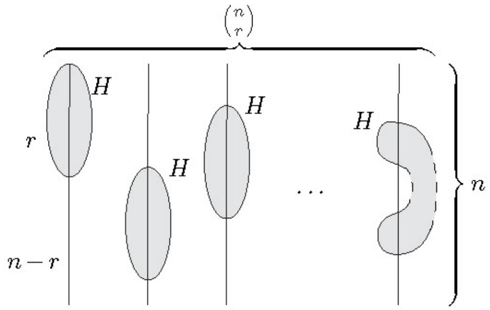

Fig. 9.3.5. The graph G^0

$$E(G^0) := \bigcup_{j=1}^{\binom{n}{r}} \left\{ (i,j)(i',j) \mid ii' \in E(H_j) \right\}.$$

The idea of the proof now is as follows. Our aim is to reduce the general case of the theorem to the bipartite case dealt with in Lemma 9.3.3. Applying the lemma iteratively to all the pairs of rows of G^0, we construct a very large graph G such that for every edge colouring of G there is an induced copy of G^0 in G that is monochromatic on all the bipartite subgraphs induced by its pairs of rows, i.e. in which edges between the same two rows always have the same colour. The projection of this $G^0 \subseteq G$ to $\{1,\dots,n\}$ (by contracting its rows) then defines an edge colouring of K. (If the contraction does not yield all the edges of K, colour the missing edges arbitrarily.) By the choice of $|K|$, some $K^r \subseteq K$ will be monochromatic. The H_j inside this K^r then occurs with the same colouring in the jth column of our G^0, where it is an induced subgraph of G^0, and hence of G.

Formally, we shall define a sequence $G^0,\dots,G^m$ of n-partite graphs G^k, with n-partition $\{V_1^k,\dots,V_n^k\}$ say, and then let $G := G^m$. The graph G^0 has been defined above; let $V_1^0,\dots,V_n^0$ be its rows:

$$V_i^0 := \left\{ (i,j) \mid j = 1,\dots,\binom{n}{r} \right\}.$$ V_i^0

Now let $e_1,\dots,e_m$ be an enumeration of the edges of K. For $k = 0,\dots,m-1$, construct G^{k+1} from G^k as follows. If $e_{k+1} = i_1 i_2$, say, let $P = (V_{i_1}^k, V_{i_2}^k, E)$ be the bipartite subgraph of G^k induced by its i_1th and i_2th row. By Lemma 9.3.3, P has a bipartite Ramsey graph $P' = (W_1, W_2, E')$. We wish to define $G^{k+1} \supseteq P'$ in such a way that every (monochromatic) embedding $P \to P'$ can be extended to an embedding $G^k \to G^{k+1}$ respecting their n-partitions. Let $\{\varphi_1,\dots,\varphi_q\}$ be the set of all embeddings of P in P', and let

[margin: e_k, m; i_1, i_2; P; P'; W_1, W_2; φ_p, q]

$$V(G^{k+1}) := V_1^{k+1} \cup \ldots \cup V_n^{k+1},$$

where

$$V_i^{k+1} := \begin{cases} W_1 & \text{for } i = i_1 \\ W_2 & \text{for } i = i_2 \\ \bigcup_{p=1}^{q} (V_i^k \times \{p\}) & \text{for } i \notin \{i_1, i_2\}. \end{cases}$$

(Thus for $i \neq i_1, i_2$, we take as V_i^{k+1} just q disjoint copies of V_i^k.) We now define the edge set of G^{k+1} so that the obvious extensions of φ_p to all of $V(G^k)$ become embeddings of G^k in G^{k+1}: for $p = 1, \ldots, q$, let $\psi_p: V(G^k) \to V(G^{k+1})$ be defined by

$$\psi_p(v) := \begin{cases} \varphi_p(v) & \text{for } v \in P \\ (v, p) & \text{for } v \notin P \end{cases}$$

and let

$$E(G^{k+1}) := \bigcup_{p=1}^{q} \{ \psi_p(v)\psi_p(v') \mid vv' \in E(G^k) \}.$$

Now for every 2-colouring of its edges, G^{k+1} contains an induced copy $\psi_p(G^k)$ of G^k whose edges in P, i.e. those between its i_1th and i_2th row, have the same colour: just choose p so that $\varphi_p(P)$ is the monochromatic induced copy of P in P' that exists by Lemma 9.3.3.

We claim that $G := G^m$ satisfies the assertion of the theorem. So let a 2-colouring of the edges of G be given. By the construction of G^m from G^{m-1}, we can find in G^m an induced copy of G^{m-1} such that for $e_m = ii'$ all edges between the ith and the i'th row have the same colour. In the same way, we find inside this copy of G^{m-1} an induced copy of G^{m-2} whose edges between the ith and the i'th row have the same colour also for $ii' = e_{m-1}$. Continuing in this way, we finally arrive at an induced copy of G^0 in G such that, for each pair (i, i'), all the edges between V_i^0 and $V_{i'}^0$ have the same colour. As shown earlier, this G^0 contains a monochromatic induced copy H_j of H. □

9.4 Ramsey properties and connectivity

According to Ramsey's theorem, every large enough graph G has a very dense or a very sparse induced subgraph of given order, a K^r or $\overline{K^r}$. If we assume that G is connected, we can say a little more:

Proposition 9.4.1. *For every $r \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that every connected graph of order at least n contains K^r, $K_{1,r}$ or P^r as an induced subgraph.*

Proof. Let $d+1$ be the Ramsey number of r, let $n \geqslant \frac{d}{d-2}(d-1)^r$, and let G be a graph of order at least n. If G has a vertex v of degree at least $d+1$ then, by Theorem 9.1.1 and the choice of d, either $N(v)$ induces a K^r in G or $\{v\} \cup N(v)$ induces a $K_{1,r}$. On the other hand, if $\Delta(G) \leqslant d$, then by Proposition 1.3.3 G has radius $> r$, and hence contains two vertices at a distance $\geqslant r$. Any shortest path in G between these two vertices contains a P^r. □ (1.3.3)

In principle, we could now look for a similar set of 'unavoidable' k-connected subgraphs for any given connectivity k. To keep these 'unavoidable sets' small, it helps to relax the containment relation from 'induced subgraph' for $k = 1$ (as above) to 'topological minor' for $k = 2$, and on to 'minor' for $k = 3$ and $k = 4$. For larger k, no similar results are known.

Proposition 9.4.2. *For every $r \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that every 2-connected graph of order at least n contains C^r or $K_{2,r}$ as a topological minor.*

Proof. Let d be the n associated with r in Proposition 9.4.1, and let G be a 2-connected graph with at least $\frac{d}{d-2}(d-1)^r$ vertices. By Proposition 1.3.3, either G has a vertex of degree $> d$ or $\mathrm{diam}(G) \geqslant \mathrm{rad}(G) > r$. (1.3.3) (3.3.6)

In the latter case let $a, b \in G$ be two vertices at distance $> r$. By Menger's theorem (3.3.6), G contains two independent a–b paths. These form a cycle of length $> r$.

Assume now that G has a vertex v of degree $> d$. Since G is 2-connected, $G - v$ is connected and thus has a spanning tree; let T be a minimal tree in $G - v$ that contains all the neighbours of v. Then every leaf of T is a neighbour of v. By the choice of d, either T has a vertex of degree $\geqslant r$ or T contains a path of length $\geqslant r$, without loss of generality linking two leaves. Together with v, such a path forms a cycle of length $\geqslant r$. A vertex u of degree $\geqslant r$ in T can be joined to v by r independent paths through T, to form a $TK_{2,r}$. □

Theorem 9.4.3. (Oporowski, Oxley & Thomas 1993)
For every $r \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that every 3-connected graph of order at least n contains a wheel of order r or a $K_{3,r}$ as a minor.

Let us call a graph of the form $C^n * \overline{K^2}$ $(n \geqslant 4)$ a *double wheel*, the 1-skeleton of a triangulation of the cylinder as in Fig. 9.4.1 a *crown*, and the 1-skeleton of a triangulation of the Möbius strip a *Möbius crown*.

Theorem 9.4.4. (Oporowski, Oxley & Thomas 1993)
For every $r \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that every 4-connected graph with at least n vertices has a minor of order $\geqslant r$ that is a double wheel, a crown, a Möbius crown, or a $K_{4,r}$.

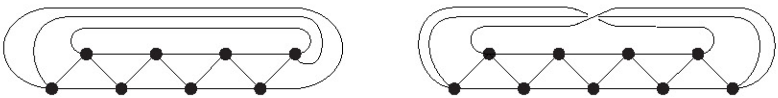

Fig. 9.4.1. A crown and a Möbius crown

Note that the graphs listed in Theorems 9.4.3 and 9.4.4 are themselves 3-connected resp. 4-connected, as required.

At first glance, the 'unavoidable' substructures presented in the four theorems above may seem to be chosen somewhat arbitrarily. In fact, the contrary is true: these sets are smallest possible, and as such unique.

non-trivial property

To make this precise, call a graph property *non-trivial* if it contains graphs of infinitely many isomorphism types. Given two such properties $\mathcal{P}, \mathcal{P}'$ and an order relation $\leqslant$ between graphs (such as the subgraph relation $\subseteq$, or the minor relation $\preccurlyeq$), write $\mathcal{P} \leqslant \mathcal{P}'$ if for every $G \in \mathcal{P}$ there is a $G' \in \mathcal{P}'$ such that $G \leqslant G'$. If $\mathcal{P} \leqslant \mathcal{P}'$ as well as $\mathcal{P} \geqslant \mathcal{P}'$, call $\mathcal{P}$ and $\mathcal{P}'$ *equivalent* and write $\mathcal{P} \sim \mathcal{P}'$. For example, if $\leqslant$ is the subgraph relation, $\mathcal{P}$ is the class of all paths, $\mathcal{P}'$ is the class of paths of even length, and $\mathcal{S}$ is the class of all subdivisions of stars, then $\mathcal{P} \sim \mathcal{P}' \leqslant \mathcal{S} \not\leqslant \mathcal{P}$.

$\leqslant$

$\sim$

Kuratowski set

Given a non-trivial graph property $\mathcal{G}$, call a finite set $\{\mathcal{P}_1, \dots, \mathcal{P}_k\}$ of non-trivial graph properties $\mathcal{P}_i \subseteq \mathcal{G}$ a *Kuratowski set* for $\mathcal{G}$ and $\leqslant$ if the $\mathcal{P}_i$ are incomparable (i.e., $\mathcal{P}_i \not\leqslant \mathcal{P}_j$ whenever $i \neq j$) and for every non-trivial graph property $\mathcal{P} \subseteq \mathcal{G}$ there is an i such that $\mathcal{P}_i \leqslant \mathcal{P}$. Such a Kuratowski set $\{\mathcal{P}_1, \dots, \mathcal{P}_k\}$ is unique up to equivalence: if $\{\mathcal{Q}_1, \dots, \mathcal{Q}_\ell\}$ is another Kuratowski set for $\mathcal{G}$ then $\ell = k$ and, with suitable enumeration, $\mathcal{Q}_i \sim \mathcal{P}_i$ for $i = 1, \dots, k$. (Why?)

The essence of our last four theorems can now be stated more comprehensively, as follows. Let us say *k-connectedness* for the class of all k-connected finite graphs, and *connectedness* for 1-connectedness.

Theorem 9.4.5.

(i) *The stars and the paths form the (2-element) Kuratowski set for connectedness and the subgraph relation.*

(ii) *The cycles and the graphs $K_{2,r}$ ($r \in \mathbb{N}$) form the (2-element) Kuratowski set for 2-connectedness and the topological minor relation.*

(iii) *The wheels and the graphs $K_{3,r}$ ($r \in \mathbb{N}$) form the (2-element) Kuratowski set for 3-connectedness and the minor relation.*

(iv) *The double wheels, the crowns, the Möbius crowns, and the graphs $K_{4,r}$ ($r \in \mathbb{N}$) form the (4-element) Kuratowski set for 4-connectedness and the minor relation.* □

Exercises

1.$^-$ Determine the Ramsey number $R(3)$.

2.$^-$ Deduce the case $k = 2$ (but c arbitrary) of Theorem 9.1.3 directly from Theorem 9.1.1.

3. An *arithmetic progression* is an increasing sequence of numbers of the form $a, a+d, a+2d, a+3d \ldots$. *Van der Waerden's theorem* says that no matter how we partition the natural numbers into two classes, one of these classes will contain arbitrarily long arithmetic progressions. Must there even be an infinite arithmetic progression in one of the classes?

4. Can you improve the exponential upper bound on the Ramsey number $R(n)$ for perfect graphs?

5.$^+$ Construct a graph on $\mathbb{R}$ that has neither a complete nor an edgeless induced subgraph on $|\mathbb{R}| = 2^{\aleph_0}$ vertices. (So Ramsey's theorem does not extend to uncountable sets.)

6.$^+$ Prove the edge version of the Erdős-Pósa theorem (2.3.2): there exists a function $g\colon \mathbb{N} \to \mathbb{R}$ such that, given $k \in \mathbb{N}$, every graph contains either k edge-disjoint cycles or a set of at most $g(k)$ edges meeting all its cycles.

   (Hint. Consider in each component a normal spanning tree T. If T has many chords xy, use any regular pattern of how the paths xTy intersect to find many edge-disjoint cycles.)

7.$^+$ Use Ramsey's theorem to show that for any $k, \ell \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that every sequence of n distinct integers has an increasing subsequence of length $k+1$ or a decreasing subsequence of length $\ell + 1$. Prove that $n = k\ell + 1$ has this property, but that $n = k\ell$ does not.

8. Show that for every $k \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that among any n points in the plane, no three of them collinear, there are k points spanning a convex k-gon, i.e. such that none of them lies in the convex hull of the others.

9. Show that for every $k \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that, for every partition of $\{1, \ldots, n\}$ into k sets, at least one of the subsets contains numbers x, y, z such that $x + y = z$.

10. Let $(X, \leqslant)$ be a totally ordered set, and let $G = (V, E)$ be the graph on $V := [X]^2$ with $E := \{(x,y)(x',y') \mid x < y = x' < y'\}$.

    (i) Show that G contains no triangle.

    (ii) Show that $\chi(G)$ will get arbitrarily large if $|X|$ is chosen large enough.

11. A family of sets is called a Δ*-system* if every two of the sets have the same intersection. Show that every infinite family of sets of the same finite cardinality contains an infinite Δ-system.

12. Prove that for every $r \in \mathbb{N}$ and every tree T there exists a $k \in \mathbb{N}$ such that every graph G with $\chi(G) \geqslant k$ and $\omega(G) < r$ contains a subdivision of T in which no two branch vertices are adjacent in G (unless they are adjacent in T).

13. Let $m, n \in \mathbb{N}$, and assume that $m-1$ divides $n-1$. Show that every tree T of order m satisfies $R(T, K_{1,n}) = m+n-1$.

14. Prove that $2^c < R(2,c,3) \leqslant 3c!$ for every $c \in \mathbb{N}$.

    (Hint. Induction on c.)

15. Explain why, in the proof of Theorem 9.2.2, choosing ϵ small enough can ensure that the regularity graph R contains a copy of K^ℓ, although some of the pairs (V_i, V_j) in G may not be ϵ-regular. Your explanation may use that $t_{\ell-1}(k) \approx \frac{\ell-2}{\ell-1}\binom{k}{2}$, but should contain no calculations.

16.$^-$ Derive the statement $(*)$ in the first proof of Theorem 9.3.1 from the theorem itself, i.e. show that $(*)$ is only formally stronger than the theorem.

17.$^-$ How is n defined in the first proof of Theorem 9.3.1? Could it be zero, and if so how does the proof work then?

18. Show that, given any two graphs H_1 and H_2, there exists a graph $G = G(H_1, H_2)$ such that, for every *vertex*-colouring of G with colours 1 and 2, there is either an induced copy of H_1 coloured 1 or an induced copy of H_2 coloured 2 in G.

19. Show that the Ramsey graph G for H constructed in the second proof of Theorem 9.3.1 does indeed satisfy $\omega(G) = \omega(H)$.

20. In the second proof of Theorem 9.3.1, is it really necessary to equip G^{k+1} for $i \notin \{i_1, i_2\}$ with separate disjoint copies of V_k^i, one for every p, or could we define G^{k+1} from G^k by just replacing P with P' and joining it to the other V_i^k in the right way?

21.$^-$ Show that any Kuratowski set for a non-trivial graph property is unique up to equivalence.

22. Deduce Theorem 9.4.5 (iii) from Theorem 9.4.3, and vice versa.

Notes

Due to increased interaction with research on random and pseudo-random[4] structures (the latter being provided, for example, by the regularity lemma), the Ramsey theory of graphs has recently seen a period of major activity and advance. Theorem 9.2.2 is an early example of this development.

For the more classical approach, the introductory text by R.L. Graham, B.L. Rothschild & J.H. Spencer, *Ramsey Theory* (2nd edn.), Wiley 1990, makes stimulating reading. This book includes a chapter on graph Ramsey theory, but is not confined to it. Surveys of finite and infinite Ramsey theory are given by J. Nešetřil and A. Hajnal in their chapters in the *Handbook of Combinatorics* (R.L. Graham, M. Grötschel & L. Lovász, eds.), North-Holland

[4] Concrete graphs whose structure resembles the structure expected of a random graph are called *pseudo-random*. For example, the bipartite graphs spanned by an ϵ-regular pair of vertex sets in a graph are pseudo-random.

1995. The Ramsey theory of infinite sets forms a substantial part of combinatorial set theory, and is treated in depth in P. Erdős, A. Hajnal, A. Máté & R. Rado, *Combinatorial Set Theory*, North-Holland 1984. An attractive collection of highlights from various branches of Ramsey theory, including applications in algebra, geometry and point-set topology, is offered in B. Bollobás, *Graph Theory*, Springer GTM 63, 1979.

Ramsey's original theorem, Theorem 9.1.1, is from F.P. Ramsey, On a problem of formal logic, *Proc. Lond. Math. Soc.* **2** (1930), 264–286. The Erdős-Hajnal conjecture is taken from P. Erdős & A. Hajnal, Ramsey-type theorems, *Discrete Appl. Math.* **25** (1989), 37–52. A survey on the state of the art a couple of years ago was given by M. Chudnovsky, The Erdős-Hajnal conjecture—a survey, *J. Graph Theory* **75** (2014), 178–190, arXiv:1606.08827.

Theorem 9.2.2 is due to V. Chvátal, V. Rödl, E. Szemerédi & W.T. Trotter, The Ramsey number of a graph with bounded maximum degree, *J. Comb. Theory B* **34** (1983), 239–243. Our proof follows the sketch in J. Komlós & M. Simonovits, Szemerédi's Regularity Lemma and its applications in graph theory, in (D. Miklós, V.T. Sós & T. Szőnyi, eds.) *Paul Erdős is 80*, Vol. 2, Proc. Colloq. Math. Soc. János Bolyai (1996). The theorem marks a breakthrough towards a conjecture of Burr and Erdős (1975), which asserts that the Ramsey numbers of graphs with bounded *average* degree in every subgraph are linear: for every $d \in \mathbb{N}$, the conjecture says, there exists a constant c such that $R(H) \leqslant c\,|H|$ for all graphs H with $d(H') \leqslant d$ for all $H' \subseteq H$. This conjecture has been verified approximately by A. Kostochka and B. Sudakov, On Ramsey numbers of sparse graphs, *Comb. Probab. Comput.* **12** (2003), 627–641, who proved that $R(H) \leqslant |H|^{1+o(1)}$.

Our first proof of Theorem 9.3.1 is based on W. Deuber, A generalization of Ramsey's theorem, in (A. Hajnal, R. Rado & V.T. Sós, eds.) *Infinite and finite sets*, North-Holland 1975. The same volume contains the alternative proof of this theorem by Erdős, Hajnal and Pósa. Rödl proved the same result in his MSc thesis at Charles University, Prague, in 1973. Our second proof of Theorem 9.3.1, which preserves the clique number of H for G, is due to J. Nešetřil & V. Rödl, Simple proof of the existence of restricted Ramsey graphs by means of a partite construction, *Combinatorica* **1** (1981), 199–202. These authors later refined their methods to obtain an even stronger version of Theorem 9.3.1, with a proof that doubles as a construction of graphs of large chromatic number and girth (Theorem 11.2.2); see J. Nešetřil & V. Rödl, Sparse Ramsey graphs, *Combinatorica* **4** (1984), 71–78.

The two theorems in Section 9.4 are based on B. Oporowski, J. Oxley & R. Thomas, Typical subgraphs of 3- and 4-connected graphs, *J. Comb. Theory B* **57** (1993), 239–257. The have been generalized to arbitrary k, but for a weaker 'global' notion of connectivity as often used in graph minor theory, by Benson Joeris, Connectivity, tree-decompositions and unavoidable minors, PhD thesis, University of Waterloo (2015).